XIIth HAND WRITTEN MATERIALS

CHAPTER – 7

APPLICATION OF DIFFERENTIAL CALCULUS

MANIKANDAN S

P.G Teacher Mathematics

9655536357

CHAPTER 7

APPLICATIONS OF DIFFERENTIAL CALCULUS

EXERCISE 7.1

1. A point moves along a straight line in such a way that after t seconds its distance from the origin is $s = 2t^2 + 3t$ metres.

(i) Find the average velocity of the points between $t=3$ and $t=6$ seconds.

Given $s = 2t^2 + 3t$

$$s(3) = 2(3)^2 + 3(3)$$
$$= 2(9) + 9 = 18 + 9$$
$$s(3) = 27\text{ m} \quad \ldots (1)$$
$$s(6) = 2(6)^2 + 3(6)$$
$$= 2(36) + 18 = 72 + 18$$
$$s(6) = 90\text{ m} \quad \ldots (2)$$

$$\text{Average velocity} = \frac{s(6) - s(3)}{6-3} \quad \text{[using (1) and (2)]}$$
$$= \frac{90-27}{3}$$
$$= \frac{63}{3} = 21\text{ m/s}$$

(ii) Find the instantaneous velocities at $t=3$ and $t=6$ seconds.

Instantaneous velocity $v(t) = \frac{ds}{dt} = 4t + 3$

[By differentiating $2t^2+3t$]

Instantaneous velocity at $t=3 \Rightarrow 4(3)+3 \Rightarrow 12+3$

$$v(3) = 15\text{ m/s}$$

Instantaneous velocity at $t=6 \Rightarrow 4(6)+3 \Rightarrow 24+3$

$$v(6) = 27\text{ m/s}$$

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2 A camera is accidently knocked off an edge of a cliff 400 ft high. The camera falls a distance of $s = 16t^2$ in t seconds.

(i) How long does the camera fall before it hits the ground?

Given $s(t) = 16t^2$, height = 400 ft

$$16t^2 = 400$$

$$t^2 = \frac{400}{16} = \frac{100}{4} = 25$$

$$t^2 = 25$$

$$t = 5 \text{ sec.}$$

(ii) What is the average velocity with which the camera falls during the last 2 seconds?

$$\text{Average velocity} = \frac{ds}{dt} = 32t$$

[∵ by diff $16t^2$]

when $t = 2$ sec,

$$\text{Average in last 2 sec} = \frac{V \text{ at } t=3 + V \text{ at } t=5}{5-3}$$

$$= \frac{32(3) + 32(5)}{2} = \frac{96+160}{2}$$

$$= \frac{256}{2}$$

$$= 128 \text{ f/sec}$$

(iii) What is the instantaneous velocity of the camera when it hits the ground?

$$\text{Instantaneous velocity} = \frac{ds}{dt} = 32t$$

when $t = 5$ sec

$$\text{velocity} = 32(5)$$

$$= 160 \text{ ft/sec.}$$

5. A particle moves along a line according to the law $S(t) = 2t^3 - 9t^2 + 12t - 4$, where $t \geq 0$ 291

(i) At what times the particle changes direction?

Given $S(t) = 2t^3 - 9t^2 + 12t - 4$, $t \geq 0$

On differentiating,

$$V(t) = 6t^2 - 18t + 12 \quad \ldots (1)$$

$$= 6(t^2 - 3t + 2)$$

$$= 6(t-1)(t-2)$$

2
-1 | -2

Now, $V(t) = 0$

$$6(t-1)(t-2) = 0$$

$$t = 1, 2.$$

The particle changes direction when $V(t)$ changes its sign

If $0 \leq t < 1$ then both $(t-1)$ and $(t-2) < 0$ $\Rightarrow V(t) > 0$

If $1 < t < 2$ then $(t-1) > 0$ and $(t-2) < 0$ $\Rightarrow V(t) < 0$

If $t > 2$ then both $(t-1)$ and $(t-2) > 0$ $\Rightarrow V(t) > 0$

$\therefore$ The particle changes direction when $t = 1$ and $t = 2$ sec.

(ii) Find the total distance travelled by the particle in the first 4 seconds.

Total distance travelled by the particle in the first 4 seconds is $|S(0) - S(1)| + |S(1) - S(2)| + |S(2) - S(4)|$

$[S(t) = 2t^3 - 9t^2 + 12t - 4]$

$S(0) = 2(0)^3 - 9(0)^2 + 12(0) - 4 = 0 - 0 + 0 - 4 \Rightarrow \boxed{S(0) = -4}$

$S(1) = 2(1)^3 - 9(1)^2 + 12(1) - 4 = 2 - 9 + 12 - 4 \Rightarrow \boxed{S(1) = 1}$

$S(2) = 2(2)^3 - 9(2)^2 + 12(2) - 4 = 2(8) - 9(4) + 24 - 4 \Rightarrow 16 - 36 + 20$

$\Rightarrow \boxed{S(2) = 0}$

$S(4) = 2(4)^3 - 9(4)^2 + 12(4) - 4 = 2(64) - 9(16) + 48 - 4 \Rightarrow 128 - 144 + 44$

$\Rightarrow \boxed{S(4) = 28}$

$\therefore |S(0) - S(1)| + |S(1) - S(2)| + |S(2) - S(4)|$

$\Rightarrow |-4-1| + |1-0| + |0-28| \Rightarrow |-5| + |1| + |-28|$

$\Rightarrow 5 + 1 + 28 = 34$ m.

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(iii) Find the particle acceleration each time the velocity is zero.

Acceleration $A = 12t - 18$

[acceleration $= \frac{dv}{dt}$] [From (1)]

when $t=1$, $A = 12(1)-18 = -6\, m/s^2$

when $t=2$, $A = 12(2)-18 = 6\, m/s^2$

4. If the volume of a cube of side length x is $V=x^3$. Find the rate of change of the volume with respect to x when $x=5$ units.

Given $V=x^3$

Differentiating with respect to x we get,

$$\frac{dV}{dx} = 3x^2$$

when $x=5$,

$$\frac{dV}{dx} = 3(5^2) = 75$$

$\therefore \frac{dV}{dx}$ when $x=5$ is 75 units.

5. If the mass $m(x)$ (in kilograms) of a thin rod of length x (in metres) is given by, $m(x) = \sqrt{3x}$ then what is the rate of change of mass with respect to the length when it is $x=3$ and $x=27$ metres.

Given $m(x) = \sqrt{3x} = \sqrt{3}.x^{1/2}$

Differentiating with respect to 'x' we get,

$$\frac{dm}{dx} = \sqrt{3}.\frac{1}{2}x^{1/2-1} \Rightarrow \frac{\sqrt{3}}{2}x^{-\frac{1}{2}} \Rightarrow \frac{\sqrt{3}}{2\sqrt{x}}$$

when $x=3$, $\frac{dm}{dx} = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2}$ Kg/m

when $x=27$, $\frac{dm}{dx} = \frac{\sqrt{3}}{2\sqrt{27}} = \frac{\sqrt{3}}{2.3\sqrt{3}} = \frac{1}{6}$ Kg/m

$$\begin{array}{r|l} 3 & 27 \\ 3 & 9 \\ 3 & 3 \\ & 1 \end{array}$$

6. A stone is dropped into a pond causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate at 2 cm per second. When the radius is 5cm find the rate of changing of the total area of the disturbed water? 293

Let r be the radius of the ripple and A be the area of the ripple.

Given $\frac{dr}{dt} = 2$ cm/sec and $r = 5$ cm ...(1)

We know $A = \pi r^2$

Differentiating with respect to 't' we get,

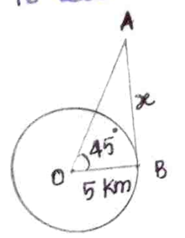
$$\frac{dA}{dt} = \pi (2r) \cdot \frac{dr}{dt}$$

$$= \pi (2)(5)(2) \quad \text{[using (1)]}$$

$$\frac{dA}{dt} = 20\pi \text{ sq.cm/sec.}$$

7. A beacon makes one revolution every 10 seconds. It is located on a ship which is anchored 5km from a straight shore line. How fast is the beam moving along the shore line when it makes an angle of 45° with the shore?

Since the beacon makes one revolution (360°) in 10 sec



$$\frac{d\theta}{dt} = \frac{2\pi}{10} = \frac{\pi}{5} \text{ rad/sec}$$

Let $AB = x$

then $\tan\theta = \frac{x}{5} \Rightarrow x = 5\tan\theta$ ...(1)

We know that velocity $= \frac{dx}{dt}$.

Differentiating (1) with respect to 't' we get.

$$\frac{dx}{dt} = 5\sec^2\theta \cdot \frac{d\theta}{dt} \Rightarrow 5(\sec^2 45°)(\pi/5) \quad [\because \frac{d\theta}{dt} = \pi/5]$$

$$\Rightarrow 5(\sqrt{2})^2(\pi/5) \Rightarrow 5(2)(\pi/5) \quad [\because \tan\theta = \sec^2\theta]$$

$$\frac{dx}{dt} = 2\pi \text{ km/sec.}$$

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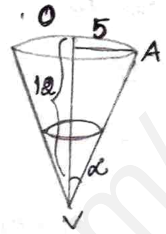
8. A conical water tank with vertex down of 12 metres height has a radius of 5 metres at the top. If water flows into the tank at a rate 10 cubic metre/min, how fast is the depth of the water increases when the water is 8 metres deep?

Given $h = 12\,m$, $r = 5\,m$



Let V be the volume of the cone and α be the semi-vertical angle of the cone.

$$\therefore \tan\alpha = \frac{OA}{VO} = \frac{5}{12} = \frac{r}{h}$$

$$\Rightarrow 12r = 5h$$

$$r = \frac{5h}{12} \quad \ldots (1)$$

We know

$$V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{1}{3}\pi\left(\frac{5h}{12}\right)^2 h \quad [\text{From}(1)]$$

$$= \frac{1}{3}\pi\left(\frac{25h^3}{144}\right)$$

$$V = \frac{25\pi h^3}{3(144)}$$

Differentiating with respect to 't' we get,

$$\frac{dV}{dt} = \frac{25\pi}{3\times 144}\, 3h^2 \frac{dh}{dt}$$

$$= \frac{25\pi}{144} h^2 \frac{dh}{dt}$$

$$10 = \frac{25\pi}{144}(8)^2 \frac{dh}{dt} \quad \left[\because h = 8, \frac{dV}{dt} = 10\,m^3/min\right]$$

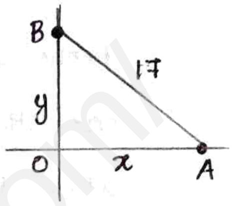
$$\frac{\cancel{10}^{\,1} \times \cancel{144}^{\,18}}{\cancel{25}_{\,5}\pi \times \cancel{64}_{\,\cancel{8}\,4}} = \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{9}{10\pi}\ m/min$$

9. A ladder 17 metre long is leaning against the wall. The base of the ladder is pulled away from the wall at a rate of 5 m/s. when the base of the ladder is 8 metres from the wall.

(i) How fast is the top of the ladder moving down the wall.

Let AB be the position of the ladder at any time t such that $OA = x$ and $OB = y$.



Then $OA^2 + OB^2 = AB^2$

$$\Rightarrow x^2 + y^2 = 17^2 \quad \ldots (1)$$

Given $\frac{dx}{dt} = 5$ and $x = 8$

when $x = 8 \Rightarrow 8^2 + y^2 = 17^2$ [∵ From (1)]

$$y^2 = 17^2 - 8^2 \Rightarrow 289 - 64 = 225$$

$$y = 15$$

Differentiating (1) with respect to 't' we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow 8(5) + 15\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-40}{15}$$

$$\Rightarrow \frac{dy}{dt} = \frac{-8}{3} \text{ m/sec.}$$

∴ The rate of top of the ladder moving down the wall is $-8/3$ m/sec.

(ii) At what rate, the area of the triangle formed by the ladder, wall and the floor is changing?

These three forms a right angled triangle

∴ Area $= \frac{1}{2}xy$ [∵ to diff. take $\frac{1}{2}$ common, diff xy in uv method]

Differentiating with respect to 't' we get, $[uv = uv' + vu']$

$$\frac{dA}{dt} = \frac{1}{2}\left[x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt}\right] \Rightarrow \frac{1}{2}\left[8 \cdot \left(\frac{-8}{3}\right) + 15(5)\right]$$

[From (i)]

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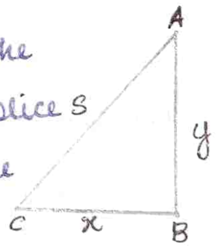
$$= \frac{1}{2}\left[\frac{-64}{3} + 75\right] = \frac{1}{2}\left[\frac{-64+225}{3}\right]$$

$$= \frac{1}{2}\left(\frac{161}{3}\right)$$

$$\frac{dA}{dt} = 26.83 \text{ sq.m/sec.}$$

10. A police jeep, approaching an orthogonal intersection from the northern direction, is chasing a speeding car that has turned and moving straight east. When the jeep is 0.6 km north of the intersection and the car is 0.8 km to the east. The police determine with the radar that the distance between them and the car is increasing at 20 km/hr. If the jeep is moving at 60 km/hr at the instant of measurement what is the speed of the car?



Let x represent the distance covered by the car, y represent the dist covered by police jeep and s represent the dist between the car and jeep.

$\therefore$ Given $x = 0.8$ km, $y = 0.6$ km, $\frac{dy}{dt} = -60$ km/hr, $\frac{ds}{dt} = 20$ km/hr.

In $\triangle ABC$, $s^2 = x^2 + y^2 \quad \ldots (1)$

$$s^2 = (0.8)^2 + (0.6)^2 \Rightarrow 0.64 + 0.36$$

$$s^2 = 1 \Rightarrow \boxed{s = 1}$$

Differentiating (1) with respect to 't' we get,

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

[÷ by 2]

$$\Rightarrow 1\left(\frac{ds}{dt}\right) = (0.8)\left(\frac{dx}{dt}\right) + (0.6)(-60)$$

$$\Rightarrow 1(20) = (0.8)\frac{dx}{dt} - 36 \Rightarrow 20 + 36 = 0.8\left(\frac{dx}{dt}\right)$$

$$\Rightarrow 56 = 0.8\left(\frac{dx}{dt}\right) \Rightarrow \frac{dx}{dt} = \frac{56}{0.8} = 70 \text{ km/hr.}$$

$\therefore$ Speed of the car is 70 km/hr.

EXERCISE 7.2

S. MANIKANDAN P.G. Asst in MATHEMATICS - 9655536357

1. Find the slope of the tangent to the curves at the respective given points:

(i) $y = x^4 + 2x^2 - x$ at $x = 1$.

Given $y = x^4 + 2x^2 - x$

$$\frac{dy}{dx} = 4x^3 + 4x - 1$$

slope of the tangent at $x=1$ is

$$m = \left(\frac{dy}{dx}\right)_{(x=1)}$$

$$= (4)(1)^3 + 4(1) - 1 \Rightarrow 4 + 4 - 1 \Rightarrow 8 - 1$$

$\therefore m = 7.$

(ii) $x = a\cos^3 t$, $y = b\sin^3 t$ at $t = \pi/2$.

$x = a\cos^3 t$ $\qquad$ $y = b\sin^3 t$

$\frac{dx}{dt} = -3a\cos^2 t \sin t$ $\qquad$ $\frac{dy}{dt} = 3b\sin^2 t \cos t$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3b\sin^2 t\cos t}{-3a\cos^2 t\sin t} = \frac{-b}{a}\frac{\sin t}{\cos t}$$

$$\therefore \frac{dy}{dx} = \frac{-b}{a}\tan t \qquad [\because \sin t/\cos t = \tan t]$$

slope of the tangent at $t = \pi/2$ is

$$m = \left(\frac{dy}{dx}\right)_{(t=\pi/2)}$$

$$= \frac{-b}{a}\tan \pi/2 = \frac{-b}{a}(\infty)$$

$\therefore m = \infty.$

2. Find the point on the curve $y = x^2 - 5x + 4$ at which the tangent is parallel to the line $3x + y = 7$.

Given curve is $y = x^2 - 5x + 4$ and the line is $3x + y = 7$.

$\therefore$ slope of tangent of the curve, $m_1 = \frac{dy}{dx} = 2x - 5$

$\therefore$ slope of line $m_2 = \frac{dy}{dx} = -3$

$$\left[\because m = \frac{-\text{co-eff of } x}{\text{co-eff of } y}\right]$$

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Since the tangents of the curve and the lines are parallel, their slopes are equal

$\therefore m_1 = m_2$

$2x - 5 = -3$

$2x = -3+5$

$2x = 2$

$x = 2/2$

$\boxed{x=1}$

Substituting $x=1$ in $y = x^2 - 5x+4$ we get

$y = (1)^2 - 5(1)+4 \Rightarrow 1-5+4$

$\boxed{y=0}$

$\therefore$ The required point is $(1,0)$.

3. Find the points on the curve $y = x^3 - 6x^2 + x + 3$ where the normal is parallel to the line $x+y=1729$.

Given curve is $y = x^3 - 6x^2 + x + 3$ and the line is $x+y=1729$.

Slope of the tangent to the curve is

$$\frac{dy}{dx} = 3x^2 - 12x + 1$$

Slope of the normal to the curve is

$$m_1 = \frac{-1}{3x^2-12x+1}$$

Slope of the line is

$$m_2 = \frac{-1}{1} = -1$$

Since the slope of the normal to the curve and the lines are parallel, $m_1 = m_2$

$$\frac{-1}{3x^2-12x+1} = -1 \Rightarrow -1 = -1(3x^2-12x+1)$$

$3x^2-12x+1 = 1 \Rightarrow 3x^2-12 = 1-1$

$3x^2 - 12 = 0$

$3x(x-4)=0$

$\Rightarrow 3x=0,\ x-4=0 \Rightarrow \boxed{x=0,\ x=4}$

When $x=0$,

$$y=(0)^3-6(0)^2+0+3 \Rightarrow y=0-0+0+3$$

$$\boxed{y=3}$$

when $x=4$,

$$y=(4)^3-6(4)^2+4+3 \Rightarrow y=64-96+4+3$$

$$\boxed{y=-25}$$

$\therefore$ $(0,3)$ and $(4,-25)$ are the required points.

4. Find the points on the curve $y^2-4xy=x^2+5$ for which the tangents is horizontal.

Equation of the curve is $y^2-4xy=x^2+5 \quad \ldots (1)$

Differentiating with respect to 'x' we get

$$\Rightarrow \quad 2y\frac{dy}{dx}-4\left[x\frac{dy}{dx}+y(1)\right]=2x$$

$$\Rightarrow \quad 2y\frac{dy}{dx}-4x\frac{dy}{dx}-4y=2x \quad [\because \text{Taking common } \tfrac{dy}{dx}]$$

$$\Rightarrow \quad \frac{dy}{dx}(2y-4x)=2x+4y$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{2x+4y}{2y-4x}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{x+2y}{y-2x}.$$

Since the tangent to the curve is horizontal, $\frac{dy}{dx}=0$

$$\therefore \frac{x+2y}{y-2x}=0$$

$$x+2y=0$$

$$\boxed{x=-2y} \quad \ldots (2)$$

Subs (2) in (1) we get,

$$\Rightarrow \quad y^2-4(-2y)y=(-2y)^2+5$$

$$\Rightarrow \quad y^2+8y^2=4y^2+5$$

$$\Rightarrow \quad 9y^2=4y^2+5 \Rightarrow 9y^2-4y^2=5$$

$$\Rightarrow \quad 5y^2=5 \Rightarrow y^2=5/5$$

$$y^2=1 \Rightarrow \boxed{y=\pm 1}$$

From (2): when $y=1, x=-2$, when $y=-1, x=2$

$\therefore$ The required points are $(2,-1)$ and $(-2,1)$.

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5. Find the tangent and normal to the following curves at the given points on the curve:

(i) $y = x^2 - x^4$ at $(\underset{x_1}{1}, \underset{y_1}{0})$.

Given $y = x^2 - x^4$

$$\frac{dy}{dx} = 2x - 4x^3$$

$\therefore$ Slope, $m = \left(\frac{dy}{dx}\right)_{(1,0)} \Rightarrow 2(1) - 4(1)^3$

$\Rightarrow 2 - 4 = -2$

$\boxed{m = -2}$

Equation of the tangent is $y - y_1 = m(x - x_1)$

$y - 0 = -2(x - 1)$

$y = -2x + 2$

$\boxed{2x + y = 2}$

Equation of the normal is $y - y_1 = \frac{-1}{m}(x - x_1)$

$y - 0 = \frac{-1}{-2}(x - 1)$

$y = \frac{1}{2}(x - 1)$

$2y = x - 1$

$\boxed{x - 2y = 1}$

(ii) $y = x^4 + 2e^x$ at $(\underset{x_1}{0}, \underset{y_1}{2})$.

Given $y = x^4 + 2e^x$

$$\frac{dy}{dx} = 4x^3 + 2e^x$$

$\therefore$ Slope, $m = \left(\frac{dy}{dx}\right)_{(0,2)} \Rightarrow 4(0)^3 + 2e^0 \Rightarrow 0 + 2(1)$

$[e^0 = 1]$

$\boxed{m = 2}$

Equation of the tangent is $y - y_1 = m(x - x_1)$

$y - 2 = 2(x - 0)$

$y - 2 = 2x$

$\boxed{2x - y = -2}$

Equation of the normal is $y - y_1 = \frac{-1}{m}(x - x_1)$

$y - 2 = \frac{-1}{2}(x - 0)$

$2y - 4 = -x$

$\boxed{x + 2y = 4}$

(iii) $y = x \sin x$ at $(\pi/2, \pi/2)$.

Equation of given curve is $y = x \sin x$ $[\because uv = uv' + vu']$

$$\frac{dy}{dx} = x \cos x + \sin x$$

$\therefore$ slope $m_1 = \left(\frac{dy}{dx}\right)_{(\pi/2, \pi/2)} \Rightarrow \frac{\pi}{2} \cos\frac{\pi}{2} + \sin\frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2}(0) + 1 = 0 + 1$

$\boxed{m = 1}$

Equation of the tangent is $y - y_1 = m(x - x_1)$

$$y - \pi/2 = 1(x - \pi/2)$$

$$\frac{2y - \pi}{\cancel{2}} = \frac{2x - \pi}{\cancel{2}}$$

$$2y = 2x$$

$$2x - 2y = 0$$

$$\boxed{x - y = 0}$$

Equation of the normal is $y - y_1 = \frac{-1}{m}(x - x_1)$

$$y - \pi/2 = \frac{-1}{1}(x - \pi/2) \Rightarrow \frac{-1}{1}\left(\frac{2x - \pi}{2}\right)$$

$$\frac{2y - \pi}{2} = -\left(\frac{2x - \pi}{2}\right)$$

$$2y - \pi = -2x + \pi$$

$$2y + 2x = 2\pi$$

$$\boxed{x + y = \pi}$$

(iv) $x = \cos t$, $y = 2\sin t^2$ at $t = \frac{\pi}{3}$.

Equation of the given curve is $x = \cos t$; $y = 2\sin t^2$

$$\frac{dx}{dt} = -\sin t \; ; \; \frac{dy}{dt} = 4\sin t \cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4\sin t \cos t}{-\sin t}$$

$$= -4\cos t$$

$\therefore$ slope, $m = \left(\frac{dy}{dx}\right)_{t = \pi/3}$

$$= -4\cos(\pi/3) = -4\left(\frac{1}{2}\right)$$

$$\boxed{m = -2.}$$

When $t = \pi/3$, $x = \cos \pi/3 = \frac{1}{2}$ $[\because \cos\frac{\pi}{3} = \frac{1}{2}, \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}]$

when $t = \pi/3$. $y = 2(\sin \pi/3)^2 = 2\left(\frac{\sqrt{3}}{2}\right)^2 = 2 \times \frac{3}{4} = \frac{3}{2}$

$\left(\frac{1}{2}, \frac{3}{2}\right)$
$x_1 \quad y_1$

Equation of the tangent is $y - y_1 = m(x - x_1)$

$$y - \frac{3}{2} = -2\left(x - \frac{1}{2}\right)$$

$$\frac{2y-3}{2} = -2\left(\frac{2x-1}{2}\right)$$

$$2y - 3 = -2(2x-1)$$

$$2y - 3 = -4x + 2$$

$$\boxed{4x + 2y = 5}$$

Equation of the normal is $y - y_1 = \frac{-1}{m}(x - x_1)$

$$y - \frac{3}{2} = \frac{-1}{-2}\left(x - \frac{1}{2}\right)$$

$$\frac{2y-3}{2} = \frac{1}{2}\left(\frac{2x-1}{2}\right)$$

$$2y - 3 = \frac{1}{2}(2x-1)$$

$$4y - 6 = 2x - 1$$

$$\boxed{2x - 4y = -5}$$

6. Find the equations of the tangents to the curve $y = 1 + x^3$ for which the tangent is orthogonal with the line $x + 12y = 12$.

Given equation of the curve is $y = 1 + x^3$ and the line is $x + 12y = 12$.

slope of the tangent to the curve, $m_1 = \frac{dy}{dx}$

$$m_1 = 3x^2$$

slope of the line, $m_2 = \frac{-1}{12}$

$$\left[\because \frac{-\text{co-eff of } x}{\text{co-eff of } y}\right]$$

since the slope of the tangent to the curve and the line are orthogonal.

$$\therefore m_1 m_2 = -1$$

$$3x^2\left(\frac{-1}{12}\right) = -1$$

$$\frac{-x^2}{4} = -1 \Rightarrow \frac{x^2}{4} = 1 \Rightarrow x^2 = 4$$

$$\therefore x = \pm 2$$

when $x = 2$, $y = 1 + 2^3 = 1 + 8 = 9$

$x = -2$ $y = 1 + (-2)^3 = 1 - 8 = -7$

let we have two points $(2, 9)$ $(-2, -7)$

$\therefore$ Equation of tangent at $(2, 9)$ is

$$y - y_1 = m(x - x_1)$$

$$y - 9 = 12(x - 2)$$

$$y - 9 = 12x - 24$$

$$\boxed{12x - y = 15}$$

$m_1 = 3(x^2)$

$\Rightarrow 3(2^2)$

$= 3(4)$

$m = 12$

$\therefore$ Equation of tangent at $(-2, -7)$

$$y - y_1 = m(x - x_1)$$

$$y + 7 = 12(x + 2)$$

$$y + 7 = 12x + 24$$

$$\boxed{12x - y = -17}$$

7 Find the equations of the tangents to the curve $y = \frac{x+1}{x-1}$ which are parallel to the line $x + 2y = 6$.

Given equation of the curve is $y = \frac{x+1}{x-1}$ and the line $x + 2y = 6$.

slope of the tangent to the curve $m_1 = \frac{dy}{dx}$

$\left[\frac{u}{v} = \frac{vu' - uv'}{v^2}\right] \Rightarrow$

$$= \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}$$

$$= \frac{(x-1) - (x+1)}{(x-1)^2}$$

$$= \frac{x - 1 - x - 1}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

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Slope of the line $m_2 = \frac{-1}{2}$ $\left[\because \frac{-\text{co-eff of } x}{\text{co-eff of } y}\right]$

since the tangent to the curve and the lines are parallel, $m_1 = m_2$

$$\frac{-2}{(x-1)^2} = \frac{-1}{2}$$

$$-4 = -(x-1)^2 \Rightarrow 4 = (x-1)^2$$

$$\Rightarrow x-1 = \pm 2$$

$$\Rightarrow x-1 = 2 \text{ (or) } x-1 = -2$$

$$\Rightarrow x = 3 \text{ or } x = -1$$

when $x=3$, $y = \frac{3+1}{3-1} = \frac{4}{2} = 2$

when $x=-1$, $y = \frac{-1+1}{-1-1} = \frac{0}{-2} = 0.$

Let we have two points $(3,2)(-1,0)$.

$\therefore$ Equation of the tangent at $(3,2)$ is

$$\Rightarrow y-y_1 = m(x-x_1)$$

$$\Rightarrow y-2 = \frac{-1}{2}(x-3)$$

$$\Rightarrow 2y-4 = -x+3$$

$$\Rightarrow \boxed{x+2y=7}$$

$\therefore$ Equation of the tangent at $(-1,0)$ is

$$\Rightarrow y-y_1 = m(x-x_1)$$

$$\Rightarrow y-0 = \frac{-1}{2}(x+1)$$

$$\Rightarrow 2y = -x-1$$

$$\Rightarrow \boxed{x+2y=-1}$$

8. Find the equation of tangent and normal to the curve given by $x = 7\cos t$ and $y = 2\sin t$, $t \in R$ at any point on the curve 305

Equation of the given curve is $x = 7\cos t$, $y = 2\sin t$

$$\frac{dx}{dt} = -7\sin t, \quad \frac{dy}{dt} = 2\cos t$$

$$\text{slope} = m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-7\sin t}$$

$\therefore$ Equation of the tangent at any point on the curve is

$$\Rightarrow y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 2\sin t = \left(\frac{+2\cos t}{-7\sin t}\right)(x - 7\cos t)$$

$$\Rightarrow (-7\sin t)y + 14\sin^2 t = (2\cos t\, x) \mp 14\cos^2 t$$

$$\Rightarrow 2\cos t\, x + 7\sin t\, y = 14(\sin^2 t + \cos^2 t)$$

$$= 14(1)$$

$$\Rightarrow (2\cos t)x + (7\sin t)y = 14.$$

$\therefore$ Equation of the normal at any point on the curve is

$$\Rightarrow y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 2\sin t = \left(\frac{7\sin t}{2\cos t}\right)(x - 7\cos t)$$

$$\Rightarrow 2\cos t\, y - 4\sin t\cos t = (7\sin t)x - 49\sin t\cos t$$

$$\Rightarrow (7\sin t)x - (2\cos t\, y) = 49\sin t\cos t - 4\sin t\cos t$$

$$\Rightarrow (7\sin t)x - (2\cos t\, y) = 45\sin t\cos t.$$

9. Find the angle between the rectangular hyperbola $xy = 2$ and the parabola $x^2 + 4y = 0$.

Equation of the rectangular hyperbola is $xy = 2$

$$y = \frac{2}{x} \quad \dots (1)$$

$$x \cdot \frac{dy}{dx} + y(1) = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}.$$

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slope of the tangent to the curve $m_1 = \frac{-y}{x}$

Equation of the parabola is $x^2+4y=0$ . . . (2)

$$2x+4\frac{dy}{dx}=0$$

$$\frac{dy}{dx} = \frac{-2x}{4} = \frac{-x}{2}$$

slope of the tangent to the curve $m_2 = -x/2$.

subs (1) in (2) we get,

$$x^2+4\left(\frac{2}{x}\right)=0. \Rightarrow x^2+\frac{8}{x}=0$$

$$\Rightarrow x^3+8=0$$

$$x^3=-8 \Rightarrow (-2)^3$$

$$\Rightarrow x=-2$$

when $x=-2$, $y=\frac{2}{-2}=-1 \Rightarrow y=-1$

$\therefore$ The point of intersection of RH and parabola is $(-2,-1)$

$$\therefore m_1 = \frac{-y}{x} = \frac{-(-1)}{-2} = \frac{-1}{2}$$

$$m_2 = \frac{-x}{2} = \frac{-(-2)}{2} = 1$$

Let θ be the angle between rectangular hyperbola and parabola.

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right| = \left|\frac{-1/2-1}{1+(-1/2)(1)}\right|$$

$$= \left|\frac{-3/2}{1-1/2}\right| = \left|\frac{-3/2}{1/2}\right|$$

$$= \left|-\frac{3}{2}\times\frac{2}{1}\right| = |-3| = 3$$

$$\therefore \theta = \tan^{-1}(3).$$

10. Show that the two curves $x^2-y^2=r^2$ and $xy=c^2$ where c, r are constants, cut orthogonally. 307

Equation of the given curves are $x^2-y^2=r^2$ and $xy=c^2$

$$x^2-y^2=r^2$$

$$2x-2y\frac{dy}{dx}=0$$

$$2x=2y\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{2x}{2y}=\frac{x}{y}.$$

let (x_1, y_1) be the point of intersection of the given curves.

$\therefore$ slope of tangent to the first curve $m_1=\frac{x_1}{y_1}$ ... (1)

$$xy=c^2$$

$$x.\frac{dy}{dx}+y(1)=0$$

$$x.\frac{dy}{dx}=-y$$

$$\frac{dy}{dx}=\frac{-y}{x}$$

$\therefore$ slope of tangent to the first curve $m_2=\frac{-y_1}{x_1}$ ... (2)

consider $m_1m_2=\left(\frac{x_1}{y_1}\right)\left(\frac{-y_1}{x_1}\right)=-1$

since $m_1m_2=-1$, the given two curves cut orthogonally.

EXERCISE 7.3

1. Explain why Rolle's theorem is not applicable to the following functions in the respective intervals.

(i) $f(x) = \left|\frac{1}{x}\right|, x \in [-1,1]$

Given $f(x) = \left|\frac{1}{x}\right|, x \in [-1,1]$

Rolle's theorem is not applicable since $f(x) = \frac{1}{x}$ is not continuous at $x=0$ in $[-1,1]$ and not differentiable in $(-1,1)$.

(ii) $f(x) = \tan x, x \in [0,\pi]$.

Given $f(x) = \tan x, x \in [0,\pi]$

Rolle's theorem is not applicable since $\tan x$ is not continuous at $x = \frac{\pi}{2}$ $\quad [\because \tan\frac{\pi}{2} = \infty]$

(iii) $f(x) = x - 2\log x, x \in [2,7]$.

Given $f(x) = x - 2\log x, x \in [2,7]$

(i) $f(x)$ is continuous in $[2,7]$

(ii) $f(x)$ is differentiable in $(2,7)$

$$f(2) = 2 - 2\log 2$$
$$= 2 - \log 2^2 = 2 - \log 4$$
$$f(7) = 7 - 2\log 7$$
$$= 7 - \log 7^2 = 7 - \log 49.$$

Since $f(2) \neq f(7)$, Rolle's theorem is not applicable.

2. Using the Rolle's theorem, determine the values of x at which the tangent is parallel to the x-axis for the following functions:

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(i) $f(x) = x^2 - x, \; x \in [0,1]$

Given $f(x) = x^2 - x, x \in [0,1]$

(i) $f(x)$ is continuous in $[0,1]$

(ii) $f(x)$ is differentiable in $(0,1)$

(iii) $f(0) = 0^2 - 0 = 0$

$f(1) = 1^2 - 1 = 1 - 1 = 0$

$\therefore f(0) = f(1)$

By Rolle's theorem, there exists $c \in [0,1]$ such that

$$f'(c) = 0$$

$$2c - 1 = 0$$

$$2c = 1$$

$$c = 1/2 \in [0,1].$$

(ii) $f(x) = \dfrac{x^2 - 2x}{x+2}, x \in [-1,6]$

(i) $f(x)$ is continuous in $[-1,6]$

(ii) $f(x)$ is differentiable in $(-1,6)$

(iii) $f(-1) = \dfrac{(-1)^2 - 2(-1)}{-1+2} = \dfrac{1+2}{1} = 3$

$f(6) = \dfrac{(6)^2 - 2(6)}{6+2} = \dfrac{36-12}{8} = \dfrac{24}{8} = 3$

$\therefore f(-1) = f(6) = 3$

By Rolle's theorem, there exists $c \in [-1,6]$ such that

$$f'(c) = 0$$

$$\frac{(c+2)(2c-2) - (c^2-2c)(1)}{(c+2)^2} = 0$$

$$2c^2 + 4c - 2c - 4 - c^2 + 2c = 0$$

$$c^2 + 4c - 4 = 0$$

$$c = \frac{-4 \pm \sqrt{4^2 - 4(1)(-4)}}{2(1)}$$

$\left[\because x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$

$$c = \frac{-4+\sqrt{16+16}}{2(1)} = \frac{-4 \pm \sqrt{2(16)}}{2}$$

$$= \frac{-4 \pm 4\sqrt{2}}{2} = \frac{2(-2 \pm 2\sqrt{2})}{2}$$

$$c = -2 \pm 2\sqrt{2}$$

$$\therefore c = -2+2\sqrt{2} \in [-1,6] \quad [\because -2-2\sqrt{2} \notin [-1,6]]$$

(iii) $f(x) = \sqrt{x} - \frac{x}{3}$, $x \in [0,9]$

(i) $f(x)$ is continuous in $[0,9]$

(ii) $f(x)$ is differentiable in $(0,9)$

(iii) $f(0) = \sqrt{0} - \frac{0}{3} = 0-0 = 0$

$f(9) = \sqrt{9} - 9/3 = \sqrt{9} - 3 = 3-3 = 0$

$\therefore f(0) = f(9)$.

$\therefore$ By Rolle's theorem, there exists $c \in [0,9]$ such that $f'(c) = 0$

$$\frac{1}{2}c^{\frac{1}{2}-1} - \frac{1}{3} = 0 \Rightarrow \frac{1}{2}c^{-\frac{1}{2}} = \frac{1}{3}$$

$$\Rightarrow \frac{1}{2\sqrt{c}} = \frac{1}{3}$$

$$\Rightarrow \frac{1}{\sqrt{c}} = \frac{2}{3} \Rightarrow \sqrt{c} = \frac{3}{2}$$

Squaring on both sides we get,

$$c = \frac{9}{4} \in [0,9]$$

3. Explain why Lagrange's mean value theorem is not applicable to the following functions in the respective intervals:

(i) $f(x) = \frac{x+1}{x}$, $x \in [-1,2]$

Lagrange's mean value theorem is not applicable since $f(x)$ is not continuous at $x=0$.

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(ii) $f(x) = |3x+1|,\ x \in [-1,3]$.

$f(x)$ is continuous in $[-1,3]$ but $f(x)$ is not differentiable at $x = -1/3$.

4. Using the lagrange's mean value theorem determine the values of x at which the tangent is parallel to the secant line at the end points of the given interval:

(i) $f(x) = x^3 - 3x + 2,\ x \in [-2,2]$

(a) $f(x)$ is continuous in $[-2,2]$

(b) $f(x)$ is differentiable in $(-2,2)$

(c) $f(-2) = (-2)^3 - 3(-2) + 2 = -8+6+2 = -8+8 = 0$

$f(2) = 2^3 - 3(2) + 2 = 8-6+2 = 8-4 = 4$

By lagrange's mean value theorem, there exists $c \in [-2,2]$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

$3c^2 - 3 = \frac{4-0}{2-(-2)} = \frac{4}{2+2} = \frac{4}{4} = 1$

$3c^2 - 3 = 1 \Rightarrow 3c^2 = 1+3 \Rightarrow 3c^2 = 4$

$c^2 = 4/3$

$c = \pm \frac{2}{\sqrt{3}} \in [-2,2]$.

(ii) $f(x) = (x-2)(x-7),\ x \in [3,11]$

(a) $f(x)$ is continuous in $[3,11]$

(b) $f(x)$ is differentiable in $(3,11)$

(c) $f(3) = (3-2)(3-7) = (1)(-4) = -4$

$f(11) = (11-2)(11-7) = (9)(4) = 36$

By lagrange mean value theorem, there exists $c \in [3,11]$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\boxed{\begin{aligned} f(x) &= (x-2)(x-7) \\ &= x^2 - 7x - 2x + 14 \\ &= x^2 - 9x + 14 \end{aligned}}$$

$2c - 9 = \frac{36+4}{11-3}$

$2c - 9 = 40/8 = 5 \Rightarrow 2c - 9 = 5 \Rightarrow 2c = 5+9$

$2c = 14 \Rightarrow c = 7 \in [3,11]$

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5. Show that the value in the conclusion of the mean value theorem for

(i) $f(x) = \frac{1}{x}$ on the closed interval of positive numbers $[a,b]$ is $\sqrt{ab}$.

Given $f(x) = \frac{1}{x}, x \in [a,b]$

a) $f(x)$ is continuous in $[a,b]$

b) $f(x)$ is differentiable in (a,b)

c) $f(a) = \frac{1}{a}, f(b) = \frac{1}{b}$

Using mean value theorem, there exists $c \in [a,b]$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\frac{-1}{c^2} = \frac{1/b - 1/a}{b-a}$$

$$= \frac{a-b/ba}{b-a} = \frac{a-b}{ab(b-a)}$$

$$= \frac{-(b-a)}{ab(b-a)} = \frac{-1}{ab}$$

$$\boxed{\begin{aligned} \frac{1}{x} &= \frac{vu'-uv'}{v^2} \\ &= \frac{x(0)-(1)(1)}{x^2} \\ &= \frac{0-1}{x^2} = \frac{-1}{x^2} \end{aligned}}$$

$$\frac{-1}{c^2} = \frac{-1}{ab} \Rightarrow c^2 = ab \Rightarrow c = \pm\sqrt{ab}$$

$$\therefore c = \sqrt{ab} \in [a,b] \quad [\because c = -\sqrt{ab} \notin [a,b]]$$

(ii) $f(x) = Ax^2+Bx+C$ on any interval $[a,b]$ is $\frac{a+b}{2}$.

Given $f(x) = Ax^2+Bx+C, x \in [a,b]$

a) $f(x)$ is continuous in $[a,b]$

b) $f(x)$ is differentiable in (a,b)

c) $f(a) = Aa^2+Ba+C, f(b) = Ab^2+Bb+C$

Using mean value theorem, there exists $c \in [a,b]$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

$$2Ac+B = \frac{(Ab^2+Bb+C)-(Aa^2+Ba+C)}{b-a}$$

$$= \frac{(Ab^2+Bb+\cancel{C}-Aa^2-Ba-\cancel{C})}{b-a}$$

$$= \frac{A(b^2-a^2)+B(b-a)}{b-a}$$

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$$= \frac{A(b+a)(b-a) + B(b-a)}{b-a}$$

$$= \frac{(b-a)[A(b+a)+B]}{b-a}$$

$$2Ac+B = A(a+b)+B \Rightarrow 2Ac = A(a+b)$$

$$c = \frac{a+b}{2} \in [a,b].$$

6. A race car driver is racing at 20th km. If his speed never exceeds 150 km/hr, what is the maximum distance he can cover in the next two hours.

Let $f(t)$ represents the distance covered at 't' hour.

Given $f(0) = 20$ and $f(2) = ?$

Also speed $= f'(t) \geq 150$

This distance function is continuous as well as differentiable.

By lagrange's theorem, there exists c such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$f'(c) = \frac{f(2) - 20}{2-0} \geq 150$$

[Max speed 150 km/hr and $f'(c)$ represent speed]

$$\frac{f(2)-20}{2-0} \geq 150$$

$$f(2) - 20 \geq 150 \times 2$$

$$f(2) - 20 \geq 300$$

$$f(2) \geq 320$$

Hence in the next two hours he can cover 320 km.

7. Suppose that for a function $f(x)$, $f'(x) \leq 1$ for all $1 \leq x \leq 4$. Show that $f(4) - f(1) \leq 3$.

$f'(x) \leq 1$ for all $1 \leq x \leq 4$

(a) $f(x)$ is continuous in $[1,4]$

(b) $f(x)$ is differentiable in $(1,4)$.

$$f'(x) = \frac{f(b)-f(a)}{b-a}$$

$$f'(x) = \frac{f(4)-f(1)}{4-1}$$

$$\frac{f(4)-f(1)}{3} = f'(x)$$

$$\frac{f(4)-f(1)}{3} \leq 1 \qquad [\because f'(x) \leq 1]$$

$$f(4)-f(1) \leq 3$$

8. Does there exists a differentiable function $f(x)$ such that $f(0)=-1$, $f(2)=4$ and $f'(x) \leq 2$ for all x. Justify your answer.

Given $f(0)=-1$, $f(2)=4$

$\therefore f(x)$ is continuous function in $[0,2]$

By lagrange mean value theorem,

$$f'(x) = \frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(0)}{2-0}$$

$$= \frac{4-(-1)}{2} = \frac{5}{2} = 2.5$$

$$= 2.5 \not\in [0,2]$$

since $f'(x)$ cannot be 2.5 at any point in $[0,2]$, there does not exist a differentiable function $f(x)$.

9. show that there lies a point on the curve $f(x) = x(x+3)e^{-\pi/2}$, $-3 \leq x \leq 0$ where tangent drawn is parallel to the x-axis.

Given $f(x) = x(x+3)e^{-\pi/2}$, $-3 \leq x \leq 0$

(a) $f(x)$ is continuous in $[-3,0]$

b) $f(x)$ is differentiable in $(-3,0)$

c) $f(0) = 0(0+3)e^{-\pi/2} = 0$

$f(-3) = -3(-3+3)e^{-\pi/2} = -3(0)e^{-\pi/2} = 0$

$\therefore$ By Rolle's theorem, there exists $c \in [-3,0]$ such that $f'(c)=0$.

$(2c+3)e^{-\pi/2} = 0.$ $[\because f(x) = x(x+3)e^{-\pi/2} = x(x^2+3x)e^{-\pi/2}$

$f'(x) \Rightarrow (2x+3)e^{-\pi/2}]$

$2c+3 = 0$

$2c = -3$

$c = -3/2 \in [-3,0]$

Hence there exist a point on the curve $f(x) = x(x+3)e^{-\pi/2}, -3 \le x \le 0$ where the tangent is parallel to the x-axis.

10. Using mean value theorem prove that for, $a>0, b>0$, $|e^{-a} - e^{-b}| < |a-b|$.

Let $f(x) = e^{-x}, x \in [a,b]$

a) e^{-x} is continuous in $[a,b]$

b) e^{-x} is differentiable in (a,b)

c) $f(a) = e^{-a}, \quad f(b) = e^{-b}$.

By lagrange mean value theorem, there exists $c \in [a,b]$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$$\Rightarrow \quad -e^{-c} = \frac{e^{-b}-e^{-a}}{b-a} = \frac{e^{-a}-e^{-b}}{a-b}$$

$$\Rightarrow \quad |-e^{-c}| = \left|\frac{e^{-a}-e^{-b}}{a-b}\right|$$

$$\Rightarrow \quad \left|\frac{e^{-a}-e^{-b}}{a-b}\right| < 1$$

$[\because |-e^{-c}| < 1$ for all $c \in [a,b], a>0, b>0]$

$$\Rightarrow \quad \frac{|e^{-a}-e^{-b}|}{|a-b|} < 1$$

$$\Rightarrow \quad |e^{-a}-e^{-b}| < |a-b|$$

Hence proved.

EXERCISE 7.4

1. Write the Maclaurin series expansion of the following functions:

(i) e^x

Functions & its distribution	e^x and its derivatives	Value at $x=0$
$f(x)$	e^x	$e^0=1$
$f^{I}(x)$	e^x	$e^0=1$
$f^{II}(x)$	e^x	$e^0=1$

$$\Rightarrow f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots$$

$\therefore$ Maclaurin's series expansion of

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \dots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

(ii) $\sin x$

Let $f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$

$f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -\cos 0 = -1$

$f^{IV}(x) = \sin x \Rightarrow f^{IV}(0) = \sin 0 = 0$

$f^{V}(x) = \cos x = f^{V}(0) = \cos 0 = 1$

$f^{VI}(x) = -\sin x = f^{VI}(0) = -\sin 0 = 0$

$f^{VII}(x) = -\cos x = f^{VII}(0) = -\cos 0 = -1$

$\therefore$ Maclaurin series expansion

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{IV}(0)}{4!}x^4 + \dots$$

$$\sin x = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \left(\frac{-1}{3!}\right)x^3 + 0 + \frac{1}{5!}x^5 + 0 + \left(\frac{-1}{7!}\right)x^7$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

(iii) $\cos x$ 317

Let,

$f(x) = \cos x \Rightarrow f(0) = \cos 0 = 1$

$f'(x) = -\sin x \Rightarrow f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x = f'''(0) = \sin 0 = 0$

$f^{IV}(x) = \cos x = f^{IV}(0) = \cos 0 = 1$

$f^{V}(x) = -\sin x = f^{V}(0) = -\sin 0 = 0$

$f^{VI}(x) = -\cos x = f^{VI}(0) = -\cos 0 = -1$

$\therefore$ Maclaurin series expansion of

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$\therefore \cos x = 1 + 0 + \frac{-1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 + \frac{-1}{6!}x^6 + \dots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

(iv) $\log(1-x)$; $-1 \le x < 1$.

Functions & its derivatives	$\log(1-x)$ and its derivatives	value at $x=0$
$f(x)$	$\log(1-x)$	$\log 1 = 0$
$f'(x)$	$\frac{-1}{1-x} = -1(1-x)^{-1}$	$-1(1)^{-1} = \frac{-1}{1} = -1$
$f''(x)$	$-1(1-x)^{-2}$	$-1(1)^{-2} = -1$
$f'''(x)$	$-2(1-x)^{-3}$	$-2(1)^{-3} = -2$
$f^{IV}(x)$	$-6(1-x)^{-4}$	$-6(1)^{-4} = -6$
$f^{V}(x)$	$-24(1-x)^{-5}$	$-24(1)^{-5} = -24$

Maclaurin series expansion

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$\log(1-x) = 0 + \frac{-1}{1!}x + \frac{-1}{2!}x^2 + \frac{-2}{3!}x^3 + \frac{-6}{4!}x^4 + \frac{-24}{5!}x^5$$

$$= 0 - x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots \text{ (or) } -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots\right)$$

(v) $\tan^{-1}(x); -1 \le x \le 1.$

$f(x) = \tan^{-1}(x) \Rightarrow f(0) = \tan^{-1}(0) = 0$ $\boxed{\therefore f(0) = 0}$

$f'(x) = \frac{1}{1+x^2} \Rightarrow (1+x^2)^{-1} \Rightarrow f'(0) = (1+0^2)^{-1} = 1$ $\boxed{\therefore f'(0) = 1}$

$f''(x) = -1(1+x^2)^{-2}.2x \Rightarrow$

$\Rightarrow -2x(1+x^2)^{-2} \Rightarrow f''(0) = -2(0)(1+0^2)^{-2} = 0$

$\boxed{\therefore f''(0) = 0}$

$f'''(x) = [-2x(-2(1+x^2)^{-3}.2x] + [(1+x^2)^{-2}.-2]$

$[\because uv = uv' + vu']$

$= -2[-4x^2(1+x^2)^{-3} + (1+x^2)^{-2}]$

$f'''(0) = -2[-4(0)(1+0)^{-3} + (1+0)^{-2}] \Rightarrow -2[0+1]$

$\boxed{\therefore f'''(0) = -2.}$

$f^{IV}(x) = -2[-4x^2(-3)(1+x^2)^{-4}(2x) + (1+x^2)^{-3}(-8x)] +$

$[-2(1+x^2)^{-3}.\cancel{-4x}(2x)]$

$= -2[24x^3(1+x^2)^{-4} - 8x(1+x^2)^{-3} - 4x(1+x^2)^{-3}]$

$= -2[24x^3(1+x^2)^{-4} - 12x(1+x^2)^{-3}]$ $[\because uv]$

$= -24[2x^3(1+x^2)^{-4} - x(1+x^2)^{-3}]$

$f^{IV}(0) = -24[2(0)(1+0^2)^{-4} - 0(1+0^2)^{-3}] \Rightarrow -24[0-0] = 0$

$\boxed{\therefore f^{IV}(0) = 0.}$

$f^{V}(x) = -24[2x^3.(-4)(1+x^2)^{-5}(2x) + (1+x^2)^{-4}.6x^2]$ $[\because uv]$

$[-x.(-3)(1+x^2)^{-4}(2x) - (1+x^2)^{-3}]$

$= -24[16x^4(1+x^2)^{-5} + 6x^2(1+x^2)^{-4} + 6x^2(1+x^2)^{-3}]$

$= -24[16x^4(1+x^2)^{-5} + 12x^2(1+x^2)^{-4} - (1+x^2)^{-3}]$

$f^{V}(0) = -24[16(0)(1+0)^{-5} + 12(0)(1+0)^{-4} - (1+0)^{-3}]$

$= -24[-1] \Rightarrow$ $\boxed{\therefore f^{V}(0) = 24.}$

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$\therefore$ Maclaurin series expansion

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$\tan^{-1}(x) = 0 + \frac{1}{1!}x + 0 + \frac{-2}{3!}x^3 + 0 + \frac{24}{5!}x^5 + \dots$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

(vi) $\cos^2 x$

Let $f(x) = \cos^2 x \Rightarrow f(0) = \cos^2(0) = 1$

$f'(x) = 2\cos x(-\sin x)$

$= -\sin 2x \Rightarrow f'(0) = -\sin 2(0) = 0$

$f''(x) = -2\cos 2x \Rightarrow f''(0) = -2\cos 2(0) = -2(1) = -2$

$f'''(x) = 4\sin 2x \Rightarrow f'''(0) = 4\sin 2(0) = 0$

$f^{IV}(x) = 8\cos 2x \Rightarrow f^{IV}(0) = 8\cos 2(0) = 8$

$f^{V}(x) = -16\sin 2x \Rightarrow f^{V}(0) = -16\sin 2(0) = 0$

$f^{VI}(x) = -32\cos 2x \Rightarrow f^{VI}(0) = -32\cos 2(0) = -32$

$\therefore$ Maclaurin series expansion,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$\therefore \cos^2 x = 1 - 0 + \frac{-2}{2!}x^2 + 0 + \frac{8}{4!}x^4 + 0 + \frac{-32}{6!}x^6 + \dots$$

$$= 1 - \frac{2x^2}{2!} + \frac{2^3 x^4}{4!} - \frac{2^5 x^6}{6!}$$

2. Write down the taylor series expansion, of the function $\log x$ about $x=1$ upto three non-zero terms for $x>0$.

Let $f(x) = \log x \Rightarrow f(1) = \log 1 = 0$

$f'(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(1) = \frac{1}{1} = 1$

$f''(x) = -1x^{-2} \Rightarrow f''(1) = -1$

$f'''(x) = 2x^{-3} \Rightarrow f'''(1) = 2$

$f^{IV}(x) = -6x^{-4} \Rightarrow f^{IV}(1) = -6.$

Taylor series of f(x) at $x=1$ is

$$f(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \dots$$

$$\log x = 0 + \frac{1}{1!}(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{-6}{4!}(x-1)^4 + \dots$$

$$= (x-1) - \frac{1}{2!}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots$$

3. Explain $\sin x$ in ascending powers $x - \frac{\pi}{4}$ upto three non-zero terms.

Let $f(x) = \sin x \Rightarrow f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x \Rightarrow f'(\pi/4) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x \Rightarrow f''\left(\frac{\pi}{4}\right) = -\sin\pi/4 = -\frac{1}{\sqrt{2}}$

$f'''(x) = -\cos x \Rightarrow f'''(\pi/4) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

Taylor series for f(x) at $x = \frac{\pi}{4}$ is

$$f(x) = f(\pi/4) + \frac{f'(\pi/4)}{1!}\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{f'''(\pi/4)}{3!}(x-\pi/4)^3 + \dots$$

$$\sin x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2}} \cdot \frac{(x-\pi/4)^2}{2!} - \frac{1}{\sqrt{2}} \frac{(x-\pi/4)^3}{3!} + \dots$$

$$= \frac{1}{\sqrt{2}}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \dots\right]$$

[x and ÷ by $\sqrt{2}$]

$$= \frac{\sqrt{2}}{2}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \dots\right]$$

4. Expand the polynomial $f(x) = x^2 - 3x + 2$ in powers of $x-1$.

$f(x) = x^2 - 3x + 2 \Rightarrow f(1) = 1^2 - 3(1) + 2 = 1 - 3 + 2 = 0$

$f'(x) = 2x - 3 \Rightarrow f'(1) = 2(1) - 3 = -1$

$f''(x) = 2 \Rightarrow f''(1) = 2 = 2$

Taylor series for f(x) at $x=1$ is

$$f(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \dots$$

$$= 0 - \frac{1}{1!}(x-1) + \frac{2}{2!}(x-1)^2 + \dots$$

$\therefore f(x) = -(x-1) + (x-1)^2 + \dots$

EXERCISE 7.5

Evaluate the following limits, if necessary use l'Hôpital rule:

1. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$

$\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} = \dfrac{1-\cos 0}{0} = \dfrac{1-1}{0} = \dfrac{0}{0}$

∴ Indeterminate form

Applying l' Hôpital rule we get,

$\lim\limits_{x\to 0} \dfrac{\sin x}{2x} \Rightarrow \dfrac{1}{2} \lim\limits_{x\to 0} \dfrac{\sin x}{x}$ $\left[\because \lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1\right]$

$= \dfrac{1}{2} \times 1$

$= \dfrac{1}{2}$

2. $\lim\limits_{x\to\infty} \dfrac{2x^2-3}{x^2-5x+3}$

$\lim\limits_{x\to\infty} \dfrac{2x^2-3}{x^2-5x+3} \Rightarrow \lim\limits_{\frac{1}{x}\to 0} \dfrac{2-\frac{3}{x^2}}{1-\frac{5}{x}+\frac{3}{x^2}}$

[÷ numerator & denominator by x^2]

$= \dfrac{2-0}{1-0+0} = \dfrac{2}{1} = 2.$

3. $\lim\limits_{x\to\infty} \dfrac{x}{\log x}$

$\lim\limits_{x\to\infty} \dfrac{x}{\log x} = \dfrac{\infty}{\infty}$

∴ It is in indeterminate form.

Applying l' Hôpital rule,

$\lim\limits_{x\to\infty} \dfrac{1}{1/x} = \lim\limits_{x\to\infty} x = \infty.$

S.MANIKANDAN P.G.Asst in MATHEMATICS- 9655536357

4. $\lim\limits_{x\to\frac{\pi}{2}^-} \dfrac{\sec x}{\tan x}$

$\lim\limits_{x\to\frac{\pi}{2}^-} \dfrac{\sec x}{\tan x} \Rightarrow \lim\limits_{x\to\frac{\pi}{2}^-} \dfrac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}} \Rightarrow$

$\Rightarrow \lim\limits_{x\to\frac{\pi}{2}^-} \dfrac{1}{\sin x} \Rightarrow \dfrac{1}{\sin\frac{\pi}{2}} = \dfrac{1}{1} = 1.$

5. $\lim\limits_{x\to\infty} e^{-x}\sqrt{x}.$

$\lim\limits_{x\to\infty} e^{-x}\sqrt{x} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{e^{x}} = \dfrac{\infty}{\infty}$

$\therefore$ It is in indeterminate form

Applying L' Hôpital rule we get

$\lim\limits_{x\to\infty} \dfrac{\frac{1}{2}x^{\frac{1}{2}-1}}{e^{x}} \Rightarrow \dfrac{1}{2}\lim\limits_{x\to\infty} \dfrac{x^{-1/2}}{e^{x}} \Rightarrow \dfrac{1}{2}\lim\limits_{x\to\infty} \dfrac{e^{-x}}{\sqrt{x}}$

$\Rightarrow \dfrac{1}{2}\lim\limits_{x\to\infty} e^{-x}\sqrt{\dfrac{1}{x}}$

[$\because$ when $x\to\infty$, $\frac{1}{x}\to 0$, $e^{-\infty}=0$]

$\Rightarrow \dfrac{1}{2}e^{-\infty}(0) = 0$

6. $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$

$\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right) \Rightarrow \lim\limits_{x\to 0}\left(\dfrac{x-\sin x}{x\sin x}\right) = \dfrac{0}{0}$

$\therefore$ It is in indeterminate form.

Applying L' Hôpital rule we get,

$\lim\limits_{x\to 0} \dfrac{1-\cos x}{x\cos x+\sin x} = \dfrac{1-\cos 0}{0+\sin 0} = \dfrac{1-1}{0} = \dfrac{0}{0}$

7. $\lim\limits_{x\to 1^+}\left(\dfrac{2}{x^2-1} - \dfrac{x}{x-1}\right)$

$\lim\limits_{x\to 1^+}\left(\dfrac{2}{x^2-1} - \dfrac{x}{x-1}\right) \Rightarrow \lim\limits_{x\to 1^+}\left(\dfrac{2-x(x+1)}{x^2-1}\right) \Rightarrow \lim\limits_{x\to 1^+}\left(\dfrac{2-x^2-x}{x^2-1}\right) = \dfrac{0}{0}$

$\therefore$ It is indeterminate, Applying L' Hôpital rule,

$\lim\limits_{x\to 1^+}\left(\dfrac{-2x-1}{2x}\right) = \dfrac{-2-1}{2(1)} = \dfrac{-3}{2}$

8. $\lim_{x\to 0^+} x^x$

$\lim_{x\to 0^+} x^x$

$\therefore$ This is an indeterminate form of 0^0

Let $g(x) = x^x$

Taking logarithm we get

$\log g(x) = \log x^x \Rightarrow x \log x \Rightarrow \frac{\log x}{1/x}$

$\therefore \lim_{x\to 0^+} \log g(x) = \lim_{x\to 0^+} \left[\frac{\log x}{1/x}\right] = \frac{\infty}{\infty}$ form

By L'Hôpital rule,

$= \lim_{x\to 0^+} \left[\frac{1/x}{-1/x^2}\right]$

$= \lim_{x\to 0^+} -\frac{1}{x} \times \frac{x^2}{1} \Rightarrow \lim_{x\to 0^+} -x = 0.$

But $\lim_{x\to 0^+} \log(g(x)) = \log\left(\lim_{x\to 0^+} g(x)\right)$

$\therefore \log\left(\lim_{x\to 0^+} g(x)\right) = 0$

$\Rightarrow e^{\log\left(\lim_{x\to 0^+} g(x)\right)} = e^0 \Rightarrow \lim_{x\to 0^+} g(x) = 1$ $[\because g(x) = x^x]$

$\therefore \lim_{x\to 0^+} x^x = 1$

9. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$

$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$. This is an indeterminate form 1^∞

Let $g(x) = \left(1+\frac{1}{x}\right)^x$

Taking logarithm, we get

$\log(g(x)) = \log\left(1+\frac{1}{x}\right)^x \Rightarrow x \log\left(1+\frac{1}{x}\right)$

$\Rightarrow \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$

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$\therefore \lim_{x\to\infty} \log(g(x)) = \lim_{x\to\infty} \dfrac{\log(1+\frac{1}{x})}{\frac{1}{x}} = \frac{0}{0}$ form.

$$= \lim_{x\to\infty} \dfrac{\frac{\cancel{-1/x^2}}{1+1/x}}{\cancel{-1/x^2}}$$

$$= \lim_{x\to\infty} \dfrac{1}{1+\frac{1}{x}} = 1.$$

But $\lim_{x\to\infty} \log(g(x)) = \log \lim_{x\to\infty} g(x)$

$\therefore \log\left(\lim_{x\to\infty} g(x)\right) = 1$

$e^{\log\left(\lim_{x\to\infty} g(x)\right)} = e^1 = e$

$\lim_{x\to\infty} g(x) = e$ $\quad [\because g(x) = (1+\frac{1}{x})^x]$

$\Rightarrow \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$

---

10. $\lim_{x\to\pi/2} (\sin x)^{\tan x}$

This is an indeterminate of the form 1^∞

Let $g(x) = (\sin x)^{\tan x}$

Taking logarithm we get

$\log(g(x)) = \log(\sin x)^{\tan x}$

$= \tan x \cdot \log(\sin x) = \dfrac{\log(\sin x)}{\cot x}$

Now, $\lim_{x\to\pi/2} \log(g(x)) = \lim_{x\to\pi/2} \dfrac{\log(\sin x)}{\cot x} = \frac{0}{0}$ form

[By L'Hôpital rule] $= \lim_{x\to\pi/2} \dfrac{\frac{1}{\sin x} \times \cos x}{-\operatorname{cosec}^2 x}$

$= \lim_{x\to\pi/2} \dfrac{\cos x}{\sin x \cdot \frac{1}{\sin^2 x}}$ $\quad [\operatorname{cosec}^2 x = \frac{1}{\sin^2 x}]$

$= \lim_{x\to\frac{\pi}{2}} \cos x \sin x = 0$

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But $\lim_{x\to\pi/2} \log(g(x)) = \log\left(\lim_{x\to\pi/2} g(x)\right)$

$\therefore \log\left(\lim_{x\to\pi/2} g(x)\right) = \log 0 = 1$

$\Rightarrow e^{\log\left(\lim_{x\to\pi/2} g(x)\right)} = e^1$

$[\because g(x) = (\sin x)^{\tan x}]$

$\Rightarrow \lim_{x\to\pi/2} g(x) = e \Rightarrow \lim_{x\to\pi/2} (\sin x)^{\tan x} = e$

11. $\lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}}$

This is an indeterminate of the form 1^∞

Let $g(x) = (\cos x)^{\frac{1}{x^2}}$

Taking logarithm we get

$$\log(g(x)) = \log(\cos x)^{\frac{1}{x^2}}$$

$$= \frac{\log(\cos x)}{x^2}$$

Now, $\lim_{x\to 0^+} \log(g(x)) = \lim_{x\to 0^+} \frac{\log(\cos x)}{x^2} = \frac{0}{0}$ form.

$$\Rightarrow \lim_{x\to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{2x}$$

[L'Hôpital rule]

$$\Rightarrow \lim_{x\to 0^+} -\frac{\tan x}{2x} \Rightarrow \frac{-1}{2} \lim_{x\to 0^+} \frac{\tan x}{x}$$

$$\Rightarrow \frac{-1}{2}(1) = \frac{-1}{2}$$

But $\lim_{x\to 0^+} \log(g(x)) = \log_e \log\left(\lim_{x\to 0^+} g(x)\right)$

$\therefore \log\left(\lim_{x\to 0^+} g(x)\right) = -\frac{1}{2}$

$\Rightarrow e^{\log\left(\lim_{x\to 0^+} g(x)\right)} = e^{-\frac{1}{2}}$

$\Rightarrow \lim_{x\to 0^+} g(x) = e^{-1/2} = \frac{1}{\sqrt{e}}$

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The amount after t years $(A) = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt}$

This is an indeterminate of the form 1^{∞}

$$\text{let } g(x) = \left(1+\frac{r}{n}\right)^{nt}$$

Taking logarithm we get,

$$\log(g(x)) = \log\left(1+\frac{r}{n}\right)^{nt} \Rightarrow \frac{\log\left(1+\frac{r}{n}\right)}{1/nt}$$

$$\therefore \lim_{n\to\infty} \log(g(x)) = \lim_{n\to\infty} \frac{\log(+r/n)}{1/nt} = \frac{0}{0} \text{ form}$$

By L'Hôpital rule

$$= \lim_{n\to\infty} \frac{\frac{1}{1+r/n}\left(\frac{-r}{n^2}\right)}{-\frac{1}{n^2 t}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{r}{n}}\left(\frac{-r}{n^2}\right)\left(-\frac{n^2 t}{1}\right)$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{r}{n}}\, rt \Rightarrow \left(\frac{1}{1+0}\right) rt$$

$$\Rightarrow A_0\, rt$$

But $\lim_{n\to\infty} \log(g(x)) = \log\left(\lim_{n\to\infty} g(x)\right)$

$$\therefore \log\left(\lim_{n\to\infty} g(x)\right) = rt$$

$$\Rightarrow e^{\log\left(\lim_{n\to\infty} g(x)\right)} = e^{rt} \Rightarrow \lim_{n\to\infty} (g(x)) = e^{rt}$$

$$\Rightarrow \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt} = A_0 \cdot e^{rt}$$

$$A = A_0 \cdot e^{rt}$$

EXERCISE 7.6

1. Find the absolute extreme of the following functions on the given closed interval:

(i) $f(x) = x^2 - 12x + 10$; $[1, 2]$

Given $f(x) = x^2 - 12x + 10$

$f'(x) = 2x - 12 \quad (f'(x) = 0)$

$\therefore 2x - 12 = 0$

$2x = 12$

$x = 6$

$\therefore$ The critical number is 6.

Evaluating $f(x)$ at the end points $x=1$, $x=2$ and at the critical number $x=6$ we get

$f(1) = 1^2 - 12(1) + 10 \Rightarrow 1 - 12 + 10 = -1$

$f(2) = 2^2 - 12(2) + 10 \Rightarrow 4 - 24 + 10 = -10$

$f(6) = 6^2 - 12(6) + 10 \Rightarrow 36 - 72 + 10 = -26$

From these values, the absolute maximum is -1 which occurs at $x=1$ and the absolute minimum is -26 which occurs at $x=6$.

(ii) $f(x) = 3x^4 - 4x^3$; $[-1, 2]$

Given $f(x) = 3x^4 - 4x^3$

$f'(x) = 12x^3 - 12x^2 \quad (f'(x) = 0)$

$\therefore 12x^3 - 12x^2 = 0$

$12x^2(x-1) = 0$

$\therefore x = 0$ or $x = 1$

Evaluating $f(x)$ at the end points $x=-1$, $x=2$ and at the critical number $x=0$, $x=1$ we get,

$f(-1) = 3(-1)^4 - 4(-1)^3 \Rightarrow 3(1) - 4(-1) \Rightarrow 3 + 4 = 7$

$f(2) = 3(2)^4 - 4(2)^3 \Rightarrow 3(16) - 4(8) \Rightarrow 48 - 32 = 16.$

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$f(0) = 3(0)^4 - 4(0)^3 = 0 - 0 = 0$

$f(1) = 3(1)^4 - 4(1)^3 = 3 - 4 = -1$

From these values, the absolute maximum is 16 at $x=2$ and the absolute minimum is -1 which occurs at $x=1$.

(iii) $f(x) = 6x^{4/3} - 3x^{1/3}$; $[-1, 1]$

Given $f(x) = 6x^{4/3} - 3x^{1/3}$

$f'(x) = \frac{4}{3} \times 6x^{4/3 - 1} - \frac{1}{3} \times 3x^{1/3 - 1}$

$= 8x^{1/3} - x^{-2/3}$ $\quad (\because f'(x) = 0)$

$\therefore 8x^{1/3} - \frac{1}{x^{2/3}} = 0$

$\Rightarrow \frac{8x^{1/3} \cdot x^{2/3} - 1}{x^{2/3}} = 0 \Rightarrow \frac{8x - 1}{x^{2/3}} = 0$

$[\because 8x^{1/3} \cdot x^{2/3}$
$\Rightarrow \frac{1}{3} + \frac{2}{3} = \frac{3}{3}$
$= 1]$ $[\because 8x^1]$

$\Rightarrow 8x - 1 = 0 \Rightarrow 8x = 1 \Rightarrow x = \frac{1}{8}$

$\therefore$ The critical number is $\frac{1}{8}$

Evaluating $f(x)$ at the end points $x = -1$, $x = 1$ and at the critical number $x = \frac{1}{8}$, we get

$f(-1) = 6(-1)^{4/3} - 3(-1)^{1/3} \Rightarrow 6(1) - 3(-1) \Rightarrow 6 + 3 = 9$

$f(1) = 6(1)^{4/3} - 3(1)^{1/3} \Rightarrow 6(1) - 3(1) \Rightarrow 6 - 3 = 3$

$f(\frac{1}{8}) = 6(\frac{1}{8})^{4/3} - 3(\frac{1}{8})^{1/3} \Rightarrow 6(2^{-3})^{4/3} - 3(2^{-3})^{1/3}$

$\Rightarrow 6(2^{-4}) - 3(2^{-1})$

$\Rightarrow \frac{6}{16} - \frac{3}{2} \Rightarrow \frac{3}{8} - \frac{3}{2}$

$\Rightarrow \frac{3 - 12}{4} \Rightarrow \frac{-9}{8}$

From these values, the absolute maximum is 9 which occurs at $x = -1$ and the absolute minimum is $-\frac{9}{8}$ which occurs at $x = \frac{1}{8}$.

(iv) $f(x) = 2\cos x + \sin 2x$; $[0, \pi/2]$

$$f'(x) = -2\sin x + 2\cos 2x \quad [\because f'(x) = 0]$$

$\therefore -2\sin x + 2\cos 2x = 0$

$\Rightarrow -2\sin x + 2(1 - 2\sin^2 x) = 0$ $\quad [\because \cos 2x = 1 - 2\sin^2 x]$

$\Rightarrow -2\sin x + 2 - 4\sin^2 x = 0$ $\quad [\times \text{ by } -]$

$\Rightarrow 2\sin x - 2 + 4\sin^2 x = 0$

$\Rightarrow 4\sin^2 x + 2\sin x - 2 = 0$ $\quad [\div \text{ by } 2]$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0$

$\Rightarrow (\sin x + 1)(2\sin x - 1) = 0$

$\Rightarrow \sin x = -1$ (or) $\sin x = \frac{1}{2}$

$\Rightarrow \sin x = -\sin\frac{\pi}{2}$ (or) $\sin x = \sin\frac{\pi}{6}$

$\Rightarrow \sin x = \sin -\frac{\pi}{2}$ (or) $\sin x = \sin\frac{\pi}{6}$

$\Rightarrow x = -\frac{\pi}{2}$ (or) $x = \frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}$ $\quad [\because x = -\frac{\pi}{2} \notin [0, \pi/2]]$

$\therefore$ The critical number is $\pi/6$.

Evaluating $f(x)$ at the end points $x = 0, x = \frac{\pi}{2}$ and at the critical number $x = \frac{\pi}{6}$ we get,

$f(0) = 2\cos 0 + \sin 0 = 2(1) + 0 = 2$

$f(\frac{\pi}{2}) = 2\cos\frac{\pi}{2} + \sin 2\frac{\pi}{2} = 2\cos\frac{\pi}{2} + \sin\pi = 2(0) + 0 = 0$

$f(\frac{\pi}{6}) = 2\cos\frac{\pi}{6} + \sin . 2\frac{\pi}{6} = 2\cos\frac{\pi}{6} + \sin\frac{\pi}{3}$

$\Rightarrow 2\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

From these values, the absolute maximum is $\frac{3\sqrt{3}}{2}$ which occurs at $x = \frac{\pi}{6}$ and the absolute minimum is 0 which occurs at $x = \frac{\pi}{2}$.

2. Find the intervals of monotonocities and hence find the local extremum for the following functions:

(i) $f(x) = 2x^3 + 3x^2 - 12x$.

Given $f(x) = 2x^3 + 3x^2 - 12x$

The given function is defined and differentiable for all $x \in (-\infty, \infty)$

$$f'(x) = 6x^2 + 6x - 12$$

The stationary points are given by $6x^2 + 6x - 12 = 0$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

$$\boxed{x = -2, 1}$$

Hence the intervals of monotonecity are $(-\infty, -2)$, $(-2, 1)$ and $(1, \infty)$

Interval	$(-\infty, -2)$	$(-2, 1)$	$(1, \infty)$
sign of $f'(x)$	Say $x = -3$ $f'(x) =$ $= 6(-3)^2 + 6(-3) - 12$ $= 6(9) - 18 - 12$ $= 54 - 30 = 24$ $(+ve)$	Say $x = 0$ $f'(0)$ $= 6(0) + 6(0) - 12$ $= 0 - 12 = -12$ $(-ve)$	Say $x = 2$ $f'(2)$ $= 6(2)^2 + 6(2) - 12$ $= 6(4) + 12 - 12$ $= 24$ $(+ve)$
Monotonocity	strictly increasing	strictly decreasing	strictly increasing

$\therefore f(x)$ is strictly increasing on $(-\infty, -2)$ $(1, \infty)$ and strictly decreasing on $(-2, 1)$

Since $f'(x)$ changes from +ve to -ve at $x = -2$, there is a local maximum at $x = -2$

$\therefore f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) \Rightarrow 2(-8) + 3(4) + 24$

$\Rightarrow -16 + 12 + 24 \quad \Rightarrow 20.$

Also $f'(x)$ changes from negative to positive at $x = 1$ there is a local minimum at $x = 1$

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$\therefore f(1) = 2(1)^3 + 3(1)^2 - 12(1) \Rightarrow 2+3-12 \Rightarrow 5-12 = -7$

(ii) $f(x) = \frac{x}{x-5}$.

Given $f(x) = \frac{x}{x-5}$

$f(x)$ is defined and differentiable for all $x \in R - \{5\}$

$\therefore f'(x) = \frac{(x-5)(1) - x(1)}{(x-5)^2} \Rightarrow \frac{x-5-x}{(x-5)^2}$ [by $\frac{u}{v}$ method]

$\Rightarrow \frac{-5}{(x-5)^2}$

$\therefore f'(x)=0$ but, $\frac{-5}{(x-5)^2} \neq 0$.

There is no stationary point. The possible intervals are $(-\infty, 5)$ and $(5, \infty)$

Interval	$(-\infty, 5)$	$(5, \infty)$
sign of $f'(x)$	say $x = 0$ $f'(0) = \frac{-5}{-5^2} = \frac{-5}{25}$ (-ve)	say $x = 6$ $f'(6) = \frac{-5}{(6-5)^2} = \frac{-5}{1^2}$ $= -5$ (-ve)
Monotonocity	strictly decreasing	strictly decreasing

$\therefore f(x)$ is strictly decreasing on $(-\infty, 5)$ and $(5, \infty)$ since there is no stationary point, the curve does not changes its position.

Hence there is no local extremum.

(iii) $f(x) = \frac{e^x}{1-e^x}$

Given, $f(x)$ is defined and differentiable in $(-\infty, \infty)$

$\therefore f'(x) = \frac{(1-e^x)(e^x) - (e^x)(-e^x)}{(1-e^x)^2} \Rightarrow \frac{1 - e^{2x} + e^{2x}}{(1-e^x)^2} \Rightarrow \frac{1}{(1-e^x)^2}$

$\therefore f'(x) = 0$, but $\frac{1}{(1-e^x)^2} \neq 0$.

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Thus there is no stationary point

Since $f'(x) = \frac{1}{(1-e^x)^2} > 0$ for all $x \in (-\infty, \infty)$

$f(x)$ is strictly increasing on $(-\infty, \infty)$.

Since there is no stationary point, the curve does not change its position.

Hence there is no local extremum.

(iv) $f(x) = \frac{x^3}{3} - \log x$.

$f(x)$ is defined and differentiable for all $x \in (0, \infty)$

$\therefore f'(x) = \frac{3x^2}{3} - \frac{1}{x} \Rightarrow x^2 - \frac{1}{x}$

$\therefore f'(x) = 0$, Here $x^2 - \frac{1}{x} = 0 \Rightarrow \frac{x^3-1}{x} = 0 \Rightarrow x^3 - 1 = 0$

$\Rightarrow x^3 = 1$

$\Rightarrow \boxed{x = 1}$

$\therefore$ The stationary point is at $x = 1$

The possible intervals are $(0,1)$ and $(1, \infty)$

Interval	$(0,1)$	$(1,\infty)$
sign of $f'(x)$	say $x = \frac{1}{2}$ $f'(x) = ((\frac{1}{2})^2 - \frac{1}{1/2})$ $= (1/2)^2 - 2 \Rightarrow \frac{1}{4} - 2$ $\Rightarrow \frac{1-8}{4} = \frac{-7}{4}$ $(-ve)$	say $x = 2$ $f'(x) = ((2)^2 - (\frac{1}{2}))$ $= 4 - \frac{1}{2} \Rightarrow \frac{8-1}{2}$ $= \frac{7}{2}$ $(+ve)$
Monotonocity.	strictly decreasing	strictly decreasing

Since $f'(x)$ changes from $-ve$ to $+ve$ at $x = 1$, there is a local minimum at $x = 1$.

$\therefore f(1) = \frac{1^3}{3} - \log(1) = \frac{1}{3} - 0 = \frac{1}{3}$.

(v) $f(x) = \sin x \cos x + 5, \ x \in (0, 2\pi)$. 333

$f(x)$ is defined and differentiable for all $x \in (0, 2\pi)$

$f'(x) = \sin x(-\sin x) + \cos x(\cos x)$

$= \cos^2 x - \sin^2 x = \cos 2x$

$\therefore f'(x) = 0$, Here $\cos 2x = 0$

$\Rightarrow \cos x = 0 \Rightarrow \cos\frac{\pi}{2}, \cos\frac{3\pi}{2}, \cos\frac{5\pi}{2}, \cos\frac{7\pi}{2}$

$\Rightarrow 2x = \pi/2, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

The stationary points are at

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

The possible intervals are

$(0, \frac{\pi}{4})(\frac{\pi}{4}, \frac{3\pi}{4})(\frac{3\pi}{4}, \frac{5\pi}{4})(\frac{5\pi}{4}, \frac{7\pi}{4})(\frac{7\pi}{4}, 2\pi)$

Interval	$(0, \frac{\pi}{4})$	$(\frac{\pi}{4}, \frac{3\pi}{4})$	$(\frac{3\pi}{4}, \frac{5\pi}{4})$	$(\frac{5\pi}{4}, \frac{7\pi}{4})$	$(\frac{7\pi}{4}, 2\pi)$
sign of $f'(x)$	say $x = \frac{\pi}{6}$ $f'(\pi/6) =$ $= \cos 2(\frac{\pi}{6})$ $= \cos \pi/3$ $= 1/2$ (+ve)	say $x = \pi/2$ $f'(\frac{\pi}{2})$ $= \cos 2(\frac{\pi}{2})$ $= \cos \pi$ $= -1$ (−ve)	say $x = \pi$ $f'(\pi)$ $= \cos 2\pi$ $= 1$ (+ve)	say $x = \frac{3\pi}{2}$ $f'(\frac{3\pi}{2})$ $= \cos 2(\frac{3\pi}{2})$ $= \cos 3\pi$ $= -1$ (−ve)	say $x = 320^\circ$ $f'(320^\circ)$ $= \cos 2(320^\circ)$ $= \cos 640^\circ$ $= \cos(360^\circ + 280^\circ)$ $= \cos 280^\circ$ $= \cos(270^\circ + 10^\circ)$ $= \sin 10^\circ$ $=$ (+ve)
Monotonocity	s. increasing	s. decreasing	s. increase	s. decrease	s. increase

$\therefore f(x)$ is strictly increasing in $(0, \frac{\pi}{4})(\frac{3\pi}{4}, \frac{5\pi}{4})(\frac{7\pi}{4}, 2\pi)$

and strictly decreasing in $(\frac{\pi}{4}, \frac{3\pi}{4})(\frac{5\pi}{4}, \frac{7\pi}{4})$

since $f'(x)$ changes its position from +ve to −ve at $x = \frac{\pi}{4}, \frac{5\pi}{4}$, there is a local maximum at $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

$f(\frac{\pi}{4}) = \sin\frac{\pi}{4}\cos\frac{\pi}{4} + 5 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + 5 = \frac{1}{2} + 5 = \frac{1+10}{2} = \frac{11}{2}$

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$f\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{4}\cos\frac{5\pi}{4} + 5 = \frac{-1}{\sqrt{2}}\cdot\frac{-1}{\sqrt{2}} + 5 = \frac{1}{2} + 5 = \frac{1+10}{2}$
$= 11/2$

Also $f'(x)$ changes its position from negative to positive at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$ there is a local minimum at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

$\therefore f\left(\frac{3\pi}{4}\right) = \cos\frac{3\pi}{4}\sin\frac{3\pi}{4} + 5 = \frac{-1}{\sqrt{2}}\cdot\frac{+1}{\sqrt{2}} + 5 = 5 - \frac{1}{2} = \frac{10-1}{2}$
$= 9/2$

$f\left(\frac{7\pi}{4}\right) = \cos\frac{7\pi}{4}\sin\frac{7\pi}{4} + 5 = \cos\left(2\pi - \frac{\pi}{4}\right)\sin\left(2\pi - \frac{\pi}{4}\right) + 5$
$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) + 5 = \frac{-1}{2} + 5 = 9/2$

EXERCISE 7.7

1. Find intervals of concavity and points of inflexion for the following functions:

(i) $f(x) = x(x-4)^3$

Given $f(x) = x(x-4)^3$

$f'(x) = x \cdot 3(x-4)^2 + (x-4)^3(1)$

$= 3x(x-4)^2 + (x-4)^3 = (x-4)^2 + (3x+x-4)$

$= (x-4)^2 + (4x-4) = 4[(x-1)][x-4]^2$

$f''(x) = 4[(x-1)2(x-4) + (x-4)^2(1)]$

$= 4[2(x-4)(x-1) + (x-4)^2]$

$= 4(x-4)[2x-2+x-4] = 4(x-4)(3x-6)$

$= 4(x-4)3(x-2) = 12(x-4)(x-2)$

$\therefore f''(x) = 0 \Rightarrow 12(x-4)(x-2) = 0 \Rightarrow (x-4)(x-2) = 0$

$\Rightarrow x = 2, 4$

The possible intervals are $(-\infty, 2)(2,4)$ and $(4, \infty)$

Interval	$(-\infty, 2)$	$(2,4)$	$(4, \infty)$
sign of $f''(x)$	say $x=0$ $f''(0) = 12(0-4)(0-2)$ $= 12(-4)(-2)$ $= 12(8) = 96$ $(+ve)$	say $x=3$ $f''(3) = 12(3-4)(3-2)$ $= 12(-1)(1)$ $= -12$ $(-ve)$	say $x=5$ $f''(5) =$ $12(5-4)(5-2)$ $= 12(1)(3)$ $= 36$ $(+ve)$
Concavity	Concave up	concave down	Concave up

$\therefore$ The curve is concave upward on $(-\infty, 2)(4,\infty)$ and concave downward on $(2,4)$

As $f''(x)$ changes its sign when it passes through $x=2$ and $x=4$, the points of inflection are $(2, f(2))$ and $(4, f(4))$.

$f(2) = 2(2-4)^3 \Rightarrow 2(-2)^3 \Rightarrow 2(-8) = -16$

$f(4) = 4(4-4)^3 \Rightarrow 4(0)^3 \Rightarrow 4(0) = 0$

$\therefore (2,-16)$ and $(4,0)$ are the points of inflection.

(ii) $f(x) = \sin x + \cos x,\ 0 < x < 2\pi$.

Given $f(x) = \sin x + \cos x,\ 0 < x < 2\pi$

$f'(x) = \cos x - \sin x$

$f''(x) = -\sin x - \cos x$

$\therefore f''(x) = 0$, Here $-\sin x - \cos x = 0$

$-\sin x = \cos x$

$\sin(-x) = \cos x$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}, 2\pi$

$\therefore$ The possible intervals are $(0, \frac{3\pi}{4})(\frac{3\pi}{4}, \frac{7\pi}{4})(\frac{7\pi}{4}, 2\pi)$.

Interval	$(0, \frac{3\pi}{4})$	$(\frac{3\pi}{4}, \frac{7\pi}{4})$	$(\frac{7\pi}{4}, 2\pi)$
sign of $f''(x)$	say $x = -1$ $f''(-1) = -\sin(-1) - \cos(-1)$ $= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$ $= -\frac{2}{\sqrt{2}}$ (-ve)	say $x = \pi$ $f''(\pi)$ $= -\sin\pi - \cos\pi$ $= 0 - (-1)$ $= 0 + 1 = 1$ (+ve)	say $x = 320^\circ$ $f''(320^\circ)$ $= -\sin 320^\circ - \cos 320^\circ$ $= -\sin(270^\circ + 60^\circ) - \cos(270^\circ + 60^\circ)$ $= \cos 60^\circ - \sin 60^\circ$ $= \frac{1}{2} - \frac{\sqrt{3}}{2}$ (-ve)
Concavity	concave down	concave up	concave down

$\therefore f(x)$ is concave upward in $(\frac{3\pi}{4}, \frac{7\pi}{4})$ and concave downward in $(0, \frac{3\pi}{4})(\frac{7\pi}{4}, 2\pi)$.

Since $f''(x)$ changes its sign from -ve to +ve at $\frac{3\pi}{4}$ and +ve to -ve at $\frac{7\pi}{4}$

$f(x)$ has point of inflection at $(\frac{3\pi}{4}, f(\frac{3\pi}{4}))$ & $(\frac{7\pi}{4}, f(\frac{7\pi}{4}))$

$\therefore f(\frac{3\pi}{4}) = \sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = \sin(\pi - \frac{\pi}{4}) + \cos(\pi - \frac{\pi}{4})$

$= \sin(\frac{\pi}{4}) + -\cos(\frac{\pi}{4})$

$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$

$\therefore$ The points of inflection are $(\frac{3\pi}{4}, 0)$ and $(\frac{7\pi}{4}, 0)$

(iii) $f(x) = \frac{1}{2}(e^{x} - e^{-x})$

$f(x)$ is defined and differentiable for all $x \in (-\infty, \infty)$

$f'(x) = \frac{1}{2}(e^{x} + e^{-x})$

$f''(x) = \frac{1}{2}(e^{x} - e^{-x})$

$\therefore f''(x) = 0$, Here $\frac{1}{2}(e^{x} - e^{-x}) = 0$

$\Rightarrow e^{x} - e^{-x} = 0 \Rightarrow e^{x} = e^{-x}$

$\Rightarrow e^{x} = \frac{1}{e^{x}}$

$\Rightarrow e^{2x} = 1$

$\Rightarrow e^{2x} = e^{0}$ $[\because e^{0} = 1]$

$\Rightarrow 2x = 0$

$\Rightarrow x = 0$

The possible intervals are $(-\infty, 0)$ and $(0, \infty)$

Interval	$(-\infty, 0)$	$(0, \infty)$
sign of $f''(x)$	say $x = -1$ $f''(-1) = \frac{1}{2}(e^{-1} - e^{1})$ $= (-ve)$ concave down	say $x = 1$ $f''(1) = \frac{1}{2}(e^{1} - e^{-1})$ $= (+ve)$ concave up

$\therefore f(x)$ is concave up in $(0, \infty)$ and concave down in $(-\infty, 0)$

Since $f''(x)$ changes its position from $-ve$ to $+ve$, when it passes through $x = 0$ the points of inflection is $(0, f(0))$.

$f(0) = \frac{1}{2}(e^{0} - e^{0}) = \frac{1}{2}(1 - 1) = \frac{1}{2}(0) = 0$

$\therefore (0, 0)$ is the point of inflection.

2. Find the local extrema for the following functions using second derivative test:

(i) $f(x) = -3x^5 + 5x^3$.

$$f(x) = -3x^5 + 5x^3$$

$$f'(x) = -15x^4 + 15x^2$$

$\therefore f'(x) = 0 \Rightarrow -15x^4 + 15x^2 = 0$

$$\Rightarrow -15x^2(1 - x^2) = 0$$

$$\Rightarrow -15x^2 = 0, \quad 1 - x^2 = 0$$

$$\Rightarrow x^2 = 0, \quad (1+x)(1-x) = 0$$

$$\Rightarrow x = 0, \; x = 1, \; x = -1$$

$\therefore$ The critical numbers are $0, 1, -1$.

$$f''(x) = -60x^3 + 30x$$

$$\therefore f''(0) = -60(0)^3 + 30(0) = 0 + 0 = 0$$

$$f''(1) = -60(1)^3 + 30(1) = -60 + 30 = -30$$

$$f''(-1) = -60(-1)^3 + 30(-1) = 60 - 30 = 30$$

since $f''(1) < 0$, it has a local maximum at $x = 1$

$$\therefore f(1) = -3(+1)^5 + 5(1)^3 = -3 + 5 = +2$$

since $f''(-1) > 0$, it has a local minimum at $x = -1$

$$\therefore f(-1) = -3(-1)^5 + 5(-1)^3 = 3 - 5 = -2$$

$\therefore$ Local maximum is 2 which occurs at $x = 1$ and local minimum is -2 which occurs at $x = -1$.

(ii) $f(x) = x \log x$

$$f(x) = x \log x$$

$$f'(x) = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$= 1 + \log x$$

$\therefore f'(x) = 0 \Rightarrow 1 + \log x = 0 \Rightarrow \log x = -1$

$$x = e^{-1} = \frac{1}{e}$$

$\therefore$ The critical number is $\frac{1}{e}$.

$f''(x) = \frac{1}{x}$

$\therefore f''(\frac{1}{e}) = \frac{1}{1/e} = e > 0$

since $f''(\frac{1}{e}) > 0$, there is a local minimum at $x = \frac{1}{e}$

$\therefore f(\frac{1}{e}) = \frac{1}{e} \log \frac{1}{e}$

$= \frac{1}{e} \log e^{-1} \Rightarrow \frac{-1}{e} \log_e e$

$\Rightarrow \frac{-1}{e}(1) \Rightarrow -\frac{1}{e}$

$\therefore$ Local minimum is $-\frac{1}{e}$ which occurs at $x = \frac{1}{e}$.

(iii) $f(x) = x^2 e^{-2x}$

$f(x) = x^2 e^{-2x}$

$f'(x) = x^2(-2)e^{-2x} + e^{-2x}(2x)$ [$\because$ take common $2xe^{-2x}$]

$= 2xe^{-2x}(-x+1) \Rightarrow 2xe^{-2x}(1-x)$ [$\because f'(x)$]

$\therefore f'(x) = 0 \Rightarrow 2xe^{-2x}(1-x) = 0$

$\Rightarrow 2xe^{-2x} = 0, (1-x) = 0$

$\Rightarrow x = 0, -x = -1 \Rightarrow x = 1.$

$\therefore$ The critical numbers are $x = 0, 1$. [$uvw = uvw' + uv'w + u'vw$]

$f''(x) = 2[xe^{-2x}(-1) + x(-2e^{-2x})(1-x) + 1.e^{-2x}(1-x)]$

$= 2e^{-2x}[-x(-2x(1-x)) + (1-x)]$ [Take out e^{-2x}]

$= 2e^{-2x}(-x - 2x + 2x^2 + 1 - x)$

$= 2e^{-2x}(2x^2 - 4x + 1)$

$\therefore f''(0) = 2e^{-2(0)}(2(0)^2 - 4(0) + 1) = 2e^0(0 - 0 + 1)$

$= 2(1)(1) = 2 > 0$

$f''(1) = 2e^{-2(1)}(2(1)^2 - 4(1) + 1) = 2e^{-2}(2 - 4 + 1)$

$= 2e^{-2}(-1) = -2e^{-2}$

$= \frac{-2}{e^2} > 0$

since $f''(0) > 0$, there is a local minimum at $x = 0$

$\therefore f(0) = 0^2 e^0 = 0(1) = 0$

since $f''(1) < 0$, there is a local maximum at $x = 1$

$\therefore f(1) = 1^2 e^{-2(1)} = 1.e^{-2} = \frac{1}{e^2}$

3. For the function $f(x) = 4x^3+3x^2-6x+1$ find the intervals of monotonocity, local extrema, intervals of concavity and points of inflection.

Given $f(x) = 4x^3+3x^2-6x+1$

$f'(x) = 12x^2+6x-6$

$f''(x) = 24x+6$

$\therefore f'(x)=0 \Rightarrow 12x^2+6x-6=0$

$\Rightarrow 2x^2+x-1=0$

$(x+1)(2x-1)=0$

$\therefore x=-1, \frac{1}{2}$.

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$\frac{2}{2}$ | $\frac{-1}{2}$

$\therefore$ The criticals numbers are $-1, \frac{1}{2}$.

The possible intervals of monotonocity are $(-\infty,-1)$ $(-1,\frac{1}{2})$ $(\frac{1}{2},\infty)$

Interval	$(-\infty,-1)$	$(-1,\frac{1}{2})$	$(\frac{1}{2},\infty)$
sign of $f'(x)$	say $x=-2$ $f'(-2)$ $=12(-2)^2+6(-2)-6$ $=12(4)-12-6$ $=48-18=30$ (+ve)	say $x=0$ $f'(0)$ $=12(0)+6(0)-6$ $=-6$ (-ve)	say $x=1$ $f'(1)$ $=12(1)^2+6(1)-6$ $=12+6-6=12$ (+ve)
Monotonocity	strictly increasing	strictly decreasing	strictly increasing

$\therefore f(x)$ is strictly increasing in $(-\infty,-1)$ $(\frac{1}{2},\infty)$ and strictly decreasing in $(-1,\frac{1}{2})$.

$\therefore f''(x)=0 \Rightarrow 24x+6=0 \Rightarrow 24x=-6 \Rightarrow x=-6/24$

$\Rightarrow x=-1/4$

The possible intervals of concavity are $(-\infty,\frac{-1}{4})(\frac{-1}{4},\infty)$

Interval	$(-\infty,-\frac{1}{4})$	$(-\frac{1}{4},\infty)$
sign of $f''(x)$	say $x=-1$ $f''(-1)=24(-1)+6=-24+6$ $=-18$ (-ve)	say $x=0$ $f''(0)=24(0)+6=0+6=0$ (+ve)
Concavity	concave down	concave up

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$\therefore f(x)$ is concave down in $(-\infty, -\frac{1}{4})$ and concave up in $(-\frac{1}{4}, \infty)$

As $f''(x)$ changes its sign when it passes through $x = -\frac{1}{4}$, the point of inflection is $(-\frac{1}{4}, f(-\frac{1}{4}))$

Now $f(-\frac{1}{4}) = 4(-\frac{1}{4})^3 + 3(-\frac{1}{4})^2 - 6(-\frac{1}{4}) + 1$

$= 4(\frac{-1}{64}) + 3(\frac{1}{16}) + \frac{6}{4} + 1 \Rightarrow \frac{-1}{16} + \frac{3}{16} + \frac{3}{2} + 1$

$= \frac{2}{16} + \frac{3}{2} + 1 = \frac{1}{8} + \frac{3}{2} + 1 \Rightarrow \frac{1+12+8}{8} = \frac{21}{8}$

$\therefore$ Point of inflection is $(-\frac{1}{4}, \frac{21}{8})$

since $f'(x)$ changes its sign from +ve to -ve at $x = -1$, it has a local maximum at $x = -1$.

$\therefore f(-1) = 4(-1)^3 + 3(-1)^2 - 6(-1) + 1 \Rightarrow 4(-1) + 3(1) + 6 + 1$

$= -4 + 3 + 7 = 6.$

since $f'(x)$ changes its sign from -ve to +ve at $x = \frac{1}{2}$ it has a local minimum at $x = \frac{1}{2}$

$f(\frac{1}{2}) = 4(\frac{1}{2})^3 + 3(\frac{1}{2})^2 - 6(\frac{1}{2}) + 1 \Rightarrow 4(\frac{1}{8}) + 3(\frac{1}{4}) - \frac{6}{2} + 1$

$\Rightarrow \frac{4}{8} + \frac{3}{4} - \frac{3}{1} + 1$

$\Rightarrow \frac{1}{2} + \frac{3}{4} - \frac{3}{1} + \frac{1}{1}$

$\Rightarrow \frac{2+3}{4} - 2$

$\Rightarrow \frac{5}{4} - 2$

$= \frac{5-8}{4}$

$= \frac{-3}{4}.$

EXERCISE 7.8

1. Find two positive numbers whose sum is 12 and their product is maximum.

Let the two positive numbers be x and y

Given $x+y=12$

$y=12-x \quad \ldots (1)$

Let $f(x) = xy$

$= x(12-x) = 12x - x^2$

$f'(x) = 12-2x$, $f''(x) = -2$

$\therefore f'(x) = 0 \Rightarrow 12-2x = 0$

$12 = 2x \Rightarrow \frac{12}{2} = x \Rightarrow \boxed{x=6}$

$\therefore$ The critical number is 6.

$f''(x) = -2$

Since $f'' = -2 < 0$, $f(x)$ is maximum when $x=6$.

The product of two numbers is maximum when $x=6$.

If $x=6$,

$y = 12-6 \Rightarrow \boxed{y=6}$ [From (1)]

Hence the required numbers are 6, 6.

2. Find two positive numbers whose product is 20 and their sum is minimum.

Let the two positive numbers be x and y.

Given $xy = 20$

$y = \frac{20}{x} \quad \ldots (1)$

Let $f(x) = x+y \Rightarrow x + \frac{20}{x}$

$f'(x) = 1 - \frac{20}{x^2}$, $f''(x) = \frac{40}{x^3}$

$\therefore f'(x) = 0 \Rightarrow 1 - \frac{20}{x^2} = 0 \Rightarrow 1 = \frac{20}{x^2} \Rightarrow x^2 = 20$

$\Rightarrow x = \sqrt{20}$

$\boxed{x = \pm 2\sqrt{5}}$

$\therefore$ The critical numbers are $2\sqrt{5}, -2\sqrt{5}$.

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$f''(x) = \frac{40}{x^3}$

when $x = 2\sqrt{5}$, $f''(x) = \frac{40}{(2\sqrt{5})^3} > 0$

$f(x)$ is minimum when $x = 2\sqrt{5}$.

[From (1)]

when $x = 2\sqrt{5}$, $y = \frac{20}{2\sqrt{5}} = \frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$ [conjugate multiply]

$= \frac{10\sqrt{5}}{5} = \boxed{2\sqrt{5}}$

Hence the required positive numbers are $2\sqrt{5}, 2\sqrt{5}$.

3. Find the smallest possible value of x^2+y^2 given that $x+y=10$.

Given $x+y=10$

$y = 10-x \quad \ldots (1)$

Let $f(x) = x^2+y^2 \Rightarrow x^2+(10-x)^2$ $[(a-b)^2 = a^2-2ab+b^2]$

$\Rightarrow x^2+100-20x+x^2$

$f(x) = 2x^2-20x+100$

$f'(x) = 4x-20, \; f''(x) = 4.$

$\therefore f'(x)=0 \Rightarrow 4x-20=0 \Rightarrow 4x=20 \Rightarrow x = \frac{20}{4} \Rightarrow \boxed{x=5}$

$\therefore$ The critical number is 5.

$f''(x) = 4$

$\therefore f''(5) = 4 > 0$

$\therefore f(x)$ is minimum when $x=5$

when $x=5$, $y = 10-5 \Rightarrow \boxed{y=5}$

$\therefore$ smallest possible value of x^2+y^2

$= 5^2+5^2 = 25+25 = 50$

$\therefore x^2+y^2 = 50.$

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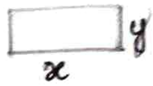
4. A garden is to be laid out in a rectangular area and protected by wire fence. What is the largest possible area of the fenced garden with 40 metres of wire?



Let x be the length of the garden and y be the breadth of the garden.

Given $2(x+y)=40$.

[Length of wire $=40$, perimeter 40]

$(x+y)=20$

$y=20-x \quad \ldots (1)$

Let $f(x)=xy=x(20-x)=20x-x^2$

$\therefore f'(x)=20-2x$, $f''(x)=-2$.

Here $f'(x)=0 \Rightarrow 20-2x=0 \Rightarrow -2x=-20$

$x=\frac{20}{2} \Rightarrow \boxed{x=10}$

$\therefore$ The critical number is 10.

$f''(x)=-2$

Now $f''(10)=-2<0$

$\therefore f(x)$ is minimum at $x=10$.

when $x=10$,

$y=20-10 \Rightarrow \boxed{y=10}$

$\therefore$ Area $=f(x)=xy$

$=10(10)=100\ m^2$

$\therefore$ Largest possible area of the garden $=100\ m^2$

5. A rectangular page is to contain $24\ cm^2$ of print. The margins at the top and bottom of the page are 1.5 cm and the margins at other sides of the page is 1 cm. What should be the dimensions of the page so that the area of the paper used is minimum.

Let x and y be the length, breadth of the printed rectangular page respectively.

Given $xy = 24$

$y = \frac{24}{x}$ ... (1)

Length of the page with margin

$= x+1+1 = x+2$

Breadth of the page with margin

$= y+1.5+1.5 = y+3$

Area of the page $= (x+2)(y+3)$

Let $f(x) = (x+2)(y+3)$ $[\because y = \frac{24}{x}]$

$= (x+2)\left(\frac{24}{x}+3\right)$

$= 3x + \frac{48}{x} + 24 + 6$

$= 3x + \frac{48}{x} + 30$

$f'(x) = 3 - \frac{48}{x^2}$,

$\therefore f'(x) = 0 \Rightarrow 3 - \frac{48}{x^2} = 0 \Rightarrow 3 = \frac{48}{x^2} \Rightarrow x^2 = 16 \Rightarrow \boxed{x = \pm 4}$

$\therefore$ The critical numbers are $4, -4$.

$f''(x) = -48\left(\frac{-2}{x^3}\right) \Rightarrow \frac{96}{x^3}$

$f''(4) = \frac{96}{4^3} = \frac{96}{64} > 0$

$\therefore f(x)$ is minimum when $x = 4$

If $x=4$, $y = \frac{24}{4} \Rightarrow \boxed{y = 6}$

$\therefore$ Length of the page $= x+2 \Rightarrow 4+2 = 6$ cm

Breadth of the page $= y+3 \Rightarrow 6+3 = 9$ cm.

6. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 1,80,000 sq.mtrs. in order to provide enough grass for herds. No fencing is needed along the river. What is the length of the minimum needed fencing material?

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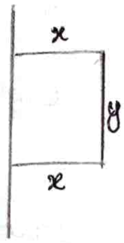
Let x be the length of the rectangular pasture and y be the breadth of the pasture



Given $xy = 1,80,000$

$$y = \frac{1,80,000}{x} \quad \ldots (1)$$

since fencing is not needed along the river side,

perimeter $= 2x + y$

Let $f(x) = 2x + \frac{1,80,000}{x}$

$$f'(x) = 2 - \frac{1,80,000}{x^2}$$

$f'(x) = 0 \Rightarrow \therefore 2 - \frac{1,80,000}{x^2} = 0 \Rightarrow 2 = \frac{1,80,000}{x^2}$

$\Rightarrow x^2 = 90,000 \Rightarrow \boxed{x = \pm 300}$

$\therefore$ The critical number is $300, -300$.

$f''(x) = -1,80,000\left(\frac{-2}{x^3}\right) = \frac{3,60,000}{x^3}$

$f''(300) = \frac{3,60,000}{(300)^3} > 0$

$\therefore f(x)$ is minimum when $x = 300$

From (1), when $x = 300$,

$y = \frac{1,80,000}{300} = 600 \Rightarrow \boxed{y = 600}$

$\therefore$ Length of the minimum needed fencing material

$= 2x + y$

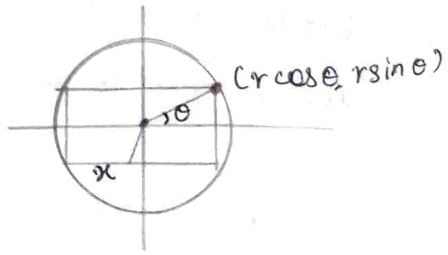
$= 2(300) + 600$

$= 600 + 600$

$\boxed{= 1200\text{ m}}$

7. Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10 cm.

Let the point on the circle be



$x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow x = 10\cos\theta$, $y = 10\sin\theta$

$\therefore$ Length of the rectangle $= 2x$ and

Breadth of the rectangle $= 2y$

$\therefore 2x = 20\cos\theta$, $2y = 20\sin\theta$

$\therefore$ Area $= f(\theta) = 2x(2y)$

$= 400\cos\theta\sin\theta$

$f(\theta) = 200\sin 2\theta$

$f'(\theta) = 200(2)\cos 2\theta = 400\cos 2\theta$

[$\because 2\cos\theta\sin\theta = \sin 2\theta$]

$400\cos\theta\sin\theta$

$= 2(200)\cos\theta\sin\theta)$

$= 200\sin 2\theta$

$\therefore f'(\theta) = 0 \Rightarrow 400\cos 2\theta = 0$

$\Rightarrow \cos 2\theta = 0$

$\Rightarrow \cos 2\theta = \cos\frac{\pi}{2}$ [$\because \cos\frac{\pi}{2} = 0$]

$\Rightarrow 2\theta = \pi/2 \Rightarrow \boxed{\theta = \frac{\pi}{4}}$

$\therefore$ The critical number is $\frac{\pi}{4}$.

$f''(\theta) = 400(2) - \sin 2\theta = -800\sin 2\theta$

$f''(\frac{\pi}{4}) = -800\sin 2(\frac{\pi}{4}) = -800\sin\frac{\pi}{2}$ [$\because \sin\frac{\pi}{2} = 1$]

$= -800(1) = -800 < 0$.

$\therefore f(\theta)$ is maximum when $\theta = \frac{\pi}{4}$

$\therefore$ Length of the rectangle $= 2x$ [$\because 2x = 20\cos\theta$]

$= 20\cos\theta = 20\cos\frac{\pi}{4} = 20 \cdot \frac{1}{\sqrt{2}}$

$\therefore$ Length of the rectangle $= \boxed{\frac{20}{\sqrt{2}}}$

Breadth of the rectangle $= 2y$ [$\because 2y = 20\sin\theta$]

$= 20\sin\theta = 20\sin\frac{\pi}{4} = 20 \cdot \frac{1}{\sqrt{2}}$

$\therefore$ Breadth of the rectangle $= \boxed{\frac{20}{\sqrt{2}}}$

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8. Prove that among all the rectangles of the given perimeter, the square has the maximum area.

Let x and y be the length and breadth of the rectangle.

$\therefore$ Perimeter $P = 2x + 2y$

$$2y = P - 2x \Rightarrow y = \frac{P-2x}{2} \quad \ldots (1)$$

Let $f(x) = \text{Area} = xy = x\left(\frac{P-2x}{2}\right)$ [$\because$ From (1)]

$$f(x) = \frac{Px - 2x^2}{2}$$

$$f'(x) = \frac{1}{2}[P - 4x] \quad [\tfrac{1}{2} \rightarrow \text{constant}]$$

$\therefore f'(x) = 0 \Rightarrow \frac{1}{2}[P - 4x] = 0 \Rightarrow P - 4x = 0 \Rightarrow P = 4x$

$$\Rightarrow \boxed{x = \frac{P}{4}}$$

$\therefore$ The critical number is $\frac{P}{4}$.

$f''(x) = \frac{1}{2}[-4] = -2$

$\therefore f''\left(\frac{P}{4}\right) = -2 < 0$

$\therefore f(x)$ is maximum when $x = \frac{P}{4}$

when $x = \frac{P}{4}$, $y = \frac{P - 2\left(\frac{P}{4}\right)}{2} = \frac{P - \frac{P}{2}}{2} = \frac{\frac{2P-P}{2}}{2} = \frac{\frac{P}{2}}{2}$

$$= \frac{P}{2} \times \frac{1}{2}$$

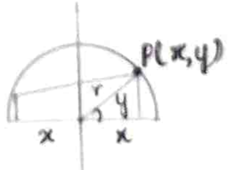
$$\boxed{y = \frac{P}{4}}$$

$\therefore x = y = \frac{P}{4}$.

$\therefore$ The rectangle is a square when area is maximum for a given perimeter.

9. Find the dimensions of the largest rectangle that can be inscribed in a semi-circle of radius r cm.

Let us take the semi circle with center (0,0) and radius r.



$x = r\cos\theta$, $y = r\sin\theta$ 349

Length of the rectangle $= 2x = 2r\cos\theta$

Breadth of the rectangle $= y = r\sin\theta$

$\therefore$ Area $= 2xy = 2r^2\cos\theta\sin\theta$

$f(\theta) = r^2\sin 2\theta$

$f'(\theta) = r^2\, 2\cos 2\theta$

$[r^2 = \text{constant}]$

$[\because 2\cos\theta\sin\theta = \sin 2\theta]$

$2r^2\cos\theta\sin\theta$

$= r^2(2\cos\theta\sin\theta)$

$= r^2\sin 2\theta$

$\therefore f'(\theta) = 0 \Rightarrow r^2\, 2\cos 2\theta = 0$

$\Rightarrow \cos 2\theta = 0$

$\Rightarrow \cos 2\theta = \cos\frac{\pi}{2}$ $[\because \cos\frac{\pi}{2} = 0]$

$\Rightarrow 2\theta = \pi/2 \Rightarrow \boxed{\theta = \frac{\pi}{4}}$

$\therefore$ The critical value is $\pi/4$.

$f''(\theta) = 2r\cdot 2(-\sin 2\theta) = -4r\sin 2\theta$

$\therefore f''(\frac{\pi}{4}) = -4r\sin 2(\pi/4) = -4r\sin\frac{\pi}{2} = 4r < 0$

$\therefore f(\theta)$ is maximum when $\theta = \frac{\pi}{4}$

$\therefore$ Length of the rectangle $(2x) = 2r\cos\theta = 2r\cos\pi/4$

$= 2r\cdot\frac{1}{\sqrt{2}}$

$= \sqrt{2}\cdot\sqrt{2}\,r\cdot\frac{1}{\sqrt{2}}$

$\boxed{= \sqrt{2}\,r.}$

$\therefore$ Breadth of the rectangle $y = r\sin\theta$

$= r\sin\frac{\pi}{4} = r\cdot\frac{1}{\sqrt{2}} = \boxed{\frac{r}{\sqrt{2}}.}$

10. A manufacturer wants to design an open box having a square base and a surface area of 108 sq.cm. Determine the dimensions of the box for the maximum value.

Since the open box has square area, let the length, breadth and height of the box be l, l and b cm respectively.

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$\therefore$ surface area $= l^2 + 4lb = 108$

$\Rightarrow l(l+4b) = 108$ [take l common]

$\Rightarrow l + 4b = \frac{108}{l}$

$\Rightarrow 4b = \frac{108}{l} - l$

$\Rightarrow b = \frac{108}{4l} - \frac{l}{4}$

$\Rightarrow b = \frac{27}{l} - \frac{l}{4} \quad \ldots (1)$

Let $f(l)$ = Volume of box $= l \times l \times b = l^2 b$

$= l^2\left(\frac{27}{l} - \frac{l}{4}\right)$ [From(1)]

$= \frac{27l^2}{l} - \frac{l^3}{4}$

$= 27l - \frac{l^3}{4}$

$\therefore f(l) = 27l - \frac{l^3}{4}$

$\therefore f'(l) = 27 - \frac{3l^2}{4}$

$\therefore f'(l) = 0 \Rightarrow 27 - \frac{3l^2}{4} = 0 \Rightarrow \frac{3l^2}{4} = 27$

$\Rightarrow l^2 = 27 \times \frac{4}{3}$

$\Rightarrow l^2 = 36 \Rightarrow l = \pm 6$

$\therefore \boxed{l = 6.}$

$\therefore$ The critical number is 6.

$f''(l) = -\frac{6l}{4} = \frac{-3l}{2}$

$\therefore f''(6) = -\frac{3(6)}{2} = \frac{-18}{2} = -9 < 0$

$\therefore f(l)$ is maximum when $l = 6$.

when $l = 6$, $b = \frac{27}{6} - \frac{6}{4} = \frac{9}{2} - \frac{3}{2} = \frac{6}{2} = \boxed{3 \text{ cm.}}$ [From(1)]

Hence the dimensions of the required box are 6 cm, 6 cm and 3 cm respectively.

11. The volume of a cylinder is given by the formula $V = \pi r^2 h$. Find the greatest and least values of V if $r+h=6$. 351

Given $r+h=6$

$h = 6-r \ldots (1)$

Let $f(x) \Rightarrow V = \pi r^2 h$

$= \pi r^2 (6-r) \Rightarrow \pi(6r^2 - r^3)$

$\therefore f(r) = \pi(6r^2 - r^3)$, $f'(r) = \pi(12r - 3r^2)$

$\therefore f'(r) = 0 \Rightarrow \pi(12r - 3r^2) = 0 \Rightarrow 12r - 3r^2 = 0$

$\Rightarrow 3r(4-r) = 0$

$\Rightarrow 3r = 0, 4-r = 0$

$\Rightarrow r = 0, r = 4.$

$\therefore$ The critical numbers are 0, 4.

$\therefore f''(r) = \pi(12-6r)$

$f''(4) = \pi(12-6(4)) = \pi(12-24) < 0$

$\therefore f(r) =$ maximum when $r=4$.

When $r=4$, $h = 6-4 = 2$ [using 1]

$r=0$, $h = 6-0 = 6$

$\therefore$ volume of cylinder $V = \pi r^2 h = \pi(4)^2(2) = \pi(16)(2)$

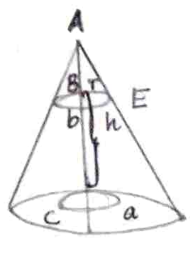
$= 32\pi$ cu. units

[or]

Volume of cylinder $V = \pi r^2 h = \pi(0^2)(6) = \pi(0)(6)$

$= 0$ cu. units.

12. A hollow cone with base radius a cm and height b cm is placed on a table. Show that the volume of the largest cylinder that can be hidden underneath is $\frac{4}{9}$ times volume of the cone.



Given $AC = b$ and $CD = a$

Let h and r be the height and radius of the cylinder.

$\therefore AB = AC - BC = b - h.$

In the diagram,

$\Delta ABE \sim \Delta ACD$

$\therefore \frac{AB}{BE} = \frac{AC}{CD} \Rightarrow \frac{b-h}{r} = \frac{b}{a}$

$\Rightarrow r = \frac{a(b-h)}{b}$ . . . (1)

Volume of the cylinder

$= \pi r^2 h$

$= \pi \left(\frac{a^2}{b^2}(b-h)^2 h \right)$ [From (1)]

$= \pi \frac{a^2}{b^2}(b^2+h^2-2bh)h$

$V = \frac{\pi a^2}{b}(b^2h^2+h^3-2bh^2)$ . . . . (2)

Differentiating with respect to 'h'

$\frac{dV}{dh} = \frac{\pi a^2}{b^2}(b^2+3h^2-4bh)$ . . . (3)

$\therefore \frac{dV}{dh} = 0 \Rightarrow \frac{\pi a^2}{b^2}(b^2+3h^2-4bh) = 0$

$\Rightarrow b^2 - 4bh + 3h^2 = 0$

3
-1 | -3

$\Rightarrow (b-3h)(b-h) = 0$

$\Rightarrow b = 3h, h$

$\Rightarrow \boxed{h = \frac{b}{3}}$ as $b \neq h$.

$\frac{d^2V}{dh^2} = \frac{\pi a^2}{b^2}(6h-4b)$

$\therefore \frac{d^2V}{dh^2}_{(b/3)} = \frac{\pi a^2}{b^2}(6(b/3)-4b) = \frac{\pi a^2}{b^2}(2b-4b)$

$= \frac{\pi a^2}{b^2}(-2b) < 0$

$\therefore V$ is maximum at $h = \frac{b}{3}$.

$\therefore$ Maximum volume of cylinder $= V = \frac{\pi a^2}{b^2}(b-h)^2 h$. [$\because h = b/3$]

$= \frac{\pi a^2}{b^2}(b-\frac{b}{3})^2(\frac{b}{3})$

$= \frac{\pi a^2}{b^2}\left(\frac{3b-b}{3}\right)^2\left(\frac{b}{3}\right)$

$= \frac{\pi a^2}{b^2}\left(\frac{2b}{3}\right)^2\left(\frac{b}{3}\right)$

$$= \frac{\pi a^2}{b^2}\left(\frac{4b^2}{9}\right)\left(\frac{b}{3}\right) = \frac{4}{9}\left(\frac{1}{3}\pi a^2 b\right)$$

$$= \frac{4}{9}\text{ (volume of the cone)}.$$

Hence, volume of the largest cylinder that can be hidden underneath is $\frac{4}{9}$ times volume of the cone.

EXERCISE 7.9.

1. Find the asymptotes of the following curves:

(i) $f(x) = \frac{x^2}{x^2-1}$

Given $f(x) = \frac{x^2}{x^2-1}$

$$\lim_{x\to 1^+} \frac{x^2}{x^2-1} = \lim_{x\to 1^+} \frac{x^2}{(x+1)(x-1)}$$

$[x^2-1^2$ form$]$

$$= \lim_{h\to 0^+} \frac{(1+h^2)}{(1+h+1)(1+h-1)} = \lim_{h\to 0^+} \frac{(1+h^2)}{(2+h)(h)} = \infty$$

Also, $\lim_{x\to 1^-} \frac{x^2}{(x+1)(x-1)} = \lim_{h\to 0^+} \frac{(1-h^2)}{(1-h+1)(1-h-1)} = \lim_{h\to 0^+} \frac{(1-h^2)}{(2-h)(-h)} = -\infty$

$\therefore x=-1$ and $x=1$ are vertical asympotates.

Also, $\lim_{x\to\infty} \frac{x^2}{x^2-1} = \lim_{1/x\to 0} \frac{1}{1-\frac{1}{x^2}} = 1$

$[\div$ numerator & denominator by $x^2]$

$\therefore y=1$ is a horizontal asymptote.

(ii) $f(x) = \frac{x^2}{x+1}$

Given $f(x) = \frac{x^2}{x+1}$

$$\lim_{x\to -1^-} \frac{x^2}{x+1} = \lim_{h\to 0^+} \frac{(-1-h)^2}{-1-h+1} = \lim_{h\to 0^+} \frac{(1+h)^2}{-h} = -\infty$$

$$\lim_{x\to -1^+} \frac{x^2}{x+1} = \lim_{h\to 0^+} \frac{(-1+h)^2}{-1+h+1} = \lim_{h\to 0^+} \frac{(h-1)^2}{h} = \infty$$

$\therefore x=-1$ is a vertical asymptote.

since the numerator's degree is more than the denominator degree, let us divide

x^2 by $x+1$

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2 \\ & \underline{x^2+x} \\ & \quad -x \\ & \underline{-x^2-1} \\ & \quad 1 \end{array}$$

$\therefore y = x-1$ is the slanting asymptote.

(iii) $f(x) = \dfrac{3x}{\sqrt{x^2+2}}$

$$\lim_{x\to\infty} \frac{3x}{\sqrt{x^2+2}} = \lim_{\frac{1}{x}\to 0} \frac{3}{\sqrt{1+\frac{2}{x^2}}} = \frac{3}{\sqrt{1+0}} = 3$$

$\therefore y = 3$ is the horizontal asymptote

Also, $\displaystyle\lim_{x\to-\infty} \frac{3x}{\sqrt{x^2+2}} = \lim_{\frac{1}{x}\to 0^-} \frac{3}{\sqrt{1+\frac{2}{x^2}}} = -3.$

$\therefore y = -3$ is the horizontal asymptote.

(iv) $f(x) = \dfrac{x^2-6x-1}{x+3}$.

Given $f(x) = \dfrac{x^2-6x-1}{x+3}$

$$\lim_{x\to-3^+} \frac{x^2-6x-1}{x+3} = \lim_{h\to 0^+} \frac{(-3+h)^2-6(-3+h)-1}{-3+h+3}$$

$$= \lim_{h\to 0^+} \frac{9+h^2-6h+18-6h-1}{h}$$

$$= \lim_{h\to 0^+} \frac{h^2-12h+26}{h} = \infty$$

$$\lim_{x\to-3^-} \frac{x^2-6x-1}{x+3} = \lim_{h\to 0^+} \frac{(-3-h)^2-6(-3-h)-1}{-3-h+3}$$

$$= \frac{9+h^2+6h+18+6h-1}{-h}$$

$$= \frac{h^2+12h+26}{-h} = -\infty$$

$\therefore x = -3$ is a vertical asymptote.

Also,

$$\begin{array}{r|l} & x-9 \\ \hline x+3 & x^2-6x-1 \\ & x^2+3x \\ & \underline{(-)\quad(-)} \\ & -9x-1 \\ & -9x-27 \\ & \underline{(+)\quad(+)} \\ & 26 \end{array}$$

$\therefore y = x-9$ is the slanting asymptote.

(v) $f(x) = \dfrac{x^2+6x-4}{3x-6}$.

Given $f(x) = \dfrac{x^2+6x-4}{3x-6}$

$$\lim_{x\to 2^+} \frac{x^2+6x-4}{3x-6} = \lim_{h\to 0^+} \frac{(2+h)^2+6(2+h)-4}{3(2+h)-6}$$

$$= \lim_{h\to 0^+} \frac{4+h^2+2h+12+6h-4}{6+h-6}$$

$$= \lim_{h\to 0^+} \frac{h^2+8h+12}{h} = -\infty$$

$\therefore x = 2$ is the vertical asymptote

Also,

$$\begin{array}{r|l} & \frac{1}{3}x+\frac{8}{3} \\ \hline 3x-6 & x^2+6x-4 \\ & x^2-2x \\ & \underline{(-)\quad(+)} \\ & +8x-4 \\ & 8x-16 \\ & \underline{(-)\quad(+)} \\ & 12 \end{array}$$

$\therefore y = \frac{1}{3}x+\frac{8}{3}$ is the slanting asymptote.

2. Sketch the graphs of the following functions:

(i) $y = -\frac{1}{3}(x^3-3x+2)$.

Given $y = -\frac{1}{3}(x^3-3x+2)$

Factorizing the given functions, we have,

$$y = f(x) = -\frac{1}{3}(x-1)(x^2+x-2).$$

Synthetic division

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

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1) The domain and the range of the given function $f(x)$ are the entire real time.

2) Putting $y=0$, we get $x=1$. The other two roots are imaginary.

$\therefore$ x intercept is $(1,0)$

Putting $x=0$, we get $y=-\frac{2}{3}$

$\therefore$ y intercept is $(0,-\frac{2}{3})$

3) $f'(x)=\frac{-1}{3}(3x^2-3) = \frac{-1}{3}(3)(x^2-1) = -x^2+1$

$\therefore f'(x)=0 \Rightarrow -x^2+1=0$

$\Rightarrow -x^2=-1 \Rightarrow x^2=1 \Rightarrow \boxed{x \pm 1}$

$\therefore$ The critical points are $x=1,-1$.

4) $f''(x)=\frac{-1}{3}(6x) = -2x$

$f''(1) = -2(1) = -2, \quad f''(-1) = -2(-1)=2.$

$\therefore f(x)$ is local minimum at $x=1$,

$f(1)=\frac{-1}{3}(1^3-3(1)+2) = \frac{-1}{3}(1-3+2) = \frac{-1}{3}(0)=0.$

$\therefore f(x)$ is local minimum at $x=-1$

$f(-1)=\frac{-1}{3}((-1)^3-3(-1)+2) = \frac{-1}{3}(-1+3+2) = \frac{-1}{3}(4) = \frac{-4}{3}$

$\therefore$ The critical points are $x=1,-1$.

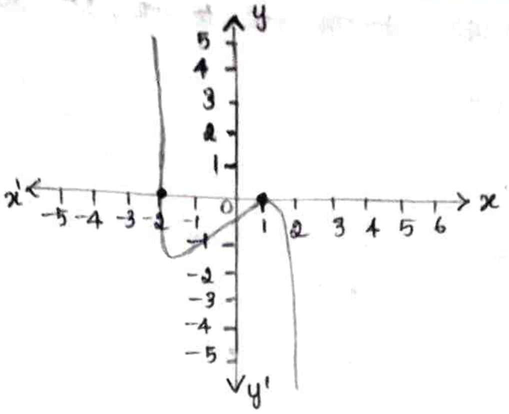
5) since $f''(x)=-2x<0 \ \forall x>0$, the function is concave downward in the positive real line and

$f''(x)=-2x>0 \ \forall x<0$, the function is concave upward in the negative real line.

6) $f''(x)=0$ at $x=0$ and $f''(x)$ changes its sign when $x=0$ the point of inflection is $(0,f(0)) = (0,-\frac{2}{3})$.

7) The curve has no asymptotes.

$\therefore$ The rough sketch of the function is



(ii) $y = x\sqrt{4-x}$

$\Rightarrow y^2 = x^2(4-x)$

1) The domain of $f(x)$ is $\{x \in R / -\infty < x \leq 4\} \Rightarrow x \in (-\infty, 4]$

2) Putting $y=0$, we get $0 = x\sqrt{4-x} \Rightarrow x = 0, 4$.

$\therefore$ The x-intercept of is $(4,0)$ and the curve passes through the origin.

3) Put $x=0$, we get $y=0$.

$\therefore$ y-intercept is $(0,0)$.

4) $f'(x) = x\left(\frac{1}{2}\right)(4-x)^{-1/2}(-1) + \sqrt{4-x}\,(1)$

$$= \frac{-x}{2\sqrt{4-x}} + \sqrt{4-x} = \frac{-x+2(4-x)}{2\sqrt{4-x}} = \frac{-x+8-2x}{2\sqrt{4-x}}$$

$[\because \sqrt{4-x}\cdot\sqrt{4-x} = 4-x]$

$$= \frac{-3x+8}{2\sqrt{4-x}}$$

$\therefore f'(x)=0 \Rightarrow \frac{-3x+8}{2\sqrt{4-x}} = 0 \Rightarrow -3x+8=0$

$\Rightarrow -3x = -8$

$\boxed{x = 8/3}$

$\therefore$ The possible intervals are $(-\infty, \frac{8}{3})(\frac{8}{3}, 4)$

Interval	$(-\infty, 8/3)$	$(8/3, 4)$
Sign of $f'(x)$	Say $x=0$ $f'(0) = \frac{-3(0)+8}{2\sqrt{4-0}} = \frac{8}{2\sqrt{4}}$ $(+ve)$	Say $x=3$ $f'(3) = \frac{-3(3)+8}{2\sqrt{4-3}}$ $= \frac{-9+8}{2(1)} = (-ve)$
Monotonocity	Increasing	Decreasing

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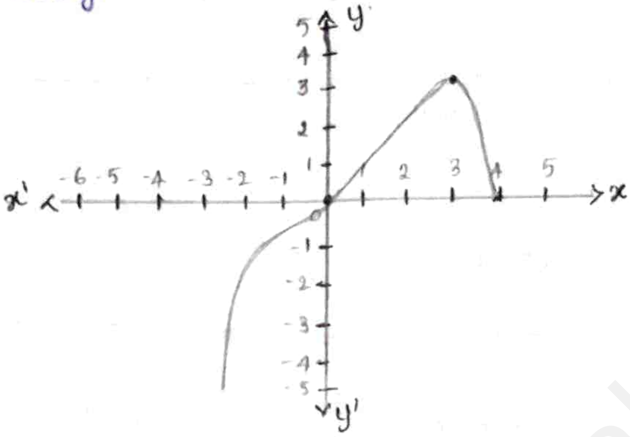
5) As $f'(x)$ changes from +ve to −ve, it has local maximum at $x = \frac{8}{3}$

6) There is no asymptote for the curve.

∴ The rough sketch of the function is



(iii) $y = \frac{x^2+1}{x^2-4}$.

1) The domain of $f(x)$ is $R / \{-2, 2\}$

2) Since $f(x,y) = f(-x,y)$ the curve is symmetric about y-axis.

3) Putting $y=0$, we get $x^2 = -1 \Rightarrow x = \pm\sqrt{-1}$ is not possible.

∴ No x-intercept.

4) Putting $x=0$, we get $y = \frac{-1}{4}$

∴ y-intercept is $(0, \frac{-1}{4})$

5) $f'(x) = \frac{(x^2-4)(2x) - (x^2+1)(2x)}{(x^2-4)^2}$

$= \frac{2x^3 - 8x - 2x^3 - 2x}{(x^2-4)^2} = \frac{-10}{(x^2-4)^2}$

$f'(x) = 0 \Rightarrow x = 0$ and the curve does not exist at $x = -2, 2$.

∴ The critical numbers are $0, -2, 2$

∴ The intervals of monotonocity are $(-\infty, -2)(-2, 0)(0, 2)(2, \infty)$

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Interval	$(-\infty,-2)$	$(-2,0)$	$(0,2)$	$(2,\infty)$
sign of $f'(x)$	Say $x=-3$ $f'(-3) = \frac{-10(-3)}{(3^2-4)^2}$ = (+ve)	Say $x=-1$ $f'(-1) = \frac{-10(-1)}{(-1^2-4)^2}$ = (+ve)	Say $x=1$ $f'(1) = \frac{-10(1)}{(1^2-4)^2}$ = (-ve)	Say $x=3$ $f'(3) = \frac{-10(+3)}{(3^2-4)^2}$ = -ve
Monotonocity	Increasing	Increasing	Decreasing	Decreasing

$\therefore f(x)$ is Increasing in $(-\infty,-2)(-2,0)$ and decreasing in $(0,2)(2,\infty)$.

6) Since $f'(x)$ changes its sign from +ve to -ve at $x=0$, it has local maximum at $x=0$.

7) $$f''(x) = -10\left[\frac{(x^2-4)^2(1) - x(2)(x^2-4)(2x)}{(x^2-4)^4}\right]$$

$$= -10\left[\frac{(x^2-4)^2 - 4x^2(x^2-4)}{(x^2-4)^4}\right]$$

$$= -10(x^2-4)\left[\frac{x^2-4-4x^2}{(x^2-4)^4}\right] = -10\left[\frac{-3x^2-4}{(x^2-4)^3}\right] = \frac{10(3x^2+4)}{(x^2-4)^3}$$

$$\therefore f''(x)=0 \Rightarrow \frac{10(3x^2+4)}{(x^2-4)^3} = 0 \Rightarrow 3x^2+4=0 \Rightarrow 3x^2=-4 \Rightarrow x^2 = \frac{-4}{3}$$

which is not possible and $f''(x)$ does not exist at $x=-2, 2$.

The intervals of concavity is tabulated:

Intervals	$(-\infty,-2)$	$(-2,2)$	$(2,\infty)$
sign of $f''(x)$	Say $x=-3$ $f''(-3) = \frac{10(3(-3)^2+4)}{(-3^2-4)^3}$ = (+ve)	Say $x=0$ $f''(0) = \frac{10(3(0)^2+4)}{(0-4)^3}$ = (-ve)	Say $x=3$ $f''(3) = \frac{10(3(3)^2+4)}{((3)^2-4)^3}$ = (-ve)
concavity	concave up	concave down	concave down

8) Since the curve does not exist at $x=2,-2$ are the vertical asymptotes and the horizontal asymptote is $y=1$

$$\begin{array}{r|l} & 1 \\ x^2-4 & x^2+1 \\ & x^2-4 \\ \hline & 5 \end{array}$$

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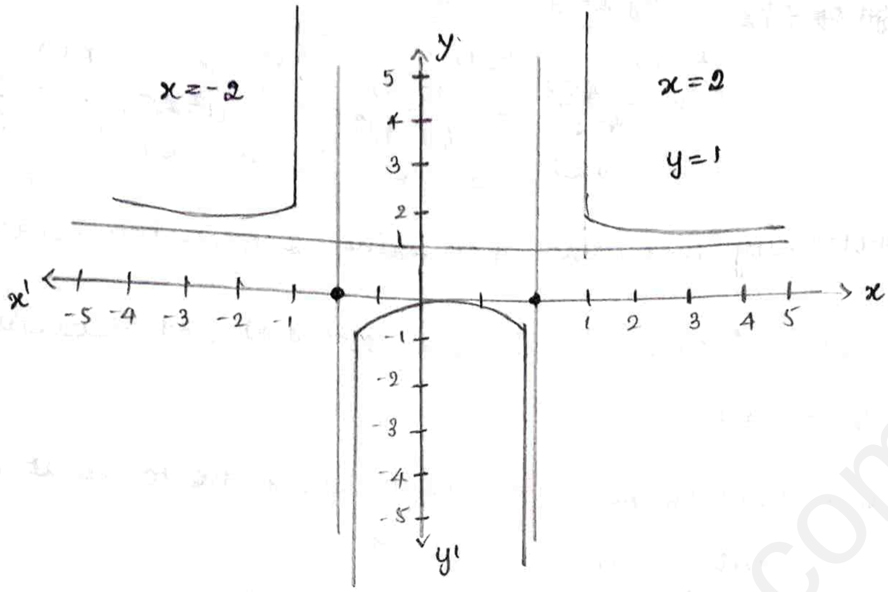
∴ The rough sketch of function



(iv) $y = \frac{1}{1+e^{-x}}$

Let $f(x) = y = \frac{1}{1+e^{-x}}$

1) The domain of y is R

2) It is not symmetric.

3) Putting $y=0$, we get x does not exist

∴ It has no x-intercept

4) Putting $x=0$, we get $y = \frac{1}{1+e^{0}} = \frac{1}{1+1} = \frac{1}{2}$

∴ y-intercept is $(0, \frac{1}{2})$

5) $f'(x) = \frac{d}{dx}(1+e^{-x})^{-1} = -1(1+e^{-x})^{-2}(-e^{-x})$

$= \frac{e^{-x}}{(1+e^{-x})^2} = e^{-x}(1+e^{-x})^{-2}$

$f'(x) = 0 \Rightarrow e^{-x}(1+e^{-x})^{-2} = 0 \Rightarrow e^{-x} = 0 \Rightarrow -x = \log 0$

$\Rightarrow x = \infty$

Also, $f(x) > 0 \ \forall x \in R$

∴ $f(x)$ is increasing in $(-\infty, \infty)$

6) Since there is no sign change in $f'(x)$, it has no local extreme.

7. $f''(x) = e^{-x}(-2)(1+e^{-x})^{-3}(-e^{-x}) + (1+e^{-x})^{-2}(-e^{-x})$

$= 2e^{-2x}(1+e^{-x})^{-3} - e^{-x}(1+e^{-x})^{-2}$ [uv = uv' + vu']

$= e^{-2x}(1+e^{-x})^{-3}[2-e^{-x}(1+e^{-x})]$

$= e^{-2x}(1+e^{-x})^{-3}[2-e^{x}-1]$

$= \dfrac{e^{-2x}(1-e^{x})}{(1+e^{x})^{3}}$

$\therefore f''(x)=0 \Rightarrow x=0$

The intervals of concavity is tabulated as follows:

Interval	$(-\infty,0)$	$(0,\infty)$
Sign of $f''(x)$	Say $x=-1$ $f''(-1)=\dfrac{e^{-2(-1)}(1-e^{-1})}{(1+e^{-1})^3}$ $=(+ve)$	Say $x=1$ $f''(1)=\dfrac{e^{-2(1)}(1-e^{(1)})}{(1+e^{1})^3}$ $=(-ve)$
Concavity.	concave up	concave down

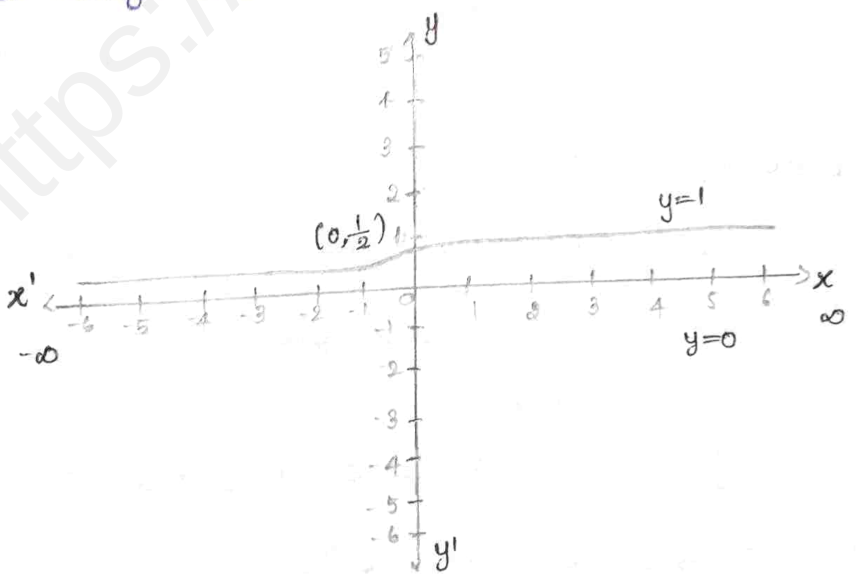
8) $\lim\limits_{x\to\infty}\dfrac{1}{1+e^{-x}}=1$

$\Rightarrow y=1$ is the horizontal asymptote and

$\lim\limits_{x\to-\infty}\dfrac{1}{1+e^{-x}}=0$

$\Rightarrow y=0$ is the horizontal asymptote.

$\therefore$ The rough sketch is



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v) $y = \frac{x^3}{24} - \log x$

Let $f(x) = y = \frac{x^3}{24} - \log x$.

1) The domain of $f(x)$ is $(0, \infty)$

2) The curve is not symmetrical.

3) Putting $x=0$, we get $y = -\infty$

It has no x-intercept

4) Putting $y=0$, we get $0 = \frac{x^3}{24} - \log x$

$\Rightarrow \log x = \frac{x^3}{24} \Rightarrow x = e^{\frac{x^3}{24}}$

$\therefore$ It has no y intercept.

5) $f'(x) = \frac{3x^2}{24} - \frac{1}{x} = \frac{x^2}{8} - \frac{1}{x}$

$\therefore f'(x) = 0 \Rightarrow \frac{x^2}{8} - \frac{1}{x} = 0 \Rightarrow \frac{x^2}{8} = \frac{1}{x} \Rightarrow x^3 = 8 \Rightarrow x = 2$

$\therefore$ The possible intervals are $(0,2)\ (2,\infty)$.

Interval	$(0,2)$	$(2,\infty)$
Sign of $f'(x)$	Say $x=1$ $f'(x) = \frac{1}{8} - \frac{1}{1}$ $= \frac{1-8}{8} = \frac{-7}{8}$ (-ve)	Say $x = 3$ $f'(3) = \frac{9}{8} - \frac{1}{3}$ $= \frac{27-8}{24} = \frac{19}{24}$ (+ve)
Monotonocity	Decreasing	Increasing

$\therefore f(x)$ is decreasing in $(0,2)$ and increasing in $(2,\infty)$

6. Since $f'(x)$ changes its sign from -ve to +ve, its has local minimum at $x=2$.

$\therefore f(2) = \frac{8}{24} - \log 2 = \frac{1}{3} - \log 2 = 0.33 - 0.30 = 0.03$.

7. $f''(x) = 0 \Rightarrow e^{-x} - e^{-2x} = 0 \Rightarrow e^{-x} = e^{-2x}$

$= \frac{1}{e^x} = \frac{1}{e^{2x}} \Rightarrow \frac{e^{2x}}{e^x} = 1$

$\Rightarrow e^x = 1$

$[e^0 = 1]$

$\Rightarrow e^{x} = e^{0}$

$\Rightarrow x = 0$

$\therefore$ The possible Interval of concavity are $(-\infty,0)(0,\infty)$

Interval	$(-\infty, 0)$	$(0, \infty)$.
Sign of $f''(x)$	Say $x = -1$ $f''(x) = \frac{e^{1}}{(1+e^{1})^3}$ $= (e^{1}-e^{2})$ $= \frac{2.72}{(3.72)^3}$ $= (2.72 - 7.39)$ $= (-ve)$	Say $x=1$ $f''(x) = \frac{e^{-1}}{(1+e^{-1})^3}$ $= (e^{-1}-e^{-2})$ $= \frac{0.37}{(1.37)^3}$ $= (0.37 - 0.14)$ $= (+ve)$
concavity	concave down	concave up.

8. Since $y=0$, does not exist, it has a horizontal asymptote at $y=0$.

9. Since $f''(x)$ changes its sign from $-ve$ to $+ve$, it has a point of inflection $(0, f(0)) = (0,\infty)$.