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Chapter 1 – Applications of Matrices & Determinants

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CH1 - Applications of Matrices

* Cofactor :-

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

* Adjoint Matrix of A

The adjoint Matrix of A is defined as the transpose of the Matrix of Cofactor of A. It is denoted by adj A.

* **Note :-** $\text{adj } A = [A_{ji}]^T = [(-1)^{i+j} M_{ij}]^T$

* For every square matrix A of order n, $A(\text{adj } A) = (\text{adj } A)A = |A| I_n$.

* If A is a singular matrix of oder n, Theo $|A| = 0$ and so $A(\text{adj } A) = (\text{adj})A = O_n$.

Ex 1.1 (3) If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ verify that $A(\text{adj } A) = (\text{adj } A)A = |A| I_3$.

$$|A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$= 8(21-16) + 6(-18+8) + 2(24-14)$

$= 8(5) + 6(-10) + 2(10)$

$= 40 - 60 + 20 = 60 - 60 = 0$.

$$(\text{adj } A) = \begin{bmatrix} (21-16) & -(-18+8) & (24-14) \\ -(18+8) & (24-4) & -(-32+12) \\ (24-14) & -(-32+12) & (56-36) \end{bmatrix}^T$$

$$= \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$$

$$\text{Now } A(\text{adj } A) = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 40-60+20 & 80-120+40 & 80-120+40 \\ -30+70-40 & -60+140-80 & -60+140-80 \\ 10-40+30 & 20-80+60 & 20-80+60 \end{bmatrix}$$

$= 0\, I_3 = |A|\, I_3$ ——— (1)

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$$(adj\,A)\,A = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 40-60+20 & -30+70-40 & 10-40+30 \\ 80-120+40 & -60+140-80 & 20-80+60 \\ 80-120+40 & -60+140-80 & 20-80+60 \end{bmatrix}$$

$$= \quad 0\,I_3 = |A|\,I_3 \quad \text{——— (2)}.$$

Hence $A(adj\,A) = (adj\,A)\,A = |A|\,I_3$.

———— ✕ ————

Inverse matrix of a square matrix :-

Let A be a square matrix of order n. If there exist a square matrix B of order n such that $AB = BA = I_n$, Then B is called an inverse of A

Note:- If a square matrix has an inverse, then it is unique.

(*) The inverse of a matrix A is denoted by A^{-1}.

* Let A be a square matrix of order n. Then A^{-1} exists iff A is non-singular.

* Inverse of A is given by

$$A^{-1} = \frac{1}{|A|}\,adj\,A.$$

Ex 1.2 / 5 If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-singular, find A^{-1}.

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ | $|A| = ad - bc \neq 0$.

$$adj\,A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adj\,A)$$

$$= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} //$$

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Ex 1.3 (5) Find the inverse of the matrix $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$

$|A| = 2(9-2) + 1(-15+3) + 3(-10+9)$

$= 2(7) + 1(-12) + 3(-1)$

$= 14 - 12 - 3 = 14 - 15 = -1 \neq 0$.

$\therefore \bar{A}^{-1}$ exist.

$\text{adj } A = \begin{bmatrix} (9-2) & -(-15+3) & (-10+9) \\ -(-3+6) & (6+9) & -(4-3) \\ (-1-9) & -(2+15) & (6-5) \end{bmatrix}^T$

$= \begin{bmatrix} 7 & 12 & -1 \\ 9 & 15 & -1 \\ -10 & -17 & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix}$

Hence $\bar{A}^{-1} = \frac{1}{|A|} \text{adj } A$

$= \frac{1}{-1} \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix}$ //

Properties of inverses of matrices:-

If A is non-singular, Then

(i) $|\bar{A}^{-1}| = \frac{1}{|A|}$

(ii) $(A^T)^{-1} = (\bar{A}^{-1})^T$

(iii) $(\lambda A)^{-1} = \frac{1}{\lambda} \bar{A}^{-1}$ where λ is scalar.

(iv) $(AB)^{-1} = \bar{B}^{-1} \bar{A}^{-1}$

(v) $(\bar{A}^{-1})^{-1} = A$

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(VI) $(adj\,A)^{-1} = adj(A^{-1}) = \frac{1}{|A|}A$

(VII) $|adj\,A| = |A|^{n-1}$

(VIII) $adj(adj\,A) = |A|^{n-2}A.$

(IX) $adj(\lambda A) = \lambda^{n-1}(adj\,A)$, λ is non zero scalar.

(X) $|adj(adj\,A)| = |A|^{(n-1)^2}$

(XI) $(adj\,A)^T = adj(A^T)$

(XII) $adj(AB) = (adj\,B)(adj\,A)$

(XIII) a) $A^{-1} = \pm \frac{1}{\sqrt{|adj\,A|}} adj\,A$

b) $A = \pm \frac{1}{\sqrt{|adj\,A|}} adj(adj\,A)$

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Ex 1.4 (9) If A is a non-singular matrix of odd order, prove that $|adj\,A|$ is positive.

Let A be a non singular matrix of odder $2m+1$.

WKT $|adj\,A| = |A|^{n-1}$ where A is non-singular matrix of order n.

Now, $|adj\,A| = |A|^{2m+1-1}$

$= |A|^{2m}$

$|A|^{2m}$ is always positive.

$\therefore |adj\,A|$ is positive.

1.5 (9) Find a matrix A if

$$(adj\,A) = \begin{bmatrix} -7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$$

$|adj\,A| = -7(77-35) - 7(-7-77) - 7(-5-121)$

$= -7(42) - 7(-84) - 7(-126)$

$= 294 + 588 + 882 = 1764 > 0$

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$$\therefore A = \pm \frac{1}{\sqrt{\text{adj}A}} \text{adj}(\text{adj}A)$$

$$= \pm \frac{1}{\sqrt{1764}} \begin{bmatrix} (77-35) & -(-7-77) & (-5-121) \\ -(49+35) & 49+77 & -(35-77) \\ (49+77) & -(49-7) & (77+7) \end{bmatrix}^T$$

$$= \pm \frac{1}{42} \begin{bmatrix} 42 & 84 & -126 \\ -84 & 126 & 42 \\ 126 & -42 & 84 \end{bmatrix}^T$$

$$= \pm \frac{1}{42} \begin{bmatrix} 42 & -84 & 126 \\ 84 & 126 & -42 \\ -126 & 42 & 84 \end{bmatrix}$$

$$= \pm \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

Ex 1.6 / 9 If $\text{adj}A = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ find A^{-1}.

XII | CH1 | 1.1 | P5

$$|\text{adj}A| = \begin{vmatrix} -1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$$

$$= -1(1-4) - 2(1-4) + 3(2-2)$$

$$= 3 + 6 + 0 = 9$$

WKT $A^{-1} = \pm \frac{1}{\sqrt{|\text{adj}A|}} \text{adj}A$.

$$= \pm \frac{1}{\sqrt{9}} \begin{bmatrix} -1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \pm \frac{1}{3} \begin{bmatrix} -1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Ex 1.7 / 10 If A is symmetric, prove that adj A is also symmetric.

Given A is symmetric

$\Rightarrow A^T = A$.

To prove adj A is symmetric is to prove $(\text{adj}A)^T = \text{adj}A$.

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$$\text{LHS} = (\text{adj } A)^T$$
$$= \text{adj}(A^T)$$
$$= \text{adj } A \quad (\because A \text{ is symmetric } \therefore A^T = A)$$
$$= \text{RHS}.$$

$\therefore$ adj A is symmetric.

Ex 1.8 (10) Verify the property $(A^T)^{-1} = (A^{-1})^T$ with $A = \begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$

Given $A = \begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$

$|A| = 14 - 9 = 5 \neq 0$.

$\therefore A^{-1}$ exists.

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{5}\begin{bmatrix} 7 & -9 \\ -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 7/5 & -9/5 \\ -1/5 & 2/5 \end{bmatrix}$$

$$\therefore (A^{-1})^T = \begin{bmatrix} 7/5 & -1/5 \\ -9/5 & 2/5 \end{bmatrix} \quad \text{---} (1)$$

$$A^T = \begin{bmatrix} 2 & 1 \\ 9 & 7 \end{bmatrix}$$

$|A^T| = 14 - 9 = 5 \neq 0$

$\therefore (A^T)^{-1}$ exists.

$$(A^T)^{-1} = \frac{1}{|A^T|}\text{adj } A^T = \frac{1}{5}\begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 7/5 & -1/5 \\ -9/5 & 2/5 \end{bmatrix} \quad \text{---} (2)$$

From (1) and (2) $(A^{-1})^T = (A^T)^{-1}$

Hence proved.

Ex 1.9 (11) Verify $(AB)^{-1} = B^{-1}A^{-1}$ with

$A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$

$$AB = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 0+0 & 0+3 \\ -2+0 & -3-4 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -2 & -7 \end{bmatrix}$$

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$|AB| = 0+6 = 6 \neq 0$.

$\therefore (AB)^{-1}$ exists

$\therefore (AB)^{-1} = \frac{1}{|AB|}(\text{adj } AB)$

$= \frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix}$ ——— (1)

$|A| = \begin{vmatrix} 0 & -3 \\ 1 & 4 \end{vmatrix} = 0+3 = 3$

$\therefore A^{-1} = \frac{1}{|A|}(\text{adj } A)$

$= \frac{1}{3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$

$B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$

$|B| = 2+0 = 2$

$\therefore B^{-1} = \frac{1}{|B|}\text{adj } B = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$

$B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} -4-3 & -3+0 \\ 0+2 & 0+0 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix}$ ——— (2)

From (1) and (2)

$(AB)^{-1} = B^{-1}A^{-1}$ is verified.

1.10 (11) If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$ find x and y such that $A^2 + xA + yI_2 = 0$ Hence find A^{-1}.

$A^2 = A \times A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$

$= \begin{bmatrix} 16+6 & 12+15 \\ 8+10 & 6+25 \end{bmatrix}$

$= \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix}$

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$A^2 + xA + yI_2 = 0$

i.e. $\begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix} + x\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} + y\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$

$\begin{bmatrix} 22+4x+y & 27+3x+0 \\ 18+2x+0 & 31+5x+y \end{bmatrix} = 0$

$\Rightarrow 4x+y+22 = 0 \Rightarrow 4x+y = -22 \quad -(1)$

$3x+27 = 0 \Rightarrow 3x = -27$

$\boxed{x = -9}$

Sub $x = -9$ in (1)

$-36+y = -22$

$y = -22+36$

$\boxed{y = 14}$

$\therefore$ we get $A^2 - 9A + 14I_2 = 0$

multiplying this eqn by A^{-1}

$A^{-1}(AA) - 9A^{-1}A + 14A^{-1}I_2 = 0$

$A - 9I_2 + 14A^{-1} = 0$

$14A^{-1} = 9I_2 - A$

$A^{-1} = \frac{1}{14}(9I_2 - A)$

$= \frac{1}{14}\left[9\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}\right]$

$= \frac{1}{14}\left[\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}\right]$

$= \frac{1}{14}\begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$

Application of Matrices to Geometry

1) A matrix A is orthogonal if $AA^T = A^TA = I$.

2) A matrix A is orthogonal iff A is non-singular and $A^{-1} = A^T$.

Ex 1.11 / 13 Prove that $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ is orthogonal.

Let $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

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$$A^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$AA^T = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$AA^T = I_2 \Rightarrow A$ is orthogonal.

1.12/13 If $A = \frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}$ is orthogonal, find a, b, c, and hence A^{-1}.

Given A is orthogonal,

Then $AA^T = A^TA = I_3$

$$AA^T = I_3$$

$$\frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix} \times \frac{1}{7}\begin{bmatrix} 6 & b & 2 \\ -3 & -2 & c \\ a & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{49}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}\begin{bmatrix} 6 & b & 2 \\ -3 & -2 & c \\ a & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 36+9+a^2 & 6b+6+6a & 12-3c+3a \\ 6b+6+6a & b^2+4+36 & 2b-2c+18 \\ 12-3c+3a & 2b-2c+18 & 4+c^2+9 \end{bmatrix} = \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$$

$a^2+45 = 49$ | $b^2+40 = 49$ | $c^2+13 = 49$

$a^2 = 4$ | $b^2 = 9$ | $c^2 = 36$

$\boxed{a = 2}$ | $\boxed{b = 3}$ | $\boxed{c = 6}$

If A is orthogonal Then

$$A^{-1} = A^T$$

$$= \frac{1}{7}\begin{bmatrix} 6 & b & 2 \\ -3 & -2 & c \\ a & 6 & 3 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 6 & 3 & 2 \\ -3 & -2 & 6 \\ 2 & 6 & 3 \end{bmatrix} //$$

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1) EXERCISE 1.1

a) 15 $\begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$ find the adjoint.

Let $A = \begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$

$\text{adj } A = \begin{bmatrix} 2 & -4 \\ -6 & -3 \end{bmatrix}$

b) 15 $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} (8-7) & -(6-3) & (21-12) \\ -(6-7) & (4-3) & -(14-9) \\ (3-4) & -(7-3) & 8-9 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix} //.$$

c) 15 $\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

Let $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

WKT $\text{adj}(kA) = k^{n-1}(\text{adj } A)$

$$\therefore \text{adj } A = \left(\frac{1}{3}\right)^{3-1} \text{adj} \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} (2+4) & -(-4-2) & (4-1) \\ -(4+2) & (4-1) & -(-4-2) \\ (4-1) & -(4+2) & (2+4) \end{bmatrix}^T$$

$$= \frac{1}{9}\begin{bmatrix} 6 & 6 & 3 \\ -6 & 3 & 6 \\ 3 & -6 & 6 \end{bmatrix}^T$$

$$= \frac{1}{9}\begin{bmatrix} 6 & -6 & 3 \\ 6 & 3 & -6 \\ 3 & 6 & 6 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix} //$$

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2) Find the inverse of the following

(i) $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

Let $A = \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

$|A| = (6-4) = 2 \neq 0$

$\therefore A^{-1}$ exists

$\therefore A^{-1} = \frac{1}{|A|}(\text{adj}A)$

$= \frac{1}{2}\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$

(ii) $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

Let $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$|A| = 5(25-1) - 1(5-1) + 1(1-5)$

$= 5(24) - 1(4) + 1(-4)$

$= 120 - 4 - 4 = 112 \neq 0$

$\therefore A^{-1}$ exists

$\therefore A^{-1} = \frac{1}{|A|}(\text{adj}A)$

$= \frac{1}{112}\begin{bmatrix} (25-1) & -(5-1) & (1-5) \\ -(5-1) & (25-1) & -(5-1) \\ (1-5) & -(5-1) & (25-1) \end{bmatrix}^T$

$= \frac{1}{112}\begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}^T$

$= \frac{1}{112}\begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$

$= \frac{4}{112}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$

$= \frac{1}{28}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$ //

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(11) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

$|A| = 2(8-7) - 3(6-3) + 1(21-12)$

$= 2(1) - 3(3) + 1(9)$

$= 2 - 9 + 9 = 2 \neq 0$

$\therefore A^{-1}$ exist

Now $A^{-1} = \frac{1}{|A|}(\text{adj } A)$

$= \frac{1}{2}\begin{bmatrix} (8-7) & -(6-3) & (21-12) \\ -(6-7) & (4-3) & -(14-9) \\ (3-4) & -(2-3) & (8-9) \end{bmatrix}^T$

$= \frac{1}{2}\begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$

$= \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$ //

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3) If $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$

Show that $[F(\alpha)]^{-1} = F(-\alpha)$

Given $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$

$|F(\alpha)| = \cos\alpha(\cos\alpha - 0) + \sin\alpha(0 + \sin\alpha)$

$= \cos^2\alpha + \sin^2\alpha = 1 \neq 0$

$\therefore [F(\alpha)]^{-1}$ exists

$[F(\alpha)]^{-1} = \frac{1}{|F(\alpha)|}[\text{adj } F(\alpha)]$

$= \frac{1}{1}\begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & \cos^2\alpha + \sin^2\alpha & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}^T$

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$$= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \quad \text{—— ①}$$

$$F(-\alpha) = \begin{bmatrix} \cos(-\alpha) & 0 & \sin(-\alpha) \\ 0 & 1 & 0 \\ -\sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \quad \text{— ②}$$

From ① and ②

$$[F(\alpha)]^{-1} = F(-\alpha)$$ Hence proved.

---

4) If $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$, show that $A^2 - 3A - 7I_2 = 0$. Hence find A^{-1}

Given $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$

12 | 1.1 | P13

$$A^2 = A \cdot A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 25-3 & 15-6 \\ -5+2 & -3+4 \end{bmatrix} = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

Now, $A^2 - 3A - 7I_2 = \Rightarrow \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - 3\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} 15 & 9 \\ -3 & -6 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 22-15-7 & 9-9-0 \\ -3+3-0 & 1+6-7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Multiplying by A^{-1}

$$A - 3I_2 - 7A^{-1} = 0$$

$$\therefore 7A^{-1} = A - 3I_2$$

$$A^{-1} = \frac{1}{7}[A - 3I_2]$$

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$$\bar{A}^{-1} = \frac{1}{7}\left[\begin{pmatrix} 5 & 3 \\ -1 & -2 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{7}\left[\begin{pmatrix} 5 & 3 \\ -1 & -2 \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}\right]$$

$$= \frac{1}{7}\begin{bmatrix} 5-3 & 3+0 \\ -1+0 & -2-3 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

---

5) If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$ prove that $\bar{A}^{-1} = A^T$.

To prove $\bar{A}^{-1} = A^T$

$A\bar{A}^{-1} = AA^T$

$AA^T = I$

It is enough to prove $AA^T = I$.

12 | 1.1 | P14

$$AA^T = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix} \times \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 64+1+16 & -32+4+28 & -8-8+16 \\ -32+4+28 & 16+16+49 & 4-32+28 \\ -8-8+16 & 4-32+28 & 1+64+16 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Hence proved.

---

6) If $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$, verify that

$A(\text{adj } A) = (\text{adj } A)A = |A|I.$

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Given $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$ $\therefore |A| = 24-20$
$= 4$ — ①

$adj A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$

$A(adj A) = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$

$= \begin{bmatrix} 24-20 & 32-32 \\ -15+15 & -20+24 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 4I_2$ — ②

$(adj A)A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$

$= \begin{bmatrix} 24-20 & -12+12 \\ 40-40 & -20+24 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= 4I_2$ — ③

from ① and ② and ③

$A(adj A) = (adj A)A = |A| I_2$

7) If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$

verify $(AB)^{-1} = B^{-1}A^{-1}$

Given $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$

$|A| = 15-14 = 1 \neq 0$

$\therefore A^{-1}$ exists

$A^{-1} = \frac{1}{|A|}(adj A)$

$= \frac{1}{1}\begin{bmatrix} 5 & -2 \\ -7 & +3 \end{bmatrix}$

$= \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$

$|B| = -2+15$
$= 13 \neq 0$

$\therefore B^{-1}$ exis.

$B^{-1} = \frac{1}{|B|}(adj B)$

$= \frac{1}{13}\begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$

$B^{-1}A^{-1} = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$= \frac{1}{13}\begin{bmatrix} 10-21 & -4+9 \\ -25+7 & 10-3 \end{bmatrix}$

$= \frac{1}{13}\begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix}$ ——— ①

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$$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3+10 & -9+4 \\ -7+25 & -21+10 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -5 \\ 18 & -11 \end{bmatrix}$$

$|AB| = -77 + 90 = 13 \neq 0$

$\therefore (AB)^{-1}$ exists.

$$(AB)^{-1} = \frac{1}{|AB|} \text{ adj } AB = \frac{1}{13}\begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix} \quad \text{— (2)}$$

From (1) and (2)

$(AB)^{-1} = B^{-1}A^{-1}$ is verified.

---

8) If $\text{adj } A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$ find A.

WKT $A = \pm \frac{1}{\sqrt{|\text{adj } A|}} \text{adj}(\text{adj } A)$

XII | 1.1 | P16

Given $\text{adj } A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$

$|\text{adj } A| = 2(24-0) + 4(-6-14) + 2(0+24)$

$= 48 - 80 + 48 = 96 - 80$

$= 16 \neq 0.$

$$\text{adj}(\text{adj } A) = \begin{bmatrix} 24 & -(-20) & 24 \\ -(-8) & (4+4) & -(8) \\ 28-24 & -(-14+6) & 24-12 \end{bmatrix}^T$$

$$= \begin{bmatrix} 24 & 20 & 24 \\ 8 & 8 & +8 \\ 4 & 8 & 12 \end{bmatrix}^T = \begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$$

$$A = \pm \frac{1}{\sqrt{16}} 4\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

$$= \pm \frac{1}{4} \times 4 \begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix} //$$

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9) If $\text{adj } A = \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$ find A^{-1}

$$\text{adj } A = \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

$|\text{adj } A| = 2(36-18) = 2(18) = 36.$

WKT $A^{-1} = \pm \frac{1}{\sqrt{|\text{adj } A|}} (\text{adj } A)$

$$= \pm \frac{1}{\sqrt{36}} \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

$$= \pm \frac{1}{6} \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

×

10) Find $\text{adj}(\text{adj } A)$ If

$$\text{adj } A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\text{adj}(\text{adj } A) = \begin{bmatrix} 2 & -0 & 2 \\ -0 & (1+1) & -0 \\ -2 & -0 & 2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

// × ×

11) $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$ Show that

$$A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$$

$$\therefore A^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$|A| = 1 + \tan^2 x \neq 0$

$\therefore A^{-1}$ exists

$$\therefore A^{-1} = \frac{1}{|A|} (\text{adj } A)$$

$$= \frac{1}{1+\tan^2 x} \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

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$$A^T A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \times \frac{1}{1+\tan^2 x} \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \frac{1}{1+\tan^2 x} \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \frac{1}{1+\tan^2 x} \begin{bmatrix} 1-\tan^2 x & -\tan x - \tan x \\ \tan x + \tan x & -\tan^2 x + 1 \end{bmatrix}$$

$$= \frac{1}{1+\tan^2 x} \begin{bmatrix} 1-\tan^2 x & -2\tan x \\ 2\tan x & 1-\tan^2 x \end{bmatrix}$$

$$= \frac{1}{\sec^2 x} \begin{bmatrix} 1-\tan^2 x & -2\tan x \\ 2\tan x & 1-\tan^2 x \end{bmatrix}$$

$$= \cos^2 x \begin{bmatrix} 1-\tan^2 x & -2\tan x \\ 2\tan x & 1-\tan^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 x - \cos^2 x \cdot \frac{\sin^2 x}{\cos^2 x} & -2 \frac{\sin x}{\cos x} \cos^2 x \\ 2 \frac{\sin x}{\cos x} \cdot \cos^2 x & \cos^2 x - \frac{\sin^2 x}{\cos^2 x} \cdot \cos^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 x - \sin^2 x & -2 \sin x \cos x \\ 2 \sin x \cos x & \cos^2 x - \sin^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$$ //

---

12) find the matrix A for which $A \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

Given $A \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

Let $B = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ & $C = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

$AB = C$

$A = CB^{-1}$ ——— ①

$|B| = -10 + 3 = -7 \neq 0$

$\therefore B^{-1}$ exists.

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$\therefore B^{-1} = \frac{1}{|B|}(\text{adj } B)$

$= \frac{1}{-7}\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$

$A = CB^{-1}$

$= \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix} \times -\frac{1}{7}\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$

$= 7\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \times \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$

$= \begin{bmatrix} 4-1 & 6-5 \\ 2-1 & 3-5 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$ //

---

13) Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ find the matrix X such that $AXB = C$

Soln: $AXB = C$

$A^{-1}AXB = A^{-1}C$

$XB = A^{-1}C$

$XBB^{-1} = A^{-1}CB^{-1}$

$X = A^{-1}CB^{-1}$

$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$

$|A| = 0+2 = +2 \neq 0$

$\therefore A^{-1}$ exists

$A^{-1} = \frac{1}{|A|}(\text{adj } A)$

$= \frac{1}{+2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$

$|B| = 3+2 = 5 \neq 0$

$\therefore B^{-1}$ exists.

$\therefore B^{-1} = \frac{1}{|B|}(\text{adj } B)$

$= \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

$\therefore X = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

$= \frac{1}{10}\begin{bmatrix} 0+2 & 0+2 \\ -2+2 & -2+2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

$= \frac{1}{10}\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

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$$= \frac{1}{10}\begin{bmatrix} 2-2 & 4+6 \\ 0+0 & 0+0 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$ //

---

14) If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ Show that

$$A^{-1} = \frac{1}{2}\left[A^2 - 3I\right]$$

To prove $A^{-1} = \frac{1}{2}\left[A^2 - 3I\right]$

$$2A^{-1} = \left[A^2 - 3I\right]$$

$$2A^{-1}A = A^3 - 3A$$

$$2I = A^3 - 3A$$

It is enough to prove $A^3 - 3A = 2I$.

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+1 & 0+0+1 & 0+1+0 \\ 0+0+1 & 1+0+1 & 1+0+0 \\ 0+1+0 & 1+0+0 & 1+1+0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+1 & 2+0+1 & 2+1+0 \\ 0+2+1 & 1+0+1 & 1+2+0 \\ 0+1+2 & 1+0+2 & 1+1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix}$$

$$A^3 - 3A = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix} - 3\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 2I$$

Hence proved.

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Row - Echelon form:-

1.13 Reduce the matrix

$$\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$$ to a row echelon form.

$$\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_1 + R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - \frac{R_2}{2}$$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to \frac{R_2}{8}$$

This is row echelon form.

Ex 1.14

Reduce the matrix to a row echelon form.

$$\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 4 & 2 & 0 & 0 \end{bmatrix} \quad R_2 \Leftrightarrow R_1$$

$$\sim \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 2 & 8 & 20 \end{bmatrix} \quad R_3 \to R_3 + 4R_1$$

$$\sim \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & 22 & 48 \end{bmatrix} \quad R_3 \to 3R_3 - 2R_2$$

Rank of a Matrix:-

The rank of a matrix A is defined as the order of a highest order non-vanishing minor of the matrix A. It is denoted by the symbol $\rho(A)$. The rank of a zero matrix is defined to be 0.

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Note:- If a matrix contains atleast one non zero element, then $\rho(A) \geq 1$.

2) The rank of the identity matrix I_n is n.

3) If the rank of the matrix is r, then there exists atleast one minor of A of order r which does not vanish and every minor of A of order $r+1$ and higher order vanishes.

4) If A is an $m \times n$ matrix, then $\rho(A) \leq \min\{m,n\}$ = minimum of m,n.

5) A square matrix A of order n is invertible if and only if $\rho(A) = n$.

Ex 1.15 find the rank of each of the following matrices

(i) $\begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix}$

Let $A = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix}$

order of matrix A is 3×3.

So, $\rho(A) \leq \min\{3,3\} = 3$.

$|A| = \begin{vmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{vmatrix} = 3(6-6) - 2(6-6) + 5(3-3) = 0$.

So, $\rho(A) < 3$

Consider the second order minor of A.

$\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 2 = 1 \neq 0$.

$\therefore$ rank of $A = \rho(A) = 2$.

(ii) Let $A = \begin{bmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{bmatrix}$

The order of A is 3×4

So rank of A $\rho(A) \leq \{\min \text{ of } 3,4\}$

$= 3$.

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$$\begin{vmatrix} 4 & 3 & 1 \\ -3 & -1 & -2 \\ 6 & 7 & -1 \end{vmatrix} = 4(1+14) - 3(3+12) + 1(-21+6)$$
$$= 4(15) - 3(15) + 1(-15)$$
$$= 60 - 45 - 15$$
$$= 60 - 60 = 0$$

$$\begin{vmatrix} 4 & 3 & -2 \\ -3 & -1 & 4 \\ 6 & 7 & 2 \end{vmatrix} = 4(-2-28) - 3(-6-24) - 2(-21+6)$$
$$= 4(-30) - 3(-30) - 2(-15)$$
$$= -120 + 90 + 30$$
$$= 120 - 120 = 0.$$

$$\begin{vmatrix} 4 & 1 & -2 \\ -3 & -2 & 4 \\ 6 & -1 & 2 \end{vmatrix} = 4(-4+4) - 1(-6-24) - 2(3+12)$$
$$= 4(0) - 1(-30) - 2(15)$$
$$= 30 - 30 = 0$$

$$\begin{vmatrix} 3 & 1 & -2 \\ -1 & -2 & 4 \\ 7 & -1 & 2 \end{vmatrix} = 3(-4+4) - 1(-2-28) - 2(1+14)$$
$$= 3(0) - 1(-30) - 2(15)$$
$$= 0 + 30 - 30 = 0.$$

$\therefore \rho(A) < 3$

Consider the second order minor of A.

$$\begin{vmatrix} 4 & 3 \\ -3 & -1 \end{vmatrix} = -4 + 9 = 5 \neq 0.$$

$\therefore \rho(A) = 2.$

Ex 1.16 Find the rank of the following matrices which are in row-echelon form.

(i) $\begin{bmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ order of A is 3×3

$\therefore \rho(A) \leq 3.$

$|A| = 2(3)(1) = 6 \neq 0$

$\therefore \rho(A) = 3.$

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(ii) $\begin{bmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Let $A = \begin{bmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ order of A is 3×3

$\therefore \rho(A) \leq 3$.

$|A| = -2(5)(0) = 0$

$\therefore \rho(A) \leq 2$

Consider the second order minor of A, $\begin{vmatrix} -2 & 2 \\ 0 & 5 \end{vmatrix} = -10 \neq 0$

$\therefore \rho(A) = 2$.

(iii) $\begin{bmatrix} 6 & 0 & -9 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Let $A = \begin{bmatrix} 6 & 0 & -9 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ order of A is 4×3

$\therefore \rho(A) \leq 3$

Consider $\begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = 12 \neq 0$

$\therefore \rho(A) = 2$ //

Note:- 1) The rank of a matrix in row echelon form is the number of non-zero rows in it.

2) The rank of a non-zero matrix is equal to the number of non-zero rows in a row echelon form of the matrix.

Ex 1.17 Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ by reducing it to a row-echelon form.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix} \quad R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 3R_1$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

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The last equivalent matrix is in row-echelon form.
It has two non-zero rows

$\therefore \rho(A) = 2$.

Ex 1.18 Find the rank of the matrix $\begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix}$ by reducing it to an echelon form.

Let $A = \begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix}$

$= \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & 2 & 8 & 7 \\ 0 & 8 & -13 & -2 \end{bmatrix}$ $R_2 \to 2R_2 + 3R_1$, $R_3 \to R_3 - 3R_1$

$= \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & 2 & 8 & 7 \\ 0 & 0 & -45 & -30 \end{bmatrix}$ $R_3 \to R_3 - 4R_2$

The last equivalent matrix is in row-echelon form.
It has three non-zero rows.

$\therefore \rho(A) = 3$.

Eg 1.20 Find the inverse of the non-singular matrix $A = \begin{bmatrix} 0 & 5 \\ -1 & 6 \end{bmatrix}$ by Gauss-Jordan method.

$[A | I_2] = \left[\begin{array}{cc|cc} 0 & 5 & 1 & 0 \\ -1 & 6 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} -1 & 6 & 0 & 1 \\ 0 & 5 & 1 & 0 \end{array}\right]$ $R_1 \leftrightarrow R_2$

$= \left[\begin{array}{cc|cc} 1 & -6 & 0 & -1 \\ 0 & 5 & 1 & 0 \end{array}\right]$ $R_1 \to (-1)R_1$

$= \left[\begin{array}{cc|cc} 1 & -6 & 0 & -1 \\ 0 & 1 & \frac{1}{5} & 0 \end{array}\right]$ $R_2 \to \frac{1}{5}R_2$

$= \left[\begin{array}{cc|cc} 1 & 0 & \frac{6}{5} & -1 \\ 0 & 1 & \frac{1}{5} & 0 \end{array}\right]$ $R_1 \to R_1 + 6R_2$

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$\therefore A^{-1} = \begin{bmatrix} 6/5 & -1 \\ 1/5 & 0 \end{bmatrix}$ or $\frac{1}{5}\begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}$

Ex 1.21 Find the inverse of

$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ by Gauss-Jordan Method

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \quad R_1 \to \frac{R_1}{2}$$

$$= \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \quad \begin{matrix} R_2 \to \\ R_2 - 3R_1 \\ R_3 \to \\ R_3 - 2R_1 \end{matrix}$$

$$= \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \quad R_2 \to 2R_2$$

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$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \quad R_1 \to R_1 - \frac{1}{2}R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \quad \begin{matrix} R_1 \to R_1 - R_3 \\ R_2 \to R_2 + R_3 \end{matrix}$$

$\therefore A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

EXERCISE

1. Find the rank of the matrices by minor method.

(i) $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

$|A| = 4 - 4 = 0$

Hence $\rho(A) \neq 0$

$\therefore \rho(A) = 1$

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(ii) $\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

Let $A = \begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$ order of A is 3×2

$\therefore \rho(A) \leq 2$.

Consider second order minor matrix

$\begin{vmatrix} -1 & 3 \\ 4 & -7 \end{vmatrix} = 7 - 12 = -5 \neq 0$.

Hence $\rho(A) = 2$

(iii) Let $A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

order of A is 2×4 $\therefore \rho(A) \leq 2$.

consider the second order matrix

$\begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} = -6 + 6 = 0$

$\begin{vmatrix} -1 & 0 \\ -3 & 1 \end{vmatrix} = -1 + 0 = -1 \neq 0$

$\therefore \rho(A) = 2$

(iv) Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & +1 & -1 \end{bmatrix}$

$|A| = 1(-4+6) + 2(-2+30) + 3(2-20)$

$= 1(2) + 2(28) + 3(-18)$

$= 2 + 56 - 54 = 4 \neq 0$

$\therefore \rho(A) = 3$

(v) Let $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

order of A is 3×4 $\therefore \rho(A) \leq 3$.

Consider the second order minor matrix

$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 0 & 2 \end{vmatrix} = 1(8-0) - 2(4-3) + 1(0-4)$

$= 8 - 2(1) - 4$

$= 8 - 2 - 4$

$= 8 - 6 = 2 \neq 0$

$\therefore \rho(A) = 3$.

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2) Find the rank of the following matrices by row reduction method.

(i) $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$ order of A is 3×4

$\therefore \rho(A) \leq 3$

$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & -6 & 2 & -4 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 5R_1$

$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$

It has two non-zero rows

$\therefore \rho(A) = 2$

(ii) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -4 & 4 \\ 0 & -3 & 2 \end{bmatrix}$ $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$

$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -1 & 1 \\ 0 & -3 & 2 \end{bmatrix}$ $R_3 \to \frac{R_3}{4}$

$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & -7 & 5 \\ 0 & -3 & 2 \end{bmatrix}$ $R_3 \leftrightarrow R_2$

$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \\ 0 & -3 & 2 \end{bmatrix}$ $R_3 \to R_3 - 7R_2$

$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix}$ $R_4 \to R_4 - 3R_2$

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$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad R_4 \to 2R_4 - R_3$$

It has three non-zero rows

$\therefore \rho(A) = 3$

(iii) Let $A = \begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

order of A is 3×4. $\therefore \rho(A) \leq 3$.

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 4 \\ 3 & -8 & 5 & 2 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & 0 \\ 0 & -2 & 14 & -4 \end{bmatrix} \quad R_2 \to R_2 + 2R_1, \; R_3 \to R_3 + 3R_1$$

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

It has three non-zero rows

$\therefore \rho(A) = 3$

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3) Find the inverse of each of the following by Gauss-Jordon method.

(i) $\begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$

$$[A/I_2] = \left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 5 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 5 & -2 & 0 & 1 \end{array}\right] \quad R_1 \to \frac{1}{2}R_1$$

$$\sim \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{array}\right] \quad R_2 \to R_2 - 5R_1$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 2 \end{array}\right] \quad R_1 \to R_1 + R_2, \; R_2 \to 2R_2$$

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$\therefore A^{-1} = \begin{vmatrix} -2 & 1 \\ -5 & 2 \end{vmatrix}$

(ii) Let $A = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{vmatrix}$

$[A/I_3] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 4 & -3 & -6 & 0 & 1 \end{array}\right]$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 6R_1$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$ $R_1 \to R_1 + R_2$, $R_3 \to R_3 - 4R_2$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$ $R_2 \to R_2 + R_3$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$ $R_1 \to R_1 + R_3$

$\therefore A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$

(iii) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

$[A/I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 0 & 1 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$ $R_1 \to R_1 + R_3$, $R_3 \to R_3 + 2R_2$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$ $R_1 \to R_1 + 8R_3$, $R_2 \to R_2 - 3R_3$, $R_3 \to (-1)R_3$

$\therefore A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$ //

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Solution to a system of Linear equations:

Note: A system of linear eqns having atleast one solution is said to be consistent. A system of linear eqns having no solution is said to be inconsistent.

Matrix inversion Method:-

This method can be applied only when the coefficient matrix is a square matrix and non-singular.

Consider the matrix eqn

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$\therefore \boxed{X = A^{-1}B}$$

Ex 1.22 (31) Solve the following system of linear eqns, using matrix inversion Method

$5x + 2y = 3, \quad 3x + 2y = 5$

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$5x + 2y = 3, \quad 3x + 2y = 5$

$$\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$AX = B \Rightarrow X = A^{-1}B$$

where $A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$|A| = 10 - 6 = 4 \neq 0$

$\therefore A^{-1}$ exists

$$A^{-1} = \frac{1}{|A|}[\text{adj } A]$$

$$= \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

Now

$$X = A^{-1}B$$

$$= \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -4 \\ 16 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

$\therefore x = -1, \quad y = 4$

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1.23 / 31 Solve, using matrix inversion Method

$$2x_1 + 3x_2 + 3x_3 = 5$$
$$x_1 - 2x_2 + x_3 = -4$$
$$3x_1 - x_2 - 2x_3 = 3$$

The Matrix form is $AX = B$.

where $A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix} = 2(4+1) - 3(-2-3) + 3(-1+6)$$
$$= 2(5) - 3(-5) + 3(5)$$
$$= 10 + 15 + 15 = 40 \neq 0$$

$\therefore A^{-1}$ exists

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$= \frac{1}{40}\begin{bmatrix} (4+1) & -(-2-3) & (-1+6) \\ -(-6+3) & (-4-9) & -(-2-9) \\ (3+6) & -(2-3) & (-4-3) \end{bmatrix}^T$$

$$= \frac{1}{40}\begin{bmatrix} 5 & 5 & 5 \\ +3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}^T$$

$$= \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$X = A^{-1}B$$
$$= \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$
$$= \frac{1}{40}\begin{bmatrix} 25 - 12 + 27 \\ 25 + 52 + 3 \\ 25 - 44 - 21 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\therefore x_1 = 1, x_2 = 2, x_3 = -1$

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Ex 1.24
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If $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ find AB and BA

and hence solve the eqns

$x-y+z=4,\ x-2y-2z=9,\ 2x+y+3z=1$

$$AB = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 8I_3$$

$$BA = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -4+7+5 & 4-1-3 & 4-3-1 \\ -4+14-10 & 4-2+6 & 4-6+2 \\ -8-7+15 & 8+1-9 & 8+3-3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 8I_3$$

$AB = 8I \Rightarrow I = \frac{1}{8}AB$

$B^{-1} = \frac{1}{8}ABB^{-1}$

$B^{-1} = \frac{1}{8}A$

The matrix form of given system of eqns

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$BX = C$ where $C = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$

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$$\therefore X = B^{-1}C$$

$$= \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$\therefore$ Solutions are $x = 3$, $y = -2$, $z = -1$

EXERCISE 1.3

1) Solve the system of linear eqns by matrix inversion method.

$2x + 5y = -2$, $x + 2y = -3$

The matrix form is $AX = B$

where $A = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$ $B = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

$AX = B \Rightarrow X = A^{-1}B$

$|A| = 4 - 5 = -1 \neq 0$

$\therefore A^{-1}$ is exists

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{-1}\begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}$$

$$X = A^{-1}B = -\begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$= -\begin{bmatrix} -4+15 \\ 2-6 \end{bmatrix} = -\begin{bmatrix} 11 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} -11 \\ 4 \end{bmatrix}$$

$\therefore x = -11$, $y = 4$.

(ii) $2x - y = 8$, $3x + 2y = -2$

The matrix form is

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix} \quad \text{i.e. } AX = B$$

$X = A^{-1}B$.

where $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$

$B = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$

$|A| = 4 + 3 = 7 \neq 0$

$\therefore A^{-1}$ exists.

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$$A^{-1} = \frac{1}{|A|}\,\text{adj}\,A = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 16-2 \\ -24-4 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 14 \\ -28 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$\therefore x = 2,\ y = -4$

(iii) $2x + 3y - z = 9$, $x + y + z = 9$

$3x - y - z = -1$.

The matrix form is $AX = B$

$X = A^{-1}B$

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

Here $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$

$|A| = 2(-1+1) - 3(-1-3) - 1(-1-3)$

$= 2(0) - 3(-4) - 1(-4)$

$12 + 4 = 16 \neq 0$

$\therefore A^{-1}$ exists.

$$A^{-1} = \frac{1}{|A|}\,\text{adj}\,A.$$

$$= \frac{1}{16}\begin{bmatrix} (-1+1) & -(-1-3) & (-1-3) \\ -(-3-1) & (-2+3) & -(-2-9) \\ (3+1) & -(2+1) & (2-3) \end{bmatrix}^T$$

$$= \frac{1}{16}\begin{bmatrix} 0 & +4 & -4 \\ +4 & 1 & +11 \\ 4 & -3 & -1 \end{bmatrix}^T$$

$$= \frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 0+36-4 \\ 36+9+3 \\ -36+99+1 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 32 \\ 48 \\ 64 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$\therefore x = 2,\ y = 3,\ z = 4.$

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(IV) $x+y+z-2=0$, $6x-4y+5z-31=0$

$5x+2y+2z=13$.

The matrix form is $AX=B$

$X=A^{-1}B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

Here $A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$|A| = 1(-8-10) - 1(12-25) + 1(12+20)$

$= 1(-18) - 1(-13) + 1(32)$

$= -18+13+32 = 27 \neq 0$

$\therefore$ A^{-1} exists.

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$= \frac{1}{27}\begin{bmatrix} (-8-10) & -(12-25) & (12+20) \\ -(2-2) & (2-5) & -(2-5) \\ (5+4) & -(5-6) & (-4-6) \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} -18 & 13 & 32 \\ 0 & -3 & 3 \\ 9 & 1 & -10 \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}\begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} -36+0+117 \\ 26-93+13 \\ 64+93-130 \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} 81 \\ -54 \\ 27 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ $x=3$, $y=-2$, $z=1$

---

2) If $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

find the product AB and BA. and hence Solve the eqns $x+y+2z=1$, $3x+2y+z=7$, $2x+y+3z=2$.

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$$AB = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5+3+6 & -5+2+3 & -10+1+9 \\ 7+3-10 & 7+2-5 & 14+1-15 \\ 1-3+2 & 1-2+1 & 2-1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 4I$$

$$BA = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5+7+2 & 1+1-2 & 3-5+2 \\ -15+14+1 & 3+2-1 & 9-10+1 \\ -10+7+3 & 2+1-3 & 6-5+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 4I$$

$AB = 4I$

$\therefore I = \frac{1}{4} AB$

$B^{-1} = \frac{1}{4} ABB^{-1} = \frac{1}{4} A$

$$= \frac{1}{4} \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$$

The matrix form of given eqn is

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$BX = C$

$X = B^{-1}C$

$$X = \frac{1}{4} \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} -5+7+6 \\ 7+7-10 \\ 1-7+2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$\therefore$ **$x = 2, y = 1, z = -1$**

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3) A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was ₹ 19,800 p.m at the end of the first month after 3 years of service and ₹ 23,400 pm at the end of the first month after 9 yrs of service, find his starting salary and his annual increment

Let x be the salary and y be the fixed increment.

$x + 3y = 19800$

$x + 9y = 23,400$

Matrix form is

$$\begin{bmatrix} 1 & 3 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19800 \\ 23400 \end{bmatrix}$$

$AX = B$

$X = A^{-1}B$

where $A = \begin{bmatrix} 1 & 3 \\ 1 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 19800 \\ 28400 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = 9 - 3 = 6 \neq 0$

$\therefore A^{-1}$ exist

$A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{6}\begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$

$X = A^{-1}B$

$= \frac{1}{6}\begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 19800 \\ 23400 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} 178200 - 70200 \\ -19800 + 23400 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} 108000 \\ 3600 \end{bmatrix}$

$= \begin{bmatrix} 18000 \\ 600 \end{bmatrix}$

$\therefore x = 18000, \; y = 600$

Starting salary = ₹ 18000

Annual Increment = ₹ 600.

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4) Four men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one women alone to finish the same work by using matrix inversion Method.

Let one man can finish the work in x days.

Let one women can finish the work in y days.

$\therefore$ A man's one day work $= \frac{1}{x}$

A Women's one day work $= \frac{1}{y}$.

In 3 days, man complete $\frac{3}{x}$ work

and In 3 days, Women complete $\frac{3}{y}$ work.

In 4 days, a man complete $\frac{4}{x}$ work and Women complete $\frac{4}{y}$ work.

$\therefore$ 4 men and 4 women complete the work

$= 4\left(\frac{3}{x}\right) + 4\left(\frac{3}{y}\right) = 1$

$\frac{12}{x} + \frac{12}{y} = 1$ put $\frac{1}{x} = a$ and $\frac{1}{y} = b$

$\therefore 12a + 12b = 1$ ——— (1)

2 men and 5 women complete the work

$= 2\left(\frac{4}{x}\right) + 5\left(\frac{4}{y}\right) = 1$

$\frac{8}{x} + \frac{20}{y} = 1$

$8a + 20b = 1$ ——— (2)

(1) $\times 2 \Rightarrow$ $24a + 24b = 2$ ——— (3)

(2) $\times 3 \Rightarrow$ $24a + 60b = 3$ ——— (4)

(4) $-$ (3) $\Rightarrow$ $36b = 1$

$b = \frac{1}{36}$

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i.e. $\frac{1}{y} = \frac{1}{36} \Rightarrow y = 36$.

Sub $b = \frac{1}{36}$ in (1)

$12a + 12\left(\frac{1}{36}\right) = 1$

$12a + \frac{1}{3} = 1 \Rightarrow 12a = 1 - \frac{1}{3}$

$12a = \frac{2}{3}$

$a = \frac{2}{3 \times 12} = \boxed{\frac{1}{18}}$

i.e. $\frac{1}{x} = \frac{1}{18} \Rightarrow \boxed{x = 18}$

∴ a man alone can do the work in 18 days.

A women alone can do the work in 36 days.

---

5) The prices of three commodities A, B and C are x, y and z respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C. person Q purchases 2 units of C and sells 3 units of A and one unit of B. person R

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purchases one unit of A and sells 3 unit of B and one unit of C. In the process, P, Q and R earn ₹ 15,000 ₹ 1,000 and ₹ 4000 respectively. Find the prices per unit of A, B and C.

Given price of A = ₹ x / unit
price of B = ₹ y / unit
price of C = ₹ z / unit.

By the given data.

$2x - 4y + 5z = 15{,}000$ — (1)
$3x + y - 2z = 1000$ — (2)
$-x + 3y + z = 4000$ — (3)

$$\begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15{,}000 \\ 1000 \\ 4000 \end{bmatrix}$$

$AX = B \Rightarrow X = A^{-1}B$

$$|A| = \begin{vmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{vmatrix} = 2(1+6) + 4(3-2) + 5(9+1)$$

$= 2(7) + 4(1) + 5(10)$

$= 14 + 4 + 50 = 68$

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$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$= \frac{1}{68}\begin{bmatrix} (1+6) & -(3-2) & (9+1) \\ -(-4-15) & (2+5) & -(6-4) \\ (8-5) & -(-4-15) & (2+12) \end{bmatrix}^T$$

$$= \frac{1}{68}\begin{bmatrix} 7 & -1 & 10 \\ 19 & 7 & -2 \\ 3 & 19 & 14 \end{bmatrix}^T$$

$$= \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}\begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$= \frac{1}{68}\begin{bmatrix} 105000 + 19000 + 12000 \\ -15000 + 7000 + 76000 \\ 150000 - 2000 + 56000 \end{bmatrix}$$

$$= \frac{1}{68}\begin{bmatrix} 136000 \\ 68000 \\ 204000 \end{bmatrix}$$

$$= \begin{bmatrix} 2000 \\ 1000 \\ 3000 \end{bmatrix}$$

$\therefore$ $x = 2000$, $y = 1000$, $z = 3000$.

$\therefore$ price of A per unit = Rs 2000
price of B per unit = Rs 1000
price of C per unit = Rs 3000.

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Cramer's rule:-

Consider the following system of equations.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Here $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$

$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$

$\therefore$ By Cramer's rule

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad x_3 = \frac{\Delta_3}{\Delta}.$$

Note: If $\Delta = 0$, cramer's rules cannot be applied.

Ex 1.25 Solve, by Cramer's rule, the system of equations.

$x_1 - x_2 = 3$, $2x_1 + 3x_2 + 4x_3 = 17$, $x_2 + 2x_3 = 7$.

$$\Delta = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix} = 1(6-4) + 1(4-0) + 0(2-0)$$
$$= 2 + 4 = 6.$$

$$\Delta_x = \begin{vmatrix} 3 & -1 & 0 \\ 17 & 3 & 4 \\ 7 & 1 & 2 \end{vmatrix} = 3(6-4) + 1(34-28) + 0.$$
$$= 3(2) + 1(6)$$
$$= 6 + 6 = 12$$

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$$\Delta_y = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 17 & 4 \\ 0 & 7 & 2 \end{vmatrix} = 1(34-28) - 3(4-0) + 0(0)$$
$$= 1(6) - 12$$
$$= -6$$

$$\Delta_z = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 17 \\ 0 & 1 & 7 \end{vmatrix} = 1(21-17) + 1(14-0) + 3(2-0)$$
$$= 1(4) + 14 + 6$$
$$= 24$$

$\therefore$ $x = \frac{\Delta_x}{\Delta} = \frac{12}{6} = 2$

$y = \frac{\Delta_y}{\Delta} = \frac{-6}{6} = -1$

$z = \frac{\Delta_z}{\Delta} = \frac{24}{6} = 4$

$\therefore$ $\boxed{x = 2,\ y = -1,\ z = 4}$

EXERCISE 1.4

Solve by Cramer's rule

(i) $5x - 2y + 16 = 0$, $x + 3y - 7 = 0$

This can be written in the form of

$5x - 2y = -16$, $x + 3y = 7$.

$$\Delta = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix} = 15 + 2 = 17$$

$$\Delta_x = \begin{vmatrix} -16 & -2 \\ 7 & 3 \end{vmatrix} = -48 + 14 = -34$$

$$\Delta_y = \begin{vmatrix} 5 & -16 \\ 1 & 7 \end{vmatrix} = 35 + 16 = 51$$

$\therefore$ By cramer rule,

$x = \frac{\Delta_x}{\Delta} = \frac{-34}{17} = -2$

$y = \frac{\Delta_y}{\Delta} = \frac{51}{17} = 3$

$\therefore$ $\boxed{x = -2,\ y = 3}$

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(ii) $\frac{3}{x} + 2y = 12$, $\frac{2}{x} + 3y = 13$

Let $\frac{1}{x} = a$,

$\therefore 3a + 2y = 12$, $2a + 3y = 13$

$$\Delta = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9 - 4 = 5$$

$$\Delta_a = \begin{vmatrix} 12 & 2 \\ 13 & 3 \end{vmatrix} = 36 - 26 = 10$$

$$\Delta_y = \begin{vmatrix} 3 & 12 \\ 2 & 13 \end{vmatrix} = 39 - 24 = 15$$

$$a = \frac{\Delta a}{\Delta} = \frac{10}{5} = 2$$

$a \Rightarrow \frac{1}{x} = 2 \Rightarrow \boxed{x = \frac{1}{2}}$

$$y = \frac{\Delta y}{\Delta} = \frac{15}{5} = 3$$

$\therefore \boxed{y = 3}$

(iii) $3x + 3y - z = 11$, $2x - y + 2z = 9$,
$4x + 3y + 2z = 25$.

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$$\Delta = \begin{vmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{vmatrix} = 3(-2-6) - 3(4-8) - 1(6+4)$$

$$= 3(-8) - 3(-4) - 1(10)$$

$$= -24 + 12 - 10$$

$$= \boxed{-22}$$

$$\Delta_x = \begin{vmatrix} 11 & 3 & -1 \\ 9 & -1 & 2 \\ 25 & 3 & 2 \end{vmatrix} = 11(-2-6) - 3(18-50) - 1(27+25)$$

$$= 11(-8) - 3(-32) - 1(52)$$

$$= -88 + 96 - 52$$

$$= 96 - 140$$

$$= \boxed{-44}$$

$$\Delta_y = \begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix} = 3(18-50) - 11(4-8) - 1(50-36)$$

$$= 3(-32) - 11(-4) - 1(14)$$

$$= -96 + 44 - 14$$

$$= 44 - 110 = \boxed{-66}$$

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$$\Delta z = \begin{vmatrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix}$$

$= 3(-25-27) - 3(50-36) + 11(6+4)$

$= 3(-52) - 3(14) + 11(10)$

$= -156 - 42 + 110$

$= -198 + 110$

$= \boxed{-88}$

By Cramer's rule

$x = \frac{\Delta x}{\Delta} = \frac{-44}{-22} = \boxed{2}$

$y = \frac{\Delta y}{\Delta} = \frac{-66}{-22} = \boxed{3}$

$z = \frac{\Delta z}{\Delta} = \frac{-88}{-22} = \boxed{4}$

$\therefore \boxed{x = 2, y = 3, z = 4}$

(iv) $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0$, $\frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0$

$\frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$.

Let $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$.

$\therefore$ $3a - 4b - 2c = 1$

$a + 2b + c = 2$

$2a - 5b - 4c = -1$

Here $\Delta = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix} = 3(-8+5) + 4(-4-2) - 2(-5-4)$

$= 3(-3) + 4(-6) - 2(-9)$

$= -9 - 24 + 18$

$= -33 + 18$

$= \boxed{-15}$

$\Delta a = \begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix}$

$= 1(-8+5) + 4(-8+1) - 2(-10+2)$

$= 1(-3) + 4(-7) - 2(-8)$

$= -3 - 28 + 16$

$= -31 + 16 = \boxed{-15}$

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$$\Delta_b = \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix}$$

$= 3(-8+1) - 1(-4-2) - 2(-1-4)$

$= 3(-7) - 1(-6) - 2(-5)$

$= -21 + 6 + 10$

$= -21 + 16 = \boxed{-5}$

$$\Delta_c = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix}$$

$= 3(-2+10) + 4(-1-4) + 1(-5-4)$

$= 3(8) + 4(-5) + 1(-9)$

$= 24 - 20 - 9 = 24 - 29$

$= \boxed{-5}$

By Cramer's rule,

$a = \frac{\Delta a}{\Delta} = \frac{-15}{-15} = \boxed{1}$

$b = \frac{\Delta b}{\Delta} = \frac{-5}{-15} = \boxed{\frac{1}{3}}$

$c = \frac{\Delta c}{\Delta} = \frac{-5}{-15} = \boxed{\frac{1}{3}}$

But $a = \frac{1}{x}$ | $b = \frac{1}{y}$ | $c = \frac{1}{z}$

i.e. $1 = \frac{1}{x}$ | $\frac{1}{3} = \frac{1}{y}$ | $\frac{1}{3} = \frac{1}{z}$

$\therefore \boxed{x = 1}$ | $\boxed{y = 3}$ | $\therefore \boxed{z = 3}$

2) In a competitive examination one mark is awarded for every correct answer while $\frac{1}{4}$ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks. How many questions did he answer correctly?

Let x be the no of questions answered correctly.

Let y be the no of questions answered wrongly.

By data, $x + y = 100$ —— (1)

$x - \frac{1}{4}y = 80$

$\frac{4x - y}{4} = 80$

$4x - y = 320$ —— (2).

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$$\Delta = \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} = -1-4 = -5$$

$$\Delta_x = \begin{vmatrix} 100 & 1 \\ 320 & -1 \end{vmatrix} = -100-320 = -420$$

$$\Delta_y = \begin{vmatrix} 1 & 100 \\ 4 & 320 \end{vmatrix} = 320-400 = -80.$$

By Cramer's rule,

$$x = \frac{\Delta x}{\Delta} = \frac{-420}{-5} = 84$$

$$y = \frac{\Delta y}{\Delta} = \frac{-80}{-5} = 16$$

∴ No of questions answered correctly = 84

---

3) A chemist has one soln which is 50% acid and another soln which is 25% acid. How much each should be mixed to make 10 litres of 40% acid soln.

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Let x litres of 50% acid and y litres of 25% acid to be mixed.

By data $x+y = 10$ — (1)

50% of x + 25% of y = 40% of $(x+y)$

$$\frac{50}{100}x + \frac{25}{100}y = \frac{40}{100} \times 10.$$

$$\frac{1}{2}x + \frac{1}{4}y = 4$$

$$\frac{2x+y}{4} = 4 \Rightarrow 2x+y = 16 \quad — (2)$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1-2 = -1$$

$$\Delta_x = \begin{vmatrix} 10 & 1 \\ 16 & 1 \end{vmatrix} = 10-16 = -6$$

$$\Delta_y = \begin{vmatrix} 1 & 10 \\ 2 & 16 \end{vmatrix} = 16-20 = -4$$

By Cramer's rule,

$$x = \frac{\Delta x}{\Delta} = \frac{-6}{-1} = \boxed{6}$$

$$y = \frac{\Delta y}{\Delta} = \frac{-4}{-1} = \boxed{4}$$

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i.e 6 litres of 50% acid and 4 litres of 25% acid to be mixed toget 10 litres of 40% acid solution.

---

4) A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself.

Let x minutes be the time taken for the pump A to fill the tank.

Let y minutes be the time taken for the pump B to fill the tank.

By data, $\frac{1}{x} + \frac{1}{y} = \frac{1}{10}$ and

$\frac{1}{x} - \frac{1}{y} = \frac{1}{30}$.

put $\frac{1}{x} = a$ and $\frac{1}{y} = b$.

$a+b = \frac{1}{10}$ | $a-b = \frac{1}{30}$

$10a+10b = 1$ | $30a-30b = 1$

$$\Delta = \begin{vmatrix} 10 & 10 \\ 30 & -30 \end{vmatrix} = -300-300 = -600$$

$$\Delta_a = \begin{vmatrix} 1 & 10 \\ 1 & -30 \end{vmatrix} = -30-10 = -40.$$

$$\Delta_b = \begin{vmatrix} 10 & 1 \\ 30 & 1 \end{vmatrix} = 10-30 = -20$$

By Cramer's rule,

$$a = \frac{\Delta a}{\Delta} = \frac{-40}{-600} = \frac{4}{60} = \boxed{\frac{1}{15}}$$

$$b = \frac{\Delta b}{\Delta} = \frac{-20}{-600} = \frac{2}{60} = \boxed{\frac{1}{30}}$$

But $a = \frac{1}{x}$ | $b = \frac{1}{y}$

i.e $\frac{1}{15} = \frac{1}{x}$ | i.e $\frac{1}{30} = \frac{1}{y} \Rightarrow \boxed{y = 30}$

$\therefore \boxed{x = 15}$

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∴ pump A can fill the tank in 15 minutes and pump B can fill the tank in 30 minutes respectively.

---

5) A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two Vadais is ₹ 150. The cost of 2 dosai two idlies and four Vadai is ₹ 200. The cost of 5 dosai 4 idlies and two Vadai is ₹ 250. The family has ₹ 350 in hand and they ate 3 dosai and 6 idlies and six Vadais. will they be able to manage to pay the bill with in the amount they had?

Let the cost of 1 dosai be x
and the " " 1 idlie be y
and the cost of 1 Vadai be z.

By data.

$$2x + 3y + 2z = 150$$
$$2x + 2y + 4z = 200$$
$$5x + 4y + 2z = 250$$

Let $\Delta = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & 4 \\ 5 & 4 & 2 \end{vmatrix}$ $= 2(4-16) - 3(4-20) + 2(8-10)$

$= 2(-12) - 3(-16) + 2(-2)$

$= -24 + 48 - 4$

$= 48 - 28$

$= \boxed{20}$

$\Delta x = \begin{vmatrix} 150 & 3 & 2 \\ 200 & 2 & 4 \\ 250 & 4 & 2 \end{vmatrix}$ $= 150(4-16) - 3(400-1000) + 2(800-500)$

$= 150(-12) - 3(-600) + 2(300)$

$= -1800 + 1800 + 600$

$= \boxed{600}$

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$$\Delta y = \begin{vmatrix} 2 & 150 & 2 \\ 2 & 200 & 4 \\ 5 & 250 & 2 \end{vmatrix}$$

$= 2(400-1000) - 150(4-20) + 2(500-1000)$

$= 2(-600) - 150(-16) + 2(-500)$

$= -1200 + 2400 - 1000$

$= 2400 - 2200$

$= \boxed{200}$

$$\Delta z = \begin{vmatrix} 2 & 3 & 150 \\ 2 & 2 & 200 \\ 5 & 4 & 250 \end{vmatrix}$$

$= 2(500-800) - 3(500-1000) + 150(8-10)$

$= 2(-300) - 3(-500) + 150(-2)$

$= -600 + 1500 - 300$

$= 1500 - 900 = \boxed{600}$

$\therefore x = \frac{\Delta x}{\Delta} = \frac{600}{20} = \boxed{30}$

$y = \frac{\Delta y}{\Delta} = \frac{200}{20} = \boxed{10}$

$z = \frac{\Delta z}{\Delta} = \frac{600}{20} = \boxed{30}$

$\therefore$ Cost of 1 dosai = Rs 30
Cost of 1 Idlie = Rs 10
Cost of 1 Vadai = Rs 30

Now the total cost of 3 dosai & 6 idlies and Six Vadai

$= 3(30) + 6(10) + 6(30)$

$= 90 + 60 + 180$

$= \boxed{330}$

$\therefore$ They will be able to manage to pay the bill since they have Rs 350.

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Gaussian Elimination Method :-

1.27 Solve by Gaussian elimination Method

$4x+3y+6z=25$, $x+5y+7z=13$, $2x+9y+z=1$.

$$\left[\begin{array}{ccc|c} 4 & 3 & 6 & 25 \\ 1 & 5 & 7 & 13 \\ 2 & 9 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 4 & 3 & 6 & 25 \\ 2 & 9 & 1 & 1 \end{array}\right] R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 0 & -17 & -22 & -27 \\ 0 & -1 & -13 & -25 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 0 & -17 & -22 & -27 \\ 0 & 0 & -199 & -398 \end{array}\right] R_3 \to 17R_3 - R_2$$

$$\Rightarrow \begin{array}{rl} x+5y+7z = 13 & \text{---} (1) \\ -17y-22z = -27 & \text{---} (2) \\ -199z = -398 & \text{---} (3) \end{array}$$

$z = \frac{-398}{-199} = 2$

Sub $z=2$ in (2)

$17y + 22(2) = 27$

$17y + 44 = 27$

$17y = 27-44$

$17y = -17$

$\boxed{y=-1}$

Sub $z=2, y=-1$ in (1)

$x+5(-1)+7(2)=13$

$x-5+14=13$

$x+9=13$

$x=13-9$

$\boxed{x=4}$

$\therefore x=4, y=-1, z=2$

EXERCISE 1.5

Solve by Gaussian elimination method.

(i) $2x-2y+3z=2$, $x+2y-z=3$, $3x-y+2z=1$.

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -2 & 3 & 2 \\ 3 & -1 & 2 & 1 \end{array}\right] R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -7 & 5 & -8 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

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$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & 0 & -5 & -20 \end{array}\right] \quad R_3 \to 6R_3 - 7R_2$$

$\Rightarrow$ $x + 2y - z = 3$ — ①

$-6y + 5z = -4$ — ②

$-5z = -20$ — ③

from ③ $-5z = -20$

$\boxed{z = 4}$

Sub $z = 4$, in ②

$-6y + 5(4) = -4$

$-6y + 20 = -4$

$-6y = -24$

$\boxed{y = 4}$

Sub $y = 4$, $z = 4$ in ①

$x + 2(4) - 4 = 3$

$x + 8 - 4 = 3$

$x + 4 = 3$

$\boxed{x = -1}$

$\therefore$ $x = -1$, $y = 4$, $z = 4$

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(11) $2x + 4y + 6z = 22$,

$3x + 8y + 5z = 27$

$-x + y + 2z = 2$

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right] \begin{array}{l} R_1 \to \frac{R_1}{2} \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 0 & 22 & 44 \end{array}\right] \quad R_3 \to 2R_3 - 3R_2$$

$\Rightarrow$ $x + 2y + 3z = 11$ — ①

$2y - 4z = -6$ — ②

$22z = 44$ — ③

from ③ $z = \frac{44}{22} = 2$

$\boxed{z = 2}$

Sub $z = 2$ in ②

$2y - 8 = -6$

$2y = 2$

$\boxed{y = 1}$

$y = 1$, $z = 2$ in ①

$x + 2 + 6 = 11$

$x + 8 = 11$

$\boxed{x = 3}$

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2) If ax^2+bx+c is divided by $x+3$, $x-5$ and $x-1$, the remainders are 21, 61 and 9 respectively find a, b, c (use Gaussian Method).

Let $f(x) = ax^2+bx+c$

we are given that

$f(-3) = 21 \Rightarrow 9a-3b+c = 21$

$f(5) = 61 \Rightarrow 25a+5b+c = 61$

$f(1) = 9 \Rightarrow a+b+c = 9$

$$\begin{bmatrix} 9 & -3 & 1 \\ 25 & 5 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 21 \\ 61 \\ 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 25 & 5 & 1 \\ 9 & -3 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 61 \\ 21 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -20 & -24 \\ 0 & -12 & -8 \end{bmatrix} \begin{bmatrix} 9 \\ -164 \\ -60 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 25R_1 \\ R_3 \to 9R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -20 & -24 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} 9 \\ -164 \\ 192 \end{bmatrix} \quad R_3 \to 5R_3 - 3R_2$$

$\Rightarrow a+b+c = 9$ ——— (1)

$-20b-24c = -164$ — (2)

$32c = 192$ — (3)

$c = \frac{192}{32} = \boxed{6}$

Sub $c=6$ in (2)

$20b+24(6) = 164$

$20b+144 = 164$

$20b = 164-144$

$20b = 20$

$\boxed{b=1}$

Sub $b=1$, $c=6$ in (1)

$a+1+6 = 9$

$a+7 = 9$

$\boxed{a=2}$

$\therefore \boxed{a=2, b=1, c=6}$

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4) A boy is walking along the path $y = ax^2 + bx + c$ through the points $(-6, 8)$, $(-2, -12)$, and $(3, 8)$. He wants to meet his friend at $P(7, 60)$. will he meet his friend? (Use Gaussian method).

Let $y = ax^2 + bx + c$

The curve passes through the points $(-6, 8)$, $(-2, -12)$, and $(3, 8)$

$\therefore$ $36a - 6b + c = 8$

$4a - 2b + c = -12$

$9a + 3b + c = 8$

Now, $\begin{bmatrix} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 9 & 3 & 1 & 8 \end{bmatrix}$

$\begin{bmatrix} 4 & -2 & 1 & -12 \\ 36 & -6 & 1 & 8 \\ 9 & 3 & 1 & 8 \end{bmatrix}$ $R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 4 & -2 & 1 & -12 \\ 0 & 12 & -8 & 116 \\ 0 & 30 & -5 & 140 \end{bmatrix}$ $R_2 \to R_2 - 9R_1$, $R_3 \to 4R_3 - 9R_1$

$\sim \begin{bmatrix} 4 & -2 & 1 & -12 \\ 0 & 3 & -2 & 29 \\ 0 & 6 & -1 & 28 \end{bmatrix}$ $R_2 \to \frac{R_2}{4}$, $R_3 \to \frac{R_3}{5}$

$\sim \begin{bmatrix} 4 & -2 & 1 & -12 \\ 0 & 3 & -2 & 29 \\ 0 & 0 & 3 & -30 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$

$\therefore$ $4a - 2b + c = -12$ — (1)

$3b - 2c = 29$ — (2)

$3c = -30$ — (3)

From (3)

$c = -10$

Sub $c = -10$ in (2)

$3b - 2(-10) = 29$

$3b + 20 = 29$

$3b = 9$

$b = 3$

Sub $b = 3$, $c = -10$ in (1)

$4a - 2(3) - 10 = -12$

$4a - 6 - 10 = -12$

$4a - 16 = -12$

$4a = 16 - 12 = 4$

$a = 1$

$\therefore$ $a = 1$, $b = 3$, $c = -10$

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$\therefore$ The eqn is $y = x^2 + 3x - 10$

At $x = 7$, $y = 49 + 3(7) - 10$

$= 49 + 21 - 10$

$= 60$

$\therefore$ The point $(7, 60)$ satisfies this eqn.

$\therefore$ The boy will meet his friend.

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Consistency of system of linear equations by Rank method:-

1) **Note:** (Rouche – capelli Th)

A system of linear eqns, written in the form as $AX = B$, is consistent iff the rank of the coefficient matrix is equal to the rank of the augmented matrix, ie $\rho(A) = \rho([A|B])$.

2) If $\rho(A) = \rho([A|B])$ = number of unknowns, then the system has unique solution.

3) If $\rho(A) = \rho([A|B]) <$ number of unknowns, then the system has infinitely many solutions.

4) If $\rho(A) \neq \rho([A|B])$ then the system is inconsistent and has no solutions.

Non homogeneous linear equations:

Test for consistency of the following system of linear eqns and solve

$x+2y-z=3$, $3x-y+2z=1$, $x-2y+3z=3$, $x-y+z+1=0$.

The matrix form is $AX = B$

where $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 1 \\ 3 \\ -1 \end{bmatrix}$

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 1 & -2 & 3 & 3 \\ 1 & -1 & 1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -4 & 4 & 0 \\ 0 & -3 & 2 & -4 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

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$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & & 3 \\ 0 & -7 & 5 & & -8 \\ 0 & 0 & 8 & & 32 \\ 0 & 0 & -4 & & -16 \end{array}\right] \quad \begin{array}{l} R_3 \to 7R_3 - 4R_2 \\ R_4 \to 4R_4 - 3R_3 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & & 3 \\ 0 & -7 & 5 & & -8 \\ 0 & 0 & 1 & & 4 \\ 0 & 0 & 1 & & 4 \end{array}\right] \quad \begin{array}{l} R_3 \to \frac{R_3}{8} \\ R_4 \to \frac{R_4}{-4} \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & & 3 \\ 0 & -7 & 5 & & -8 \\ 0 & 0 & 1 & & 4 \\ 0 & 0 & 0 & & 0 \end{array}\right] \quad R_4 \to R_4 - R_3$$

Three non zero rows in echlon form

$\therefore \rho([A|B]) = 3.$

Also, $\rho(A) = 3$

Hence $\rho(A) = \rho([A|B]) = 3.$

$\therefore$ The system is consistent and has unique solns.

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The system reduces to.

$x + 2y - z = 3$ — (1)

$-7y + 5z = -8$ — (2)

$\boxed{z = 4}$ — (3)

Sub (3) in (2) $-7y + 5(4) = -8$

$-7y + 20 = -8$

$-7y = -28$

$\boxed{y = 4}$

Sub $y = 4, z = 4$ in (1)

$x + 2(8) - 4 = 3$

$x + 16 - 4 = 3$

$x + 4 = 3$

$\boxed{x = -1}$

$\therefore \boxed{x = -1, \ y = 4, \ z = 4}$

Ex 1.30 (39) Test for consistency of the following system of linear eqns. and solve for

$4x - 2y + 6z = 8, \ x + y - 3z = -1, \ 15x - 3y + 9z = 21$

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The matrix form is $AX = B$

where $A = \begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$B = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$

The augment matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{array}\right] \quad R_2 \leftrightarrow R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 5 & -1 & 3 & 7 \end{array}\right] \quad R_2 \to R_2 - 4R_1, \quad R_3 \to \frac{R_3}{3}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -6 & 18 & 12 \end{array}\right] \quad R_3 \to R_3 - 5R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$\rho([A|B]) = 2$ and $\rho(A) = 2$.

$\rho(A) = \rho([A|B]) = 2 < 3$

$x + y - 3z = -1$ — (1)

$-6y + 18z = 12$ — (2)

put $z = t$ in (2)

$-6y + 18t = 12$

$y - 3t = -2$

$y = 3t - 2$

$x + 3t - 2 - 3t = -1$

$x = 2 - 1$

$\boxed{x = 1}$

The system is consistent and infinetly many solns

$\therefore \boxed{x = 1, \; y = 3t - 2, \; z = t}$

Ex 1.31 (40) Test for consistency if possible

Solve $x - y + z = -9$, $2x - 2y + 2z = -18$

$3x - 3y + 3z + 27 = 0$

The matrix form is $AX = B$

where $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & -3 & 3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} -9 \\ -18 \\ -27 \end{bmatrix}$

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$$[A|B] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 2 & -2 & 2 & -18 \\ 3 & -3 & 3 & -27 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$\rho(A) = \rho([A|B]) = 1 < 3$

∴ The system is consistent and has infinitely many solns.

∴ $x - y + z = -9$

put $z = t$, $y = s$

∴ $x - y + t = -9$

$x = y - t - 9$

∴

$x = y - t - 9$
$y = s$
$z = t$

Ex 1.32

Test the consistency and solve.

$x - y + z = -9$, $2x - y + z = 4$,

$3x - y + z = 6$, $4x - y + 2z = 7$.

The matrix form is $AX = B$

Here $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} -9 \\ 4 \\ 6 \\ 7 \end{bmatrix}$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 2 & -1 & 1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 2 & -2 & 33 \\ 0 & 3 & -2 & 43 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 4R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 0 & 11 \\ 0 & 0 & 1 & -23 \end{array}\right] \begin{array}{l} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 3R_2 \end{array}$$

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$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 1 & -23 \\ 0 & 0 & 0 & 11 \end{array}\right] \quad R_3 \leftrightarrow R_4$$

$\Rightarrow \rho(A) = 3, \quad \rho[A|B] = 4$

$\therefore \rho(A) \neq \rho[(A|B)]$

∴ The given system of eqns is inconsistent and has no solution.

Ex 1.34 Investigate for what values of λ and μ the system of linear eqns $x+2y+z=7$, $x+y+\lambda z=\mu$, $x+3y-5z=5$ has (i) no soln (ii) a unique soln (iii) an infinite no of solns.

The matrix form is $AX = B$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & \lambda \\ 1 & 3 & -5 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 7 \\ \mu \\ 5 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 1 & \lambda & \mu \\ 1 & 3 & -5 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 3 & -5 & 5 \\ 1 & 1 & \lambda & \mu \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & -6 & -2 \\ 0 & -1 & \lambda-1 & \mu-7 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & \lambda-7 & \mu-9 \end{array}\right] \quad R_3 \to R_3 + R_2$$

(i) The system has no soln i.e $\rho(A) = 2$ / $\rho(A|B) = 3$

∴ $\lambda - 7 = 0$, $\mu - 9 \neq 0$

$\lambda = 7$ $\mu \neq 9$

(ii) The system has unique soln, i.e $\rho(A)$ = $\rho(A|B)$ = 3.

∴ $\lambda - 7 \neq 0$ and $\mu - 9 \neq 0$

$\lambda \neq 7$ and $\mu \neq 9$

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(iii) The system is consistent and has infinite no of soln i.e $\rho(A) = \rho(A|B) = 2$.

$\therefore \lambda - 7 = 0$ and $\mu - 9 = 0$

$\lambda = 7$ and $\mu = 9$

EXERCISE 1.6

1. Test for consistency if possible solve the following.

(i) $x - y + 2z = 2$, $2x + y + 4z = 7$, $4x - y + z = 4$.

The matrix form is $AX = B$

Here $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 4 \\ 4 & -1 & 1 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 2 & 1 & 4 & 7 \\ 4 & -1 & 1 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 3 & -7 & -4 \end{array}\right] \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & -7 & -7 \end{array}\right] R_3 \to R_3 - R_2$$

Now $\rho(A) = \rho[A|B] = 3$ (No of unknowns)

$\therefore$ The system has a unique soln

$-7z = -7 \Rightarrow \boxed{z = 1}$

$3y = 3 \Rightarrow \boxed{y = 1}$

$x - y + 2z = 2$

$x - 1 + 2 = 2$

$x + 1 = 2$

$\boxed{x = 1}$

$\therefore x = 1, \; y = 1, \; z = 1$

(ii) $3x + y + z = 2$, $x - 3y + 2z = 1$, $7x - y + 4z = 5$.

The matrix form is $AX = B$

where

$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & -3 & 2 \\ 7 & -1 & 4 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$

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$$[A|B] = \left[\begin{array}{ccc|c} 3 & 1 & 1 & 2 \\ 1 & -3 & 2 & 1 \\ 7 & -1 & 4 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 3 & 1 & 1 & 2 \\ 7 & -1 & 4 & 5 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 20 & -10 & -2 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 10 & -5 & -1 \end{array}\right] \quad R_3 \to \frac{R_3}{2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$\Rightarrow \rho(A) = 2, \ \rho[A|B] = 2$

$\rho(A) = \rho[A|B] = 2 < 3$

$\therefore$ The system is consistent and has infinitely many solns.

The system reduced to

$$x - 3y + 2z = 1 \quad \text{—①}$$

$$10y - 5z = -1 \quad \text{—②}$$

put $z = t$.

$\therefore 10y - 5t = -1$

$10y = 5t - 1$

$y = \frac{5t-1}{10}$

$x - 3\left(\frac{5t-1}{10}\right) + 2t = 1$

$x - \frac{15t+3+20t}{10} = 1$

$x + \frac{5t+3}{10} = 1$

$x = 1 - \frac{5t+3}{10}$

$x = \frac{10-5t-3}{10}$

$x = \frac{7-5t}{10}$.

$\therefore x = \frac{7-5t}{10}, \quad y = \frac{5t-1}{10}, \quad z = t$

(iii) $2x + 2y + 2 = 5, \quad x - y + z = 1,$

$3x + y + 2z = 4.$

The matrix form is $AX = B$.

$$A = \left[\begin{array}{ccc} 2 & 2 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 2 \end{array}\right], \quad X = \left[\begin{array}{c} x \\ y \\ z \end{array}\right], \quad B = \left[\begin{array}{c} 5 \\ 1 \\ 4 \end{array}\right]$$

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$$[A|B] = \left[\begin{array}{ccc|c} 2 & 2 & 1 & 5 \\ 1 & -1 & 1 & 1 \\ 3 & 1 & 2 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 2 & 1 & 5 \\ 3 & 1 & 2 & 4 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 4 & -1 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & -2 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$\Rightarrow \rho(A) = 2, \quad \rho[A|B] = 3$

$\rho(A) \neq \rho[A|B]$

$\therefore$ The system is inconsistent.

(IV) $2x - y + z = 2$
$6x - 3y + 3z = 6$
$4x - 2y + 2z = 4$

The matrix form is

$AX = B$

Here $A = \left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 6 & -3 & 3 & 3 \\ 4 & -2 & 2 & 2 \end{array}\right] \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$

$$[A|B] = \left[\begin{array}{ccc|c} 2 & -1 & 1 & 2 \\ 6 & -3 & 3 & 6 \\ 4 & -2 & 2 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$\Rightarrow \rho(A) = \rho[A|B] = 1 < 3$ (unknowns)

$\therefore$ The system is consistent and has infinitely many solns.

The system reduces to

$2x - y + z = 2$

put $z = t$ and $y = s$

$\therefore 2x - s + t = 2$

$2x = s - t + 2$

$x = \frac{s - t + 2}{2}$

$x = \frac{s-t+2}{2}, \quad y = s, \quad z = t.$

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2) Find the values of k for which the eqns $kx - 2y + z = 1$, $x - 2ky + z = -2$, $x - 2y + kz = 1$ have
(i) no soln (ii) unique soln
(iii) infinitely many soln.

The matrix form is $AX = B$.

$$A = \begin{bmatrix} k & -2 & 1 \\ 1 & -2k & 1 \\ 1 & -2 & k \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} k & -2 & 1 & 1 \\ 1 & -2k & 1 & -2 \\ 1 & -2 & k & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 1 & -2k & 1 & -2 \\ k & -2 & 1 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & -3 \\ 0 & 2k-2 & 1-k^2 & 1-k \end{array}\right] \quad \begin{matrix} R_2 \to \\ R_2 - R_3 \\ R_3 \to \\ R_3 - kR_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & -3 \\ 0 & 0 & (1-k)+1-k^2 & -2-k \end{array}\right] \quad R_3 \to$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & -3 \\ 0 & 0 & (1-k)+(1-k)(1+k) & -2-k \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & -3 \\ 0 & 0 & (1-k)(2+k) & -2-k \end{array}\right]$$

(i) <u>No soln</u> $\Leftrightarrow \rho(A) \neq \rho[A|B]$

$\Leftrightarrow 1-k = 0 \Rightarrow \boxed{k = 1}$

(ii) <u>Unique soln:</u> $\Leftrightarrow \rho(A) = \rho[A|B] = 3$

$\Leftrightarrow (1-k)(2+k) \neq 0$ and $-2-k \neq 0$

$k \neq 1$ and $k \neq -2$

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(iii) **infinitely many soln:-**

ie $\rho(A) = \rho(A|B) = 2$

ie $(1-k)(2+k) = 0$ and $-2-k=0$

$k = 1$ and $k = -2$

---

3) Investigate the values of λ and μ the system of linear eqns $2x+3y+5z=9$, $7x+3y-5z=8$, $2x+3y+\lambda z=\mu$, have

(i) no soln (ii) a unique soln

(iii) an infinite no of solns.

The matrix form is $AX = B$

where

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -5 \\ 2 & 3 & \lambda \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$$[A|B] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -5 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$$

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$$\sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -45 & -47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix} \begin{matrix} \\ R_2 \to 2R_2 - 7R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

(i) **For no solution:**

ie $\rho(A) \neq \rho(A|B)$

ie $\rho(A) = 2$ and $\rho(A|B) = 3$

$\Rightarrow \lambda - 5 = 0$ | $\mu - 9 \neq 0$

$\Rightarrow \boxed{\lambda = 5}$ | $\boxed{\mu \neq 9}$.

(ii) **For unique soln:-**

$\rho(A) = \rho[A|B] = 3$

ie $\lambda - 5 \neq 0$ and $\mu - 9 \neq 0$

$\boxed{\lambda \neq 5}$ and $\boxed{\mu \neq 9}$

(iii) **For infinite solns:**

$\rho(A) = \rho[A|B] = 2$

ie $\lambda - 5 = 0$ and $\mu - 9 = 0$

$\boxed{\lambda = 5 \text{ and } \mu = 9}$

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Homogeneous system of linear eqns.

Condition	Consistency	Solutions
$\Delta \neq 0$ $\rho(A) = \rho[A\|0] = n$ n – no of unknowns	Consistency	System has a unique soln $x=0, y=0, z=0$ trivial soln
$\rho(A) = \rho[A\|0] < n$	consistency	System has infinitely many soln in addition to the trivial solution.

Ex 1.35 Solve the following system

$x+2y+3z=0$, $3x+4y+4z=0$, $7x+10y+12z=0$

The matrix form is $AX=B$

$$A=\begin{bmatrix}1&2&3\\3&4&4\\7&10&12\end{bmatrix} \quad X=\begin{bmatrix}x\\y\\z\end{bmatrix} \quad B=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$[A|B]=\left[\begin{array}{ccc|c}1&2&3&0\\3&4&4&0\\7&10&12&0\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}1&2&3&0\\0&-2&-5&0\\0&-4&-9&0\end{array}\right] \quad \begin{matrix}R_2 \to R_2-3R_1\\ R_3 \to R_3-7R_1\end{matrix}$$

$$\sim \left[\begin{array}{ccc|c}1&2&3&0\\0&2&5&0\\0&4&9&0\end{array}\right] \quad \begin{matrix}R_2 \to (-)R_2\\ R_3 \to (-)R_3\end{matrix}$$

$$\sim \left[\begin{array}{ccc|c}1&2&3&0\\0&2&5&0\\0&0&-1&0\end{array}\right] \quad R_3 \to R_3-2R_2$$

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$$\sim \begin{bmatrix} 1 & 2 & 3 & | & 0 \\ 0 & 2 & 5 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \quad R_3 \to (-1)R_3$$

$\rho(A) = \rho(A|B) = 3 =$ No of unknowns

$\therefore$ The system has only trivial soln i.e $x=0, y=0, z=0$

Ex 1.36 Solve the system

$x+3y-2z=0,\ 2x-y+4z=0,$

$x-11y+14z=0.$

The matrix form is $AX=B$

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 2 & -1 & 4 & 0 \\ 1 & -11 & 14 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & -7 & 8 & 0 \\ 0 & -14 & 16 & 0 \end{array}\right] \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & -7 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - 2R_2$$

$\rho(A) = \rho[A|B] = 2 < 3 =$ no of unknowns

$\therefore$ The system has infinitely many solns and reduces to

$x+3y-2z=0$ — (1)

$-7y+8z=0$ — (2)

Take $z=t$

$\therefore$ (2) $\Rightarrow -7y+8t=0$

$7y=8t$

$y=\frac{8}{7}t$

$y=\frac{8}{7}t,\ z=t$ in (1)

$x+3\left(\frac{8t}{7}\right)-2(t)=0$

$x+\frac{24t-14t}{7}=0$

$x+\frac{10t}{7}=0$

$x=-\frac{10t}{7}$

$\therefore\ x=\frac{-10t}{7},\ y=\frac{8t}{7},\ z=t$

1.37 Solve the system

$x+y-2z=0,\ 2x-3y+z=0,\ 3x-7y+10z=0$

$6x-9y+10z=0$

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The matrix form is $AX = B$

Here $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -3 & 1 \\ 3 & -7 & 10 \\ 6 & -9 & 10 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & -3 & 1 & 0 \\ 3 & -7 & 10 & 0 \\ 6 & -9 & 10 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -10 & 16 & 0 \\ 0 & -15 & 22 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 5 & -8 & 0 \\ 0 & -15 & 22 & 0 \end{array}\right] \begin{array}{l} R_2 \to \frac{R_2}{-5} \\ R_3 \to \frac{R_3}{-2} \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \begin{array}{l} R_3 \to R_3 - 5R_2 \\ R_4 \to R_4 + 3R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_3 \to R_3 \div (-3) \\ R_4 \to R_4 \div (-2) \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_4 \to R_4 - R_3$$

So, $\rho(A) = \rho[A|B] = 3 =$ No of unknowns

Hence the system has trivial soln

i.e. $x = 0, y = 0, z = 0$

1.38 Determine the values of λ for which the system of eqns

$(3\lambda - 8)x + 3y + 3z = 0$,

$3x + (3\lambda - 8)y + 3z = 0$

$3x + 3y + (3\lambda - 8)z = 0$.

has a non-trivial soln.

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The matrix form is $AX = B$

$$A = \begin{bmatrix} 3\lambda-8 & 3 & 3 \\ 3 & 3\lambda-8 & 3 \\ 3 & 3 & 3\lambda-8 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 3\lambda-8 & 3 & 3 & 0 \\ 3 & 3\lambda-8 & 3 & 0 \\ 3 & 3 & 3\lambda-8 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 3\lambda-2 & 3\lambda-2 & 3\lambda-2 & 0 \\ 3 & 3\lambda-8 & 3 & 0 \\ 3 & 3 & 3\lambda-8 & 0 \end{array}\right] \begin{array}{l} R_1 \to \\ R_1+R_2+R_3 \end{array}$$

$$\sim (3\lambda-2)\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 3\lambda-8 & 3 & 0 \\ 3 & 3 & 3\lambda-8 & 0 \end{array}\right]$$

$$\sim (3\lambda-2)\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 3\lambda-11 & 0 & 0 \\ 0 & 0 & 3\lambda-11 & 0 \end{array}\right] \begin{array}{l} R_1 \to \\ R_2-3R_1 \\ R_3 \to \\ R_3-3R_2 \end{array}$$

For non trivial soln

$$(3\lambda-2)(3\lambda-11) = 0$$

i.e. $3\lambda-2=0$ or $3\lambda-11=0$

$3\lambda = 2$ | $3\lambda = 11$

$\boxed{\lambda = \frac{2}{3}}$ | $\boxed{\lambda = \frac{11}{3}}$

Ex 1.20 If the system of eqns $px+by+cz=0$, $ax+qy+cz=0$, $ax+by+rz=0$ has a non trivial solution and $p \neq a$, $q \neq b$, $r \neq c$, prove that $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$.

The matrix form is $AX = B$

$$A = \begin{bmatrix} p & b & c \\ a & q & c \\ a & b & r \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} p & b & c & 0 \\ a & q & c & 0 \\ a & b & r & 0 \end{array}\right]$$

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$$\sim \begin{bmatrix} p & b & c & 0 \\ a-p & q-b & 0 & 0 \\ a-p & 0 & r-c & 0 \end{bmatrix} \quad R_2 \to R_2 - R_1, \; R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} p & b & c & 0 \\ -(p-a) & q-b & 0 & 0 \\ -(p-a) & 0 & r-c & 0 \end{bmatrix}$$

$$\sim (p-a)(q-b)(r-c) \begin{bmatrix} \frac{p}{p-a} & \frac{b}{q-b} & \frac{c}{r-c} & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

For non trivial soln $|A| = 0$.

Since $p \neq a$, $q \neq b$, $r \neq c$

$$\therefore \begin{vmatrix} \frac{p}{p-a} & \frac{b}{q-b} & \frac{c}{r-c} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$$\frac{p}{p-a}(1-0) - \frac{b}{q-b}(-1) + \frac{c}{r-c}(0+1) = 0$$

$$\frac{p}{p-a} + \frac{b}{q-b} + \frac{c}{r-c} = 0.$$

$$\frac{p}{p-a} + \frac{q-(q-b)}{q-b} + \frac{r-(r-c)}{r-c} = 0$$

$$\frac{p}{p-a} + \frac{q}{q-b} - \frac{q-b}{q-b} + \frac{r}{r-c} - \frac{r-c}{r-c} = 0$$

$$\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} - 1 - 1 = 0$$

$$\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2.$$

EXERCISE 1.7

10 / 47 Solve the following system of homogeneous eqns.

(i) $3x + 2y + 7z = 0$, $4x - 3y - 2z = 0$
$5x + 9y + 23z = 0$

The matrix form is $AX = B$

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$$A = \begin{bmatrix} 3 & 2 & 7 \\ 4 & -3 & -2 \\ 5 & 9 & 23 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 4 & -3 & -2 & 0 \\ 5 & 9 & 23 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 0 & -17 & -34 & 0 \\ 0 & 17 & 34 & 0 \end{array}\right] \quad R_2 \to 3R_2 - 4R_1, \quad R_3 \to 3R_3 - 5R_1$$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \quad R_2 \to \frac{R_2}{-17}, \quad R_3 \to \frac{R_3}{17}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$\rho(A) = 2 < 3$

The system has infinitely many solns and reduces to

$$3x + 2y + 7z = 0 \quad -(1)$$
$$y + 2z = 0 \quad -(2)$$

put $z = t$, $y + 2t = 0$

$$\boxed{y = -2t}$$

$$3x + 2(-2t) + 7t = 0$$
$$3x - 4t + 7t = 0$$
$$3x + 3t = 0$$
$$\boxed{x = -t}$$

$\therefore x = -t, \; y = -2t, \; z = t$.

(ii) $2x + 3y - z = 0, \quad x - y - 2z = 0, \quad 3x + y + 3z = 0$

The matrix form is $AX = B$

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & -1 & -2 & 0 \\ 3 & 1 & 3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 0 & -5 & -3 & 0 \\ 0 & -7 & 9 & 0 \end{array}\right] \quad R_2 \to 2R_2 - R_1, \quad R_3 \to 2R_3 - 3R_1$$

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$$\sim \begin{bmatrix} 2 & 3 & -1 & 0 \\ 0 & -5 & -3 & 0 \\ 0 & 0 & 66 & 0 \end{bmatrix} \quad R_3 \to 5R_3 - 7R_2$$

$\Rightarrow \rho(A) = 3$ and the system has trivial soln

i.e. $x = 0, y = 0, z = 0$

---

2) Determine the values of λ for which system of eqns

$x + y + 3z = 0$, $4x + 3y + \lambda z = 0$,

$2x + y + 2z = 0$ has

(i) a unique soln (ii) a non-trivial soln.

The matrix form of eqn is

$AX = B$

where

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 4 & 3 & \lambda \\ 2 & 1 & 2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A|B] = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 4 & 3 & \lambda & 0 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

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$$\sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 3 & \lambda & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -1 & \lambda - 12 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & \lambda - 8 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

For trivial soln $\rho(A) = 3$

$\therefore \lambda - 8 \neq 0 \Rightarrow \lambda \neq 0$.

For non trivial soln $\rho(A) < 3$ or $|A| = 0$

i.e. $\lambda - 8 = 0 \Rightarrow \underline{\lambda = 8}$

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Chapter 2 – Complex Numbers

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CHAPTER - 2

Complex Numbers:

Power of imaginary unit i

$i^0 = 1$

$i^1 = i$

$i^2 = \boxed{-1}$

$i^3 = i^2 \cdot i$

$= -1 \cdot i$

$= \boxed{-i}$

$i^4 = i^2 \cdot i^2$

$= (-1)(-1)$

$= \boxed{1}$

$i^{-1} = \frac{1}{i} = \frac{1}{i} \times \frac{i}{i}$

$= \frac{i}{i^2} = \frac{i}{-1}$

$= \boxed{-i}$

$i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = \boxed{-1}$

$i^{-3} = \frac{1}{i^3} = \frac{1}{-i} \times \frac{i}{i}$

$= \frac{i}{-i^2}$

$= \frac{i}{-(-1)}$

$= \boxed{i}$

$i^{-4} = \frac{1}{i^4} = \frac{1}{1} = \boxed{1}$

Note:-

$i^n = i^{4q+k} = i^{4q} \cdot i^k$

$= (i^4)^q \cdot i^k$

$= 1^q \cdot i^k = \boxed{i^k}$

Ex 2.1 Simplify

(i) $i^7 = i^{4+3}$

$= i^4 \cdot i^3$

$= 1 \cdot i^2 \cdot i$

$= 1 \cdot (-1) \cdot i$

$= \boxed{-i}$

(ii) i^{1729}

$= i^{1728+1}$

$= i^{1728} \cdot i^1$

$= 1 \cdot i = \boxed{i}$

(iii) $i^{-1924} + i^{2018}$

$= \frac{1}{i^{1924}} + i^{2018}$

$= \frac{1}{(i^4)^{481}} + (i^2)^{1009}$

$= \frac{1}{1^{481}} + (-1)^{1009}$

$= 1 + (-1)$

$= 1 - 1$

$= 0$

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XII | CH2 | 2.1 | P2

EXERCISE 2.1

Simplify:

1) $i^{1947} + i^{1950}$

$= (i^4)^{486} \cdot i^3 + (i^4)^{487} \cdot i^2$

$= 1 \cdot i^3 + 1 \cdot i^2$

$= -i + (-1)$

$= \boxed{-i - 1}$

486	487
4) 1947	4) 1950
16	16
34	35
32	32
27	30
24	28
3	2

---

2) $i^{1948} - i^{-1869}$

$= (i^4)^{487} - \frac{1}{i^{1869}}$

$= 1 - \frac{1}{(i^4)^{467} \cdot i^1}$

$= 1 - \frac{1}{i} \times \frac{i}{i}$

$= 1 - \frac{i}{i^2} = 1 - \frac{i}{-1} = \boxed{1 + i}$

(IV) $\sum_{n=1}^{102} i^n = i + i^2 + i^3 + \cdots + i^{102}$

$= (i^1 + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) + (i^9 + i^{10} + i^{11} + i^{12}) + \cdots + (i^{97} + i^{98} + i^{99} + i^{100}) + i^{101} + i^{102}$

$= (i + (-1) + (-i) + 1) + (i + (-1) + (-i) + 1) + \cdots + i^{100} \cdot i + i^{100} \cdot i^2$

$= 0 + 0 + 0 + \cdots + (i^4)^{25} \cdot i + (i^4)^{25} \cdot i^2$

$= 1 \cdot i + 1 \cdot (-1) = \boxed{i - 1}$

(V) $i \cdot i^2 \cdot i^3 \cdot \cdots \cdot i^{40}$

$= i^{1+2+3+\cdots+40}$

$= i^{\frac{40 \times 41}{2}} = i^{20 \times 41} = (i^4)^{5 \times 41}$

$= (i^4)^{205}$

$= 1^{205} = \boxed{1}$

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3) $\sum_{n=1}^{12} i^n = i^1 + i^2 + \dots + i^{12}$

$= (i^1 + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) + (i^9 + i^{10} + i^{11} + i^{12})$

$= (i - 1 - i + 1) + (i - 1 - i + 1) + (i - 1 - i + 1)$

$= 0 + 0 + 0 = 0$

---

4) $i^{59} + \frac{1}{i^{59}}$

$= (i^4)^{14} \cdot i^3 + \frac{1}{(i^4)^{14} \cdot i^3}$

$= i^3 + \frac{1}{i^3}$

$= -i + \frac{1}{-i} \times \frac{i}{i}$

$= -i + \frac{i}{-i^2} = -i + \frac{i}{-(-1)}$

$= -i + i = \boxed{0}$

---

5) $i \cdot i^2 \cdot i^3 \dots i^{2000}$

$= i^{1+2+\dots+2000}$

$= i^{\frac{2000 \times 2001}{2}}$

$= i^{1000 \times 2001}$

$= (i^{1000})^{2001}$

$= \left[(i^4)^{250}\right]^{2001}$

$= 1$

---

6) $\sum_{n=1}^{10} i^{n+50} = \sum_{n=1}^{10} i^n \cdot i^{50}$

$= \sum_{n=1}^{10} i^n \cdot i^{48} \cdot i^2$ $\quad \because i^{48} = 1$, $i^2 = -1$

$= -1 \sum_{n=1}^{10} i^n$

$= -\left[i^1 + i^2 + i^3 + \dots + i^{10}\right]$

$= -\left[(i + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) + i^9 + i^{10}\right]$

$= -\left[0 + 0 + i + i^2\right] = -[i - 1]$

$= 1 - i$ //

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Complex Numbers :-

Rectangular form :- A complex no is of the form $x+iy$, where x and y are real nos. x is called real part and y is called Imaginary part of the complex no.

Note 1 Two complex nos $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal iff $Re(z_1) = Re(z_2)$ and $Img(z_1) = Img(z_2)$. i.e. $x_1 = x_2$ and $y_1 = y_2$.

Argand plane :

In a complex $z = x+iy$ with a point (x,y), in a coordinate plane, x axis as real axis and y-axis as imaginary axis to represent a complex no, Then the xy-plane is called complex plane or Argand plane.

2) **Addition of a complex no**

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$.

3) **Subtraction of complex no**

$z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$\therefore z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$$
$$= (x_1 - x_2) + i(y_1 - y_2)$$

4) **Multiplication of a complex no**

$z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$\therefore z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= x_1 x_2 + i x_1 y_2 + i y_1 x_2 + i^2 y_1 y_2$$
$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

5) Multiplication of a complex number z by i

$$z = x + iy$$
$$iz = i(x+iy) = ix + i^2 y$$
$$= xi - y$$

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Ex 2.2 Find the value of the real numbers x and y, if the complex number $(2+i)x + (1-i)y + 2i - 3$ and $x + (-1+2i)y + 1 + i$ are equal.

Let $z_1 = (2+i)x + (1-i)y + 2i - 3$

$z_2 = x + (-1+2i)y + 1 + i$

$z_1 = 2x + ix + y - iy + 2i - 3$

$= (2x + y - 3) + i(x - y + 2)$

$z_2 = x - y + 2iy + 1 + i$

$= (x - y + 1) + i(2y + 1)$

Given $z_1 = z_2$

$\Rightarrow 2x + y - 3 = x - y + 1$

$\Rightarrow 2x - x + y + y = 4$

$x + 2y = 4$ ——— (1) and

$x - y + 2 = 2y + 1$

$x - y - 2y = 1 - 2$

$x - 3y = -1$ ——— (2)

(1) − (2) $\Rightarrow 5y = 5$

$\boxed{y = 1}$

Sub $y = 1$ in (1)

$x + 2 = 4$

$\boxed{x = 2}$

$\therefore \boxed{x = 2}, \boxed{y = 1}$

EXERCISE 2.2

1) Evaluate if $z = 5 - 2i$, $w = -1 + 3i$

(i) $z + w$

$z + w = 5 - 2i + -1 + 3i$

$= 4 + i$

(ii) $z - iw$

$z - iw = (5 - 2i) - i(-1 + 3i)$

$= 5 - 2i + i - 3i^2$

$= 5 - i - 3(-1)$

$= 5 - i + 3$

$= 8 - i$

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(iii) $2z + 3w$

$= 2(5-2i) + 3(-1+3i)$

$= 10 - 4i - 3 + 9i$

$= 7 + 5i$

(iv) zw

$zw = (5-2i)(-1+3i)$

$= -5 + 15i + 2i - 6i^2$

$= -5 + 17i - 6(-1)$

$= -5 + 17i + 6$

$= \boxed{+17i}$

(v) $z^2 + 2zw + w^2$

$= (5-2i)^2 + 2(5-2i)(-1+3i) + (-1+3i)^2$

$= 25 - 20i + (2i)^2 + (10 - 4i)(-1+3i) + (-1)^2 + 2(-1)(3i) + (3i)^2$

$= 25 - 20i + 4i^2 + (-10) + 30i + 4i - 12i^2 + 1 - 6i + 9i^2$

$= 25 - 20i - 4 - 10 + 30i + 4i + 12 + 1 - 6i - 9$

$= (25 - 4 - 10 + 12 + 1 - 9) + i(-20 + 30 + 4 - 6)$

$= 15 + 8i$

(vi) $(z+w)^2$

$= (5 - 2i + -1 + 3i)^2$

$= (4+i)^2$

$= 16 + i^2 + 8i$

$= 16 - 1 + 8i$

$= 15 + 8i$

---

2) Given the complex nos $z = 2+3i$ represent the complex nos in Argand diagram

(i) z, iz and $z+iz$

$z = 2+3i$, $(2, 3)$

$iz = i(2+3i) = 2i + 3i^2$

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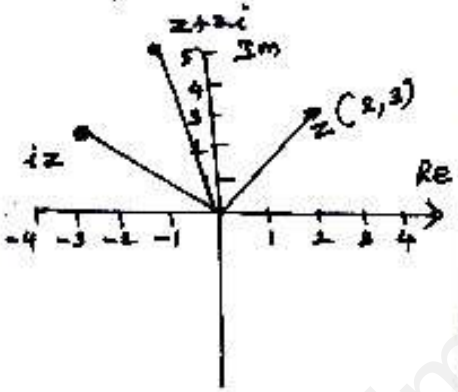
$= 2i - 3$

$-3 + 2i$, $(-3, 2)$

$z + iz = 2 + 3i - 3 + 2i$

$= -1 + 5i$, $(-1, 5)$



(ii) $z, -iz, z - iz$

$z = 2 + 3i$, $(2, 3)$

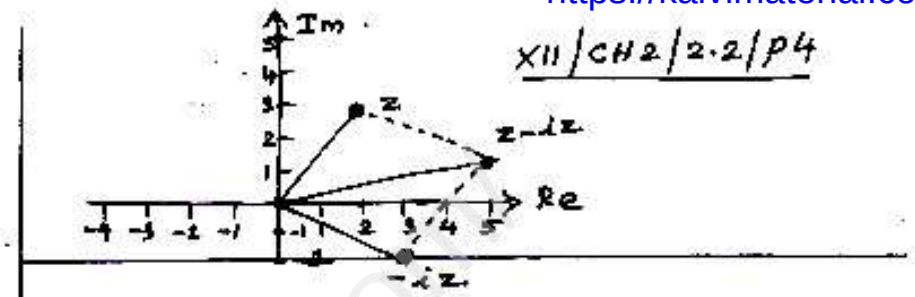
$-iz = -i(2 + 3i) = -2i - 3i^2$

$= -2i + 3 = 3 - 2i$

$= (3, -2)$

$z - iz = 2 + 3i - 2i + 3$

$= 5 + i$, $(5, 1)$



9) Find x and y if the complex nos $(3-i)x - (2-i)y + 2i + 5$ and $2x + (-1+2i)y + 3 + 2i$ are equal.

$3x - ix - 2y + iy + 2i + 5 = 2x - y + 2iy + 3 + 2i$

$(3x - 2y + 5) + i(y - x + 2) = (2x - y + 3) + i(2y + 2)$

$\Rightarrow 3x - 2y + 5 = 2x - y + 3$ | $y - x + 2 = 2y + 2$

$x - y = -2$ — (1) | $2y - y + x = 0$

$x + y = 0$ — (2)

(1) + (2) $\Rightarrow 2x = -2$

$\boxed{x = -1}$

$x = -1$ in (2)

$-1 + y = 0$

$\boxed{y = 1}$

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Basic Algebraic properties of complex Numbers.

The complex nos satisfies the following properties under **ADDITION**

For any two complex nos z_1 and z_2

(1) $z_1 + z_2$ is a complex no (closure property)

2) $z_1 + z_2 = z_2 + z_1$ commutative property

3) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ (Associative property)

4) $z + 0 = 0 + z = z$, 0 is additive identity

5) $z + (-z) = -z + z = 0$, $-z$ is called additive inverse of z.

The complex numbers satisfy the properties under **multiplication**

For any two complex nos z_1 and z_2

(1) $z_1 z_2$ is also a complex no (closure property)

(2) $z_1 z_2 = z_2 z_1$ (commutative property)

3) $z_1(z_2 z_3) = (z_1 z_2) z_3$ (associative property)

4) $z \cdot 1 = 1 z = 1$, 1 is multiplicative identity.

5) $zw = wz = 1$, then w is multiplicative inverse of z and is denoted by z^{-1}

For any three complex no z_1, z_2 and z_3

$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ and

$(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

(Distributive property).

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Lawsoft indices:-

(i) $z^m \cdot z^n = z^{m+n}$ (iii) $(z^m)^n = z^{mn}$

(ii) $\frac{z^m}{z^n} = z^{m-n}$ (iv) $(z_1 z_2)^m = z_1^m \cdot z_2^m$

EXERCISE 2.3

1. If $z_1 = 1-3i$, $z_2 = -4i$, $z_3 = 5$ show that

(i) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

$z_1 + z_2 = 1 - 3i - 4i = 1 - 7i$

$(z_1 + z_2) + z_3 = (1 - 7i) + 5$

$= 1 - 7i + 5$

$= \boxed{6 - 7i}$ —— (1)

$z_2 + z_3 = -4i + 5$

$z_1 + (z_2 + z_3) = 1 - 3i - 4i + 5$

$= \boxed{6 - 7i}$ —— (2)

from (1) and (2)
$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ verified

(ii) $(z_1 z_2) z_3 = [(1-3i)(-4i)]5$

$= [-4i + 12i^2]5$

$= -20i + 60(-1)$

$= \boxed{-60 - 20i}$ —— (1)

$z_1 (z_2 z_3) = (1-3i)[-4i(5)]$

$= (1-3i)[-20i]$

$= -20i + 60i^2$

$= -20i - 60$

$= \boxed{-60 - 20i}$ —— (2)

from (1) and (2) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

---

2) If $z_1 = 3$, $z_2 = -7i$, $z_3 = 5 + 4i$ ST

(i) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

$z_1 (z_2 + z_3) = 3[-7i + 5 + 4i]$

$= 3[5 - 3i]$

$= \boxed{15 - 9i}$ —— (1)

$z_1 z_2 + z_1 z_3 = 3(-7i) + 3(5 + 4i)$

$= -21i + 15 + 12i$

$= \boxed{15 - 9i}$ —— (2)

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From ① and ②

$z_1(z_2+z_3) = z_1z_2 + z_1z_3$ verified.

(ii) $(z_1+z_2)z_3 = z_1z_3 + z_2z_3$

$(z_1+z_2)z_3 = (3-7i)(5+4i)$

$= 15+12i-35i-28i^2$

$= 15-23i-28(-1)$

$= \boxed{43-23i}$ ——— ①

$z_1z_3 + z_2z_3 = 3(5+4i) + -7i(5+4i)$

$= 15+12i-35i-28i^2$

$= 15-23i+28$

$= \boxed{43-23i}$ ——— ②

From ① and ②

$(z_1+z_2)z_3 = z_1z_3 + z_2z_3$

---

3) If $z_1 = 2+5i$, $z_2 = -3-4i$, $z_3 = 1+i$ find the additive and multiplicative inverse of z_1, z_2, z_3.

Additive inverse of $z_1 = -z_1$

$= -(2+5i)$

$= \boxed{-2-5i}$

Multiplicative inverse of $z_1 = \frac{1}{z_1}$

$\frac{1}{z_1} = \frac{1}{2+5i} = \frac{2-5i}{(2+5i)(2-5i)}$

$= \frac{2-5i}{4+25} = \boxed{\frac{2}{29} - \frac{5}{29}i}$

additive inverse of $z_2 = -z_2$

$= -(-3-4i)$

$= \boxed{3+4i}$

Multiplicative inverse of $z_2 = \frac{1}{z_2}$

$= \boxed{-\frac{3}{25} + \frac{4}{25}i}$

additive inverse of $z_3 = -z_3$

$= -(1+i)$

$= \boxed{-1-i}$

Multiplicative inverse of $z_3 = \frac{1}{z_3}$

$\boxed{\frac{1}{2} - \frac{1}{2}i}$

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Conjugate of a Complex no:

The conjugate of a complex no $x+iy$ is $x-iy$ and denoted by $\bar{z}$

Note: $(x+iy)(x-iy) = x^2+y^2$

Properties:-

1) $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

2) $\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2$

3) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

4) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$.

5) $Re(z) = \frac{z+\bar{z}}{2}$ ⊛

6) $Img(z) = \frac{z-\bar{z}}{2i}$ ⊛

7) $\overline{z^n} = (\bar{z})^n$

8) z is real iff $z = \bar{z}$. ⊛

9) z is purely imaginary iff $z = -\bar{z}$

10) $\bar{\bar{z}} = z$.

Ex 2.9 / 62 Write $\frac{3+4i}{5-12i}$ in the $x+iy$ form, hence find its real and Imaginary parts.

$$\frac{3+4i}{5-12i} = \frac{3+4i}{5-12i} \times \frac{5+12i}{5+12i}$$

$$= \frac{(3+4i)(5+12i)}{5^2+12^2} = \frac{15+36i+20i+48i^2}{25+144}$$

$$= \frac{15+56i-48}{169} = \frac{-33+56i}{169}$$

$$= -\frac{33}{169} + \frac{56}{169}i$$

This is in the form of $x+iy$

Real part $= -\frac{33}{169}$

Imaginary part $= \frac{56}{169}$ //

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2.4 Simplify $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3$ into rectangular form.

$$\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)^2}{1^2+1^2}$$

$$= \frac{1+i^2+2i}{2} = \frac{2i}{2} = \boxed{i}$$

and $\frac{1-i}{1+i} = \left(\frac{1+i}{1-i}\right)^{-1} = (i)^{-1}$

$$= \frac{1}{i} = \frac{1}{i} \times \frac{i}{i^2}$$

$$= \frac{i}{-1} = \boxed{-i}$$

Now $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3$

$= i^3 - (-i)^3$

$= -i - (i)$

$= -i - i = -2i$

$(-i)^3$
$= -i \times -i \times -i$
$= i^2(-i)$
$= (-1)(-i)$
$= i$

Rectangular form is $\boxed{0 - 2i}$

Ex 2.5 If $\frac{z+3}{z-5i} = \frac{1+4i}{2}$ find the complex number z in the rectangular form.

$$\frac{z+3}{z-5i} = \frac{1+4i}{2}$$

$$\Rightarrow 2(z+3) = (1+4i)(z-5i)$$

$$2z+6 = z - 5i + 4zi - 20i^2$$

$$2z - z - 4zi = -5i + 20 - 6$$

$$z - 4zi = 14 - 5i$$

$$z(1-4i) = 14 - 5i$$

$$z = \frac{14-5i}{1-4i}$$

$$= \frac{14-5i}{1-4i} \times \frac{1+4i}{1+4i}$$

$$= \frac{14+56i-5i-20i^2}{1+16}$$

$$= \frac{14+51i+20}{17}$$

$$= \frac{34+51i}{17} = \frac{34}{17} + \frac{51}{17}i$$

$$= \boxed{2+3i}$$

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Ex 2.6 If $z_1 = 3-2i$ and $z_2 = 6+4i$ find $\frac{z_1}{z_2}$ in the rectangular form.

$$\frac{z_1}{z_2} = \frac{3-2i}{6+4i} = \frac{3-2i}{6+4i} \times \frac{6-4i}{6-4i}$$

$$= \frac{18-12i-12i+8i^2}{6^2+4^2}$$

$$= \frac{18-24i-8}{36+16} = \frac{10-24i}{52}$$

$$= \frac{10}{52} - \frac{24}{52}i$$

$$= \frac{5}{26} - \frac{6}{13}i$$

Ex 2.7 Find z^{-1}, if $z = (2+3i)(1-i)$

$$z = (2+3i)(1-i)$$

$$= 2-2i+3i-3i^2$$

$$= 2+i-3(-1)$$

$$= 2+i+3$$

$$= 5+i$$

$$z^{-1} = \frac{1}{z} = \frac{1}{5+i}$$

$$= \frac{1}{5+i} \times \frac{5-i}{5-i} = \frac{5-i}{25+1}$$

$$= \frac{5}{26} - \frac{1}{26}i$$

Ex 2.8

Show that (i) $(2+i\sqrt{3})^{10} + (2-i\sqrt{3})^{10}$ is real.

Let $z = (2+i\sqrt{3})^{10} + (2-i\sqrt{3})^{10}$

To prove z is real

is to prove $z = \bar{z}$.

$$\bar{z} = \overline{(2+i\sqrt{3})^{10} + (2-i\sqrt{3})^{10}}$$

$$= \overline{(2+i\sqrt{3})^{10}} + \overline{(2-i\sqrt{3})^{10}}$$

$$= \overline{(2+i\sqrt{3})}^{10} + \overline{(2-i\sqrt{3})}^{10} \quad (\because \overline{z^n} = \bar{z}^n)$$

$$= (2-i\sqrt{3})^{10} + (2+i\sqrt{3})^{10}$$

$$= z.$$

$\bar{z} = z \Rightarrow z$ is real.

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(ii) $\left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$ is purely imaginary.

Let $z = \left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$

To prove z is purely imaginary i.e to prove $\bar{z} = -z$.

Now $\frac{19+9i}{5-3i} \times \frac{5+3i}{5+3i}$

$= \frac{95+57i+45i+27i^2}{25+9}$

$= \frac{95+102i-27}{34} = \frac{68+102i}{34}$

$= \boxed{2+3i}$

$\frac{8+i}{1+2i} = \frac{8+i}{1+2i} \times \frac{1-2i}{1-2i}$

$= \frac{8-16i+i-2i^2}{1+4}$

$= \frac{8-15i+2}{5} = \frac{10-15i}{5}$

$= \boxed{2-3i}$

$\therefore z = \left(2+3i\right)^{15} - \left(2-3i\right)^{15}$

$\bar{z} = \overline{\left(2+3i\right)^{15} - \left(2-3i\right)^{15}}$

$= \overline{\left(2+3i\right)^{15}} - \overline{\left(2-3i\right)^{15}}$

$= \overline{2+3i}^{15} - \overline{(2-3i)}^{15}$

$= (2-3i)^{15} - (2+3i)^{15}$

$= -\left[(2+3i)^{15} - (2-3i)^{15}\right]$

$= -z$

$\therefore \bar{z} = -z \Rightarrow z$ is purely imaginary.

EXERCISE 2.4

1) write the following in rectangular form (or) Cartesian form.

(i) $\frac{(5+9i)+(2-4i)}{(5+9i)+(2-4i)}$

Let $z = \frac{(5+9i)+(2-4i)}{(5+9i)+(2-4i)}$

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$z = \overline{5+9i} + \overline{2-4i}$ | $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$

$= 5-9i+2+4i$

$= \boxed{7-5i}$

(ii) $\dfrac{10-5i}{6+2i}$

Let $z = \dfrac{10-5i}{6+2i}$

$= \dfrac{10-5i}{6+2i} \times \dfrac{6-2i}{6-2i}$

$= \dfrac{60-20i-30i+10i^2}{36+4}$

$= \dfrac{60-50i-10}{40} = \dfrac{50-50i}{40}$

$= \dfrac{50}{40}(1-i)$

$= \boxed{\dfrac{5}{4}-\dfrac{5}{4}i}$

(iii) $\overline{3i} + \dfrac{1}{2-i}$

Let $z = \overline{3i} + \dfrac{1}{2-i}$

$= -3i + \dfrac{1}{2-i} \times \dfrac{2+i}{2+i}$

$= -3i + \dfrac{2+i}{5}$

$= \dfrac{-15i+2+i}{5} = \dfrac{2-14i}{5}$

$= \boxed{\dfrac{2}{5}-\dfrac{14}{5}i}$

---

2) If $z = x+iy$ find in cartesian form

(i) $Re\left(\frac{1}{z}\right)$

$z = x+iy$

$\dfrac{1}{z} = \dfrac{1}{x+iy} \times \dfrac{x-iy}{x-iy} = \dfrac{x-iy}{x^2+y^2}$

$Re\left(\frac{1}{z}\right) = \dfrac{x}{x^2+y^2}$

(ii) $Re(i\bar{z})$, $z = x+iy$, $\bar{z} = x-iy$

$i\bar{z} = i(x-iy)$

$= xi - yi^2$

$= xi + y$

$= y + xi$

$Re(i\bar{z}) = y$

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(iii) $\text{Im}(3z + 4\bar{z} - 4i)$

$3z + 4\bar{z} - 4i$

$= 3(x+iy) + 4(x-iy) - 4i$

$= 3x + 3yi + 4x - 4yi - 4i$

$= 7x - yi - 4i$

$= [7x + i(-y-4)]$

$\text{Im}(3z + 4\bar{z} - 4i) = -y-4$

---

3) If $z_1 = 2-i$ and $z_2 = -4+3i$ find the inverse of $z_1 z_2$ and $\frac{z_1}{z_2}$.

$z_1 z_2 = (2-i)(-4+3i)$

$= -8 + 6i + 4i - 3i^2$

$= -8 + 10i + 3$

$= -5 + 10i$

Inverse of $z_1 z_2 = (z_1 z_2)^{-1}$

$= \frac{1}{z_1 z_2}$

$= \frac{1}{-5+10i}$

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$= \frac{1}{-5+10i} \times \frac{-5-10i}{-5-10i}$

$= \frac{-5-10i}{25+100} = \frac{-5}{125} - \frac{10}{125}i$

$= -\frac{1}{25} - \frac{2}{25}i$

$= \boxed{\frac{1}{25}(-1-2i)}$

(ii) Inverse of $\frac{z_1}{z_2} = \left(\frac{z_1}{z_2}\right)^{-1}$

$= \frac{1}{\frac{z_1}{z_2}} = \frac{z_2}{z_1} = \frac{-4+3i}{2-i}$

$= \frac{-4+3i}{2-i} \times \frac{2+i}{2+i}$

$= \frac{-8-4i+6i+3i^2}{4+1}$

$= \frac{-8+2i-3}{5}$

$= \frac{-11+2i}{5}$

$= \frac{1}{5}(-11+2i)$

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4) The complex numbers u, v and w are related by $\frac{1}{u} = \frac{1}{v} + \frac{1}{w}$. If $v = 3-4i$ and $w = 4+3i$ find u in Cartesian form.

$$\frac{1}{u} = \frac{1}{v} + \frac{1}{w}$$

$$= \frac{1}{3-4i} + \frac{1}{4+3i}$$

$$= \frac{4+3i+3-4i}{(3-4i)(4+3i)}$$

$$= \frac{7-i}{12+9i-16i-12i^2}$$

$$\frac{1}{u} = \frac{7-i}{24-7i}$$

$$\therefore u = \frac{24-7i}{7-i} = \frac{24-7i}{7-i} \times \frac{7+i}{7+i}$$

$$= \frac{168+24i-49i-7i^2}{49+1}$$

$$= \frac{175-25i}{50} = \frac{175}{50} - \frac{25}{50}i$$

$$= \frac{7}{2} - \frac{1}{2}i$$ //

5) Prove that

(i) z is real iff $z = \bar{z}$

Let $z = x+iy$ then $\bar{z} = x-iy$

$$z = \bar{z} \Leftrightarrow x+iy = x-iy$$

$$\Leftrightarrow x+iy-x+iy = 0$$

$$\Leftrightarrow 2iy = 0$$

$$\Leftrightarrow y = 0$$

$$\Leftrightarrow z \text{ is real.}$$

(ii) $Re(z) = \frac{z+\bar{z}}{2}$

$z = x+iy$, $\bar{z} = x-iy$

$$z+\bar{z} = x+iy+x-iy$$

$$= 2x$$

$$\frac{z+\bar{z}}{2} = x$$

$$= Re(z).$$

$$\therefore Re(z) = \frac{z+\bar{z}}{2}$$

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(iii) $Im(z) = \frac{z-\bar{z}}{2i}$

Let $z = x+iy$, $\bar{z} = x-iy$

$z-\bar{z} = x+iy-(x-iy)$

$= x+iy-x+iy$

$= 2iy$

$\frac{z-\bar{z}}{2i} = y = Im(z)$

$\therefore Im(z) = \frac{z-\bar{z}}{2i}$

---

6) Find the least value of the positive integer n for which $(\sqrt{3}+i)^n$ (i) real (ii) purely Imaginary.

$(\sqrt{3}+i)^2 = 3+2\sqrt{3}i+i^2$

$= 2+2\sqrt{3}i$

$(\sqrt{3}+i)^3 = (\sqrt{3}+i)^2 \cdot (\sqrt{3}+i)$

$= (2+2\sqrt{3}i)(\sqrt{3}+i)$

$= 2\sqrt{3}+2i+6i+2\sqrt{3}i^2$

$= 2\sqrt{3}+8i-2\sqrt{3} = 8i$

when $n=3$, $(\sqrt{3}+i)^n$ is purely Imaginary

$(\sqrt{3}+i)^4 = (\sqrt{3}+i)^3(\sqrt{3}+i)$

$= 8i(\sqrt{3}+i)$

$= 8\sqrt{3}i+8i^2$

$= -8+8\sqrt{3}i$

$(\sqrt{3}+i)^5 = (\sqrt{3}+i)^4(\sqrt{3}+i)$

$= (-8+8\sqrt{3}i)(\sqrt{3}+i)$

$= -8\sqrt{3}-8i+24i-8\sqrt{3}$

$= -16\sqrt{3}+16i$

$(\sqrt{3}+i)^6 = (\sqrt{3}+i)^5(\sqrt{3}+i)$

$= (-16\sqrt{3}+16i)(\sqrt{3}+i)$

$= -48-16\sqrt{3}i+16\sqrt{3}i+16i^2$

$= -48-16 = -64$

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when $n=6$, $(\sqrt{3}+i)^n$ is real.

---

7) Show that $(2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}$ is purely imaginary.

Let $z = (2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}$

To prove z is purely imaginary i.e to prove $\bar{z} = -z$.

$$\bar{z} = \overline{(2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}}$$

$$= \overline{(2+i\sqrt{3})^{10}} - \overline{(2-i\sqrt{3})^{10}}$$

$$= \overline{(2+i\sqrt{3})}^{10} - \overline{(2-i\sqrt{3})}^{10}$$

$$= (2-i\sqrt{3})^{10} - (2+i\sqrt{3})^{10}$$

$$= -\left[(2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}\right]$$

$$= -z$$

$\bar{z} = -z \Rightarrow z$ is purely Imaginary

(ii) $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real.

Let $z = \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$.

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To prove z is real.

i.e to prove $\bar{z} = z$.

$$\frac{19-7i}{9+i} = \frac{19-7i}{9+i} \times \frac{9-i}{9-i} = \frac{171-19i-63i+7i^2}{81+1}$$

$$= \frac{164-82i}{82} = \boxed{2-i}$$

$$\frac{20-5i}{7-6i} = \frac{20-5i}{7-6i} \times \frac{7+6i}{7+6i} = \frac{140+120i-35i-30i^2}{49+36}$$

$$= \frac{170+85i}{85} = \boxed{2+i}$$

$$\therefore z = (2-i)^{12} + (2+i)^{12}$$

$$\bar{z} = \overline{(2-i)^{12} + (2+i)^{12}}$$

$$= \overline{(2-i)^{12}} + \overline{(2+i)^{12}}$$

$$= \overline{(2-i)}^{12} + \overline{(2+i)}^{12}$$

$$= (2+i)^{12} + (2-i)^{12}$$

$$= z$$

$\bar{z} = z \Rightarrow z$ is real.

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Modulus of a Complex Number

If $z = x+iy$, then modulus of z, denoted by $|z|$ is defined as $|z| = \sqrt{x^2+y^2}$.

properties of modulus of a complex no

1) $|z| = |\bar{z}|$
2) $|z_1+z_2| \le |z_1|+|z_2|$ Triangle inequality
3) $|z_1 z_2| = |z_1||z_2|$
4) $|z_1-z_2| \ge \big||z_1|-|z_2|\big|$
5) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
6) $|z^n| = |z|^n$
7) $Re(z) \le |z|$
8) $Im(z) \le |z|$
9) $\big||z_1|-|z_2|\big| \le |z_1+z_2| \le |z_1|+|z_2|$

Note: i) The distance between the two points z_1 and z_2 in complex plane is $|z_1-z_2|$.

2) $|z|^2 = z\bar{z}$ (*)

For any two complex nos z_1 and z_2 prove the triangular inequality $|z_1+z_2| \le |z_1|+|z_2|$

WKT $|z|^2 = z\bar{z}$

$$\therefore |z_1+z_2|^2 = (z_1+z_2)(\bar{z_1}+\bar{z_2})$$
$$= (z_1+z_2)(\bar{z_1}+\bar{z_2})$$
$$= z_1\bar{z_1} + z_1\bar{z_2} + z_2\bar{z_1} + z_2\bar{z_2}$$
$$= z_1\bar{z_1} + z_1\bar{z_2} + \overline{z_1\bar{z}} + z_2\bar{z_2}$$
$$= |z_1|^2 + 2Re(z_1\bar{z_2}) + |z_2|^2$$
$$\le |z_1|^2 + 2|z_1\bar{z_2}| + |z_2|^2$$
$$= |z_1|^2 + 2|z_1||\bar{z_2}| + |z_2|^2$$

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$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2$

$= (|z_1| + |z_2|)^2$

$\Rightarrow |z_1+z_2|^2 \leq (|z_1|+|z_2|)^2$

$\Rightarrow |z_1+z_2| \leq |z_1| + |z_2|$.

EX 2.9 If $z_1 = 3+4i$, $z_2 = 5-12i$ and $z_3 = 6+8i$ find $|z_1|$, $|z_2|$, $|z_1+z_2|$, $|z_2-z_3|$ and $|z_1+z_3|$

$z_1 = 3+4i$, $z_2 = 5-12i$, $z_3 = 6+8i$

$|z_1| = |3+4i| = \sqrt{3^2+4^2}$

$= \sqrt{9+16}$

$= \sqrt{25}$

$= \boxed{5}$

$|z_2| = |5-12i| = \sqrt{5^2+(-12)^2}$

$= \sqrt{25+144}$

$= \sqrt{169} = \boxed{13}$

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$|z_3| = |6+8i| = \sqrt{6^2+8^2}$

$= \sqrt{36+64}$

$= \sqrt{100} = \boxed{10}$

$|z_1+z_2| = |8+4i+5-12i|$

$= |8-8i| = \sqrt{64+64}$

$= \sqrt{128}$

$= \sqrt{64 \times 2}$

$= \boxed{8\sqrt{2}}$

$|z_2-z_3| = |5-12i-(6+8i)|$

$= |5-12i-6-8i|$

$= |-1-20i|$

$= \sqrt{1+400} = \boxed{\sqrt{401}}$

$|z_1+z_3| = |3+4i+6+8i|$

$= |9+12i| = \sqrt{81+144}$

$= \sqrt{225} = \boxed{15}$

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Ex 2.10 Find (i) $\left|\frac{2+i}{-1+2i}\right|$

$$\left|\frac{2+i}{-1+2i}\right| = \frac{|2+i|}{|-1+2i|} = \frac{\sqrt{4+1}}{\sqrt{1+4}}$$

$$= \frac{\sqrt{5}}{\sqrt{5}} = 1$$

(ii) $\left|\overline{(1+i)}(2+3i)(4i-3)\right|$

$$= |1+i|\,|2+3i|\,|4i-3| \quad \because |\bar{z}| = |z|$$

$$= \sqrt{1+1}\ \sqrt{4+9}\ \sqrt{16+9}$$

$$= \sqrt{2}\ \sqrt{13}\ \sqrt{25}$$

$$= 5\sqrt{26}$$

(iii) $\left|\frac{i(2+i)^3}{(1+i)^2}\right|$

$$= \frac{|i|\,|(2+i)^3|}{|(1+i)^2|}$$

$$= \frac{1\left(\sqrt{4+1}\right)^3}{\left(\sqrt{1+1}\right)^2} = \frac{1(\sqrt{5})^3}{(\sqrt{2})^2} = \frac{5\sqrt{5}}{2}$$

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Ex 2.11 Which one of the points $i, -2+i, 3$ is farthest from the origin?

Distance betn origin to $z = i, -2+i, 3$

$|z| = |i| = \sqrt{1} = 1$

$|-2+i| = \sqrt{4+1} = \sqrt{5} = 2.236$.

$|3| = 3$

From this $1 < 2.236 < 3$

$\therefore$ 3 is farthest point from the origin.

Ex 2.12 If z_1, z_2 and z_3 are complex numbers s.t $|z_1| = |z_2| = |z_3| = |z_1 + z_2 + z_3| = 1$ find the value of $\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right|$

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$|z_1| = |z_2| = |z_3| = 1$

$|z_1| = 1 \Rightarrow |z_1|^2 = 1$

$z_1 \bar{z}_1 = 1$

$\bar{z}_1 = \frac{1}{z_1}$

$\therefore \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = \left|\bar{z}_1 + \bar{z}_2 + \bar{z}_3\right|$

$= \left|\overline{z_1 + z_2 + z_3}\right|$

$= |z_1 + z_2 + z_3|$

$= \boxed{1}$

EX 2.13 If $|z| = 2$ show that

$3 \le |z + 3 + 4i| \le 7$

$|z + 3 + 4i| \le |z| + |3 + 4i|$

$= 2 + \sqrt{9+16}$

$= 2 + \sqrt{25}$

$= 2 + 5$

$= 7$ ——— (1)

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$|z + 3 + 4i| \ge \big||z| - |3 + 4i|\big|$

$= \big|2 - \sqrt{9+16}\big|$

$= \big|2 - \sqrt{25}\big|$

$= |2 - 5| = |-3| = 3$ ——— (2)

$\therefore$ From (1) and (2)

$3 \le |z + 3 + 4i| \le 7$

——— x ——— x ——— x ———

EX 2.14 Show that the points $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$ are the vertices of an equilateral triangle.

Let $z_1 = 1$, $z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$z_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Length of sides of the triangles are

$|z_1 - z_2| = \left|1 - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right|$

$= \left|\frac{3}{2} - i\frac{\sqrt{3}}{2}\right|$

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$$= \sqrt{\left(\frac{3}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \boxed{\sqrt{3}} \quad -(1)$$

$$|z_2 - z_3| = \left|\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right|$$

$$= \left| -\frac{1}{2} + i\frac{\sqrt{3}}{2} + \frac{1}{2} + i\frac{\sqrt{3}}{2} \right|$$

$$= \left| 2i\frac{\sqrt{3}}{2} \right| = \left| i\sqrt{3} \right| = \sqrt{1(\sqrt{3})^2}$$

$$= \boxed{\sqrt{3}} \quad -(2)$$

$$|z_3 - z_1| = \left| -\frac{1}{2} - i\frac{\sqrt{3}}{2} - (1) \right|$$

$$= \left| -\frac{3}{2} - i\frac{\sqrt{3}}{2} \right|$$

$$= \sqrt{\left(-\frac{3}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \boxed{\sqrt{3}}$$

$$-(3)$$

From (1), (2), (3)

Since the sides are equal, the given points form an equilateral triangle

---

Ex 2.15 Let z_1, z_2 and z_3 be complex numbers such that $|z_1| = |z_2| = |z_3| = r$ and $z_1 + z_2 + z_3 \neq 0$ prove that

$$\left| \frac{z_1 z_2 + z_2 z_3 + z_3 z_1}{z_1 + z_2 + z_3} \right| = r.$$

Given that $|z_1| = |z_2| = |z_3| = r$

$$\Rightarrow z_1\bar{z}_1 = z_2\bar{z}_2 = z_3\bar{z}_3 = r^2$$

$$\therefore z_1 = \frac{r^2}{\bar{z}_1}, \quad z_2 = \frac{r^2}{\bar{z}_2}, \quad z_3 = \frac{r^2}{\bar{z}_3}.$$

$$\therefore z_1 + z_2 + z_3 = \frac{r^2}{\bar{z}_1} + \frac{r^2}{\bar{z}_2} + \frac{r^2}{\bar{z}_3}$$

$$= r^2\left[\frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3}\right]$$

$$= r^2\left[\frac{\bar{z}_2\bar{z}_3 + \bar{z}_1\bar{z}_3 + \bar{z}_1\bar{z}_2}{\bar{z}_1\bar{z}_2\bar{z}_3}\right]$$

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$$= r^2 \left[\frac{\overline{z_2 z_3 + z_1 z_3 + z_1 z_2}}{\overline{z_1 z_2 z_3}} \right]$$

$$|z_1 + z_2 + z_3| = |r^2| \left| \frac{\overline{z_2 z_3 + z_1 z_3 + z_1 z_2}}{\overline{z_1 z_2 z_3}} \right|$$

$$= r^2 \left| \frac{z_2 z_3 + z_1 z_3 + z_1 z_2}{|z_1|\,|z_2|\,|z_3|} \right|$$

$$= r^2 \frac{|z_2 z_3 + z_1 z_3 + z_1 z_2|}{r^3}$$

$$= \frac{|z_1 z_2 + z_2 z_3 + z_3 z_1|}{r}$$

$$\therefore \frac{|z_1 z_2 + z_2 z_3 + z_3 z_1|}{|z_1 + z_2 + z_3|} = r.$$

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Ex 2.16 Show that the eqn $z^2 = \bar{z}$ has four solutions.

$$z^2 = \bar{z}$$

$$|z^2| = |\bar{z}| = |z|$$

$$|z|^2 = |z| \Rightarrow |z|^2 - |z| = 0$$

$$\Rightarrow |z|\,[\,|z| - 1\,] = 0$$

$$\Rightarrow |z| = 0 \text{ (or) } |z| = 1$$

$|z| = 0 \Rightarrow z = 0$ is a solution.

$|z| = 1 \Rightarrow z\bar{z} = 1$

$$\Rightarrow \bar{z} = \frac{1}{z}$$

i.e. $z^2 = \frac{1}{z}$ $\quad \because (\bar{z} = z^2 \text{ given})$

$$z^3 = 1$$

It has 3 non zero solutions.
Hence including zero soln, there are four solns.

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Square root of a complex no:

Formula for finding Square root of a complex no

$$\sqrt{a+ib} = \pm\left[\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right]$$

Note: If b is negative $\frac{b}{|b|} = -1$

If b is positive $\frac{b}{|b|} = 1$.

Ex 2.17 Find the square root of $6-8i$

$$|6-8i| = \sqrt{6^2+(-8)^2}$$
$$= \sqrt{36+64} = \sqrt{100} = 10$$

$$\therefore \sqrt{6-8i} = \pm\left[\sqrt{\frac{10+6}{2}} + i\frac{-8}{8}\sqrt{\frac{10-6}{2}}\right]$$
$$= \pm\left[\sqrt{8} - i\sqrt{2}\right]$$
$$= \pm\left[2\sqrt{2} - i\sqrt{2}\right]$$

EXERCISE 2.5

1) Find the modulus of the complex numbers.

(i) $\frac{2i}{3+4i}$

let $z = \frac{2i}{3+4i}$

$$\therefore |z| = \left|\frac{2i}{3+4i}\right| = \frac{|2i|}{|3+4i|} = \frac{\sqrt{4}}{\sqrt{9+16}}$$
$$= \frac{2}{\sqrt{25}} = \boxed{\frac{2}{5}}$$

(ii) $\frac{2-i}{1+i} + \frac{1-2i}{1-i}$

let $z = \left|\frac{2-i}{1+i} + \frac{1-2i}{1-i}\right|$

$$= \left|\frac{(2-i)(1-i)+(1-2i)(1+i)}{(1+i)(1-i)}\right|$$

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$$= \left| \frac{2-2i-i+i^2+1+i-2i-2i^2}{1^2-i^2} \right|$$

$$= \left| \frac{2-3i-1+1+i-2i+2}{2} \right|$$

$$= \left| \frac{4-4i}{2} \right| = |2-2i|$$

$$= \sqrt{4+4} = \sqrt{2\times 4}$$

$$= \boxed{2\sqrt{2}}$$

(iii) $|(1-i)^{10}|$

$$= |1-i|^{10} = \left(\sqrt{1+1}\right)^{10} = (\sqrt{2})^{10}$$

$$= 2^5$$

$$= 32$$

(iv) $2i(3-4i)(4-3i)$

Let $z = 2i(3-4i)(4-3i)$

$$|z| = |2i(3-4i)(4-3i)|$$

$$= |2i|\,|3-4i|\,|4-3i|$$

$$= \sqrt{4}\ \sqrt{9+16}\ \sqrt{16+9}$$

$$= 2 \times \sqrt{25} \times \sqrt{25}$$

$$= 2\times 5\times 5 = 50.$$

---

2) For any two complex numbers z_1 and z_2 such that $|z_1| = |z_2| = 1$ and $z_1 z_2 \neq -1$ then show that $\frac{z_1+z_2}{1+z_1z_2}$ is real.

Let $z = \frac{z_1+z_2}{1+z_1z_2}$

To prove z is real, it to prove $\bar{z} = z$.

$$\bar{z} = \frac{\bar{z_1}+\bar{z_2}}{1+\bar{z_1}\bar{z_2}} = \frac{\frac{1}{z_1}+\frac{1}{z_2}}{1+\frac{1}{z_1}\cdot\frac{1}{z_2}}$$

$$= \frac{\frac{z_1+z_2}{z_1z_2}}{1+\frac{1}{z_1z_2}} = \frac{\frac{z_1+z_2}{z_1z_2}}{\frac{z_1z_2+1}{z_1z_2}}$$

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$$= \frac{z_1+z_2}{1+z_1z_2} = z$$

$\therefore \bar{z} = z \Rightarrow z$ is real.

---

3) Which one of the point $10-8i$, $11+6i$, is closed to $1+i$

Distance from $10-8i$ to $1+i$

$$= |10-8i-(1+i)|$$

$$= |10-8i-1-i| = |9-9i|$$

$$= \sqrt{81+81}$$

$$= \sqrt{2\times 81}$$

$$= 9\sqrt{2} = 9(1.414) = 12.726$$

Distance from $11+6i$ to $1+i$

$$= |11+6i-(1+i)|$$

$$= |11+6i-1-i|$$

$$= |10+5i| = \sqrt{100+25} = 125$$

$$= \sqrt{5\times 25}$$

$$= 5\sqrt{5}$$

$$= 5(2.236)$$

$= 11.180$

$\therefore 11+6i$ is closest to $1+i$

---

4) If $|z| = 3$, show that $7 \le |z+6-8i| \le 13$

$$|z+6-8i| \le |z| + |6-8i|$$

$$= 3 + \sqrt{36+64}$$

$$= 3 + \sqrt{100}$$

$$= 3+10 = 13.$$

$\therefore |z+6-8i| \le 13$ ——— (1)

$$|z+6-8i| \ge \big||z| - |6-8i|\big|$$

$$= |3 - \sqrt{36+64}|$$

$$= |3-10| = |-7|$$

$$= 7$$

$\therefore |z+6-8i| \ge 7$ ——— (2)

From (1) and (2)

$7 \le |z+6-8i| \le 13$

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5) If $|z| = 1$, show that $2 \le |z^2 - 3| \le 4$

$|z^2 - 3| \le |z^2| + |-3|$

$= |z|^2 + 3$

$= 1 + 3$

$= 4$

$\therefore |z^2 - 3| \le 4$ ——— (1)

$|z^2 - 3| \ge \left| |z|^2 - |3| \right|$

$= |1 - 3|$

$= |-2| = 2$

$\therefore |z^2 - 3| \ge 2$ ——— (2)

From (1) and (2)

$2 \le |z^2 - 3| \le 4$

——— x ——— x ———

6) If $\left| z - \frac{2}{z} \right| = 2$, show that the greatest and least value of $|z|$ are $\sqrt{3}+1$ and $\sqrt{3}-1$ respectively.

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Soln: $\left| z - \frac{2}{z} \right| = 2$.

WKT $\left| z - \frac{2}{z} \right| \ge |z| - \left| \frac{2}{z} \right|$

i.e. $2 \ge |z| - \frac{2}{|z|}$

put $|z| = t$,

$2 \ge t - \frac{2}{t}$

$2 \ge \frac{t^2 - 2}{t}$

$2t \ge t^2 - 2$

$0 \ge t^2 - 2t - 2$

i.e. $t^2 - 2t - 2 \le 0$.

$\therefore t = \frac{2 \pm \sqrt{4 - 4(1)(-2)}}{2}$

$t = \frac{2 \pm \sqrt{4+8}}{2}$

$t = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm \sqrt{4 \times 3}}{2}$

$= \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$

i.e. $|z| = 1 \pm \sqrt{3}$

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∴ The minimum value of $|z|$ is $\sqrt{3}-1$

The greatest value of $|z| = \sqrt{3}+1$

---

7) If z_1, z_2 and z_3 are three complex nos such that $|z_1|=1$, $|z_2|=2$, $|z_3|=3$ and $|z_1+z_2+z_3|=1$ Show that $|9z_1z_2+4z_1z_3+z_2z_3|=6$

$|z_1|=1 \Rightarrow |z_1|^2=1$

$\Rightarrow z_1\bar{z}_1 = 1$

$z_1 = \frac{1}{\bar{z}_1}$.

$|z_2|=2 \Rightarrow |z_2|^2 = 4$

$\Rightarrow z_2\bar{z}_2 = 4$

$z_2 = \frac{4}{\bar{z}_2}$.

$|z_3|=3 \Rightarrow |z_3|^2 = 9$

$z_3\bar{z}_3 = 9$

$z_3 = \frac{9}{\bar{z}_3}$.

Now $|z_1+z_2+z_3| = 1$

$\Rightarrow \left|\frac{1}{z_1}+\frac{4}{z_2}+\frac{9}{z_3}\right| = 1$

$\left|\frac{\bar{z}_2\bar{z}_3 + 4\bar{z}_1\bar{z}_3 + 9\bar{z}_1\bar{z}_2}{\bar{z}_1\bar{z}_2\bar{z}_3}\right| = 1$

$\Rightarrow \left|\frac{\overline{z_2z_3 + 4z_1z_3 + 9z_1z_2}}{\overline{z_1z_2z_3}}\right| = 1$

$\frac{|9z_1z_2+4z_1z_3+z_2z_3|}{|\overline{z_1z_2z_3}|} = 1$

$\frac{|9z_1z_2+4z_1z_3+z_2z_3|}{|z_1z_2z_3|} = 1$

$|9z_1z_2+4z_1z_3+z_2z_3| = |z_1z_2z_3|$

$= |z_1||z_2||z_3|$

$= 1\times2\times3$

$= 6$

Hence the proof.

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8) If the area of the triangle formed by the vertices z, iz and $z+zi$ is 50 sq units, find the value of $|z|$

The given vertices are $z, zi, z+zi$.

$\Rightarrow$ z, zi are $\perp$ to each other

$\therefore$ Area of the $\Delta = \frac{1}{2}|z||iz| = 50$

$$|z||z| = 50 \times 2$$
$$|z|^2 = 100$$
$$|z| = 10.$$

---

9) Show that the eqn $z^3 + 2\bar{z} = 0$ has five solutions.

Given $z^3 + 2\bar{z} = 0$

$$z^3 = -2\bar{z} \quad \text{---} \quad (1)$$

Taking modulus

$$|z^3| = |-2\bar{z}|$$
$$= 2|\bar{z}|$$
$$= 2|z|$$

i $|z|^3 - 2|z| = 0$

$|z|\left[|z|^2 - 2\right] = 0$

$\Rightarrow |z| = 0 \Rightarrow z = 0$ is one soln

$|z|^2 - 2 = 0$

$|z|^2 = 2$

$z\bar{z} = 2$

From (1) $\bar{z} = \frac{z^3}{-2}$

$\therefore z\left(\frac{z^3}{-2}\right) = 2$

$$z^4 = -4$$

$\Rightarrow z$ has four non zero soln.

Hence including zero, there are five solns.

---

10) Find the square root of

(i) $4+3i$

Let $z = 4+3i$

$|z| = |4+3i|$

$= \sqrt{16+9} = \sqrt{25} = 5.$

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$\sqrt{z} = \sqrt{4+3i}$

$= \pm \left[\sqrt{\frac{|z|+a}{2}} + \frac{b}{|b|} i \sqrt{\frac{|z|-a}{2}} \right]$

$= \pm \left[\sqrt{\frac{5+4}{2}} + i \frac{3}{3} \sqrt{\frac{5-4}{2}} \right]$

$= \pm \left[\frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right]$ //

(ii) Let $z = -6+8i$

$|z| = \sqrt{36+64} = \sqrt{100} = 10.$

$\sqrt{z} = \sqrt{-6+8i} = \pm \left[\sqrt{\frac{|z|+a}{2}} + i \frac{b}{|b|} \sqrt{\frac{|z|-a}{2}} \right]$

$= \pm \left[\sqrt{\frac{10-6}{2}} + i \frac{8}{8} \sqrt{\frac{10+6}{2}} \right]$

$= \pm \left[\sqrt{2} + i\sqrt{8} \right] = \pm \left(\sqrt{2} + i 2\sqrt{2} \right)$

$= \pm\sqrt{2}(1+2i)$ //

(iii) $z = -5-12i$

$|z| = \sqrt{25+144} = \sqrt{169} = 13.$

$\sqrt{z} = \sqrt{-5-12i}$

$= \pm \left[\sqrt{\frac{|z|+a}{2}} + i \frac{b}{|b|} \sqrt{\frac{|z|-a}{2}} \right]$

$= \pm \left[\sqrt{\frac{13-5}{2}} + i \frac{-12}{12} \sqrt{\frac{13+5}{2}} \right]$

$= \pm \left[\sqrt{\frac{8}{2}} - i\sqrt{9} \right]$

$= \pm \left[2 - 3i \right]$ //

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Equation of complex form of a circle:

$|z-z_0| = r$ is the complex form of the eqn of a circle.
where z_0 is a fixed complex number and r is a fixed positive real no.

Note: $|z| = r \Rightarrow \sqrt{x^2+y^2} = r$

or

$x^2+y^2 = r^2$

represents a circle centre at the origin and radius r units.

2.19 Show that $|3z-5+i| = 4$ represents a circle, and find its centre and radius.

The given eqn is $|3z-5+i| = 4$

$= 3\left|z - \frac{5}{3} + \frac{i}{3}\right| = 4$

$= 3\left|z - \left(\frac{5}{3} - \frac{1}{3}i\right)\right|$

$\left|z - \left(\frac{5}{3} - \frac{1}{3}i\right)\right| = \frac{4}{3}$

It is of the form $|z-z_0| = r$

$\therefore$ centre is $\left(+\frac{5}{3}, -\frac{1}{3}\right)$

and radius is $4/3$

Ex 2.21 obtain the cartesian form of the locus of z in each of the following cases.

(i) $|z| = |z-i|$

$|z| = |z-i|$

$|x+iy| = |x+iy-i|$

$|x+iy| = |x+i(y-1)|$

$\Rightarrow \sqrt{x^2+y^2} = \sqrt{x^2+(y-1)^2}$

Squaring, $x^2+y^2 = x^2+(y-1)^2$

$x^2+y^2 = x^2+y^2+1-2y$

$\Rightarrow 2y-1 = 0$.

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(11) $|2z-3-i| = 3$

$|2z-3-i| = 3$

$|2(x+iy)-3-i| = 3$

$|(2x-3)+i(2y-1)| = 3$

$\Rightarrow \sqrt{(2x-3)^2+(2y-1)^2} = 3$

squaring on both sides

$(2x-3)^2+(2y-1)^2 = 9$

$4x^2+9-12x+4y^2+1-4y = 9$

$4x^2+4y^2-12x-4y+1 = 0$.

EXERCISE 2.6

1) If $z = x+iy$ is a complex no such that $\left|\frac{z-4i}{z+4i}\right| = 1$ show that the locus of z is real axis.

$\left|\frac{z-4i}{z+4i}\right| = 1$

$\Rightarrow \frac{|z-4i|}{|z+4i|} = 1$

$\Rightarrow |z-4i| = |z+4i|$

$\Rightarrow |x+iy-4i| = |x+iy+4i|$

$\Rightarrow |x+i(y-4)| = |x+i(y+4)|$

$x^2+(y-4)^2 = x^2+(y+4)^2$

$x^2+y^2+16-8y = x^2+y^2+16+8y$

$16y = 0 \Rightarrow y = 0$.

$\therefore \left|\frac{z-4i}{z+4i}\right|$ is real axis.

---

2) If $z = x+iy$ is a complex number such that $\text{Im}\left(\frac{2z+1}{iz+1}\right) = 0$. Show that the locus of z is $2x^2+2y^2+x-2y = 0$

$z = x+iy$

$\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1}$

$= \frac{2x+2yi+1}{xi-y+1}$

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$$= \frac{(2x+1)+2yi}{(1-y)+ix}$$

$$= \frac{(2x+1)+2yi}{(1-y)+ix} \times \frac{(1-y)-ix}{(1-y)-ix}$$

$$= \frac{(2x+1)(1-y)-ix(2x+1)+i2y(1-y)-2xyi^2}{(1-y)^2+x^2}$$

$$= \frac{(2x+1)(1-y)+2xy+i[(2x+1)x+2y(1-y)]}{(1-y)^2+x^2}$$

$$\text{Im}\left(\frac{2z+1}{iz+1}\right) = 0$$

$$\Rightarrow -\frac{(2x+1)x+2y(1-y)}{(1-y)^2+x^2} = 0$$

$$\Rightarrow -2x^2-x+2y-2y^2 = 0$$

$$\Rightarrow 2x^2+2y^2+x-2y = 0.$$

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3) obtain the cartesian form of the locus of $z = x+iy$ in each of the following

(i) $[\text{Re}(iz)]^2 = 3$

Let $z = x+iy$

$iz = i(x+iy)$

$= ix-y$

$\text{Re}(iz) = -y.$

$[\text{Re}(iz)]^2 = 3$

$\Rightarrow (-y)^2 = 3 \Rightarrow y^2 = 3.$

(ii) $\text{Im}[(1-i)z+1] = 0$

Let $z = x+iy.$

$(1-i)z+1 = (1-i)(x+iy)+1$

$= x+iy-xi+y+1$

$= (x+y+1)+i(y-x)$

$\text{Im}[(1-i)z+1] = 0 \Rightarrow y-x = 0$

$\Rightarrow \boxed{x=y}$

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(iii) $|z+i| = |z-1|$

Let $z = x+iy$

$|z+i| = |z-1|$

$\Rightarrow |x+iy+i| = |x+iy-1|$

$\Rightarrow |x+i(y+1)| = |(x-1)+iy|$

$\Rightarrow \sqrt{x^2+(y+1)^2} = \sqrt{(x-1)^2+y^2}$

Squaring on both sides

$x^2+(y+1)^2 = (x-1)^2+y^2$

$x^2+y^2+2y+1 = x^2-2x+1+y^2$

$2y+2x = 0$

$2(x+y) = 0$

$(x+y) = 0$

(iv) $\bar{z} = z^{-1}$

Let $z = x+iy$, $\therefore \bar{z} = x-iy$

$\bar{z} = z^{-1}$

$\Rightarrow x-iy = \frac{1}{x+iy}$

$(x+iy)(x-iy) = 1$

$x^2+y^2 = 1$

---

4) Show that the following eqns represents a circle, and find its centre and radius.

(i) $|z-2-i| = 3$

$|z-2-i| = 3 \Rightarrow |z-(2+i)| = 3$

$\therefore$ centre is $(2,1)$ and radius is 3

(ii) $|2z+2-4i| = 2$

$2|z+1-2i| = 2$

$|z-(-1+2i)| = 1$

$\therefore$ centre is $(-1,2)$ and radius is 1

(iii) $|3z-6+12i| = 8$

$3|z-2+4i| = 8$

$|z-(2-4i)| = 8/3$

$\therefore$ centre is $(2,-4)$ and radius is $8/3$

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5) obtain the cartesian eqn for the locus of $z = x+iy$ in each of the following cases.

(i) $|z-4| = 16$

Let $z = x+iy$

$|z-4| = 16 \Rightarrow |x+iy-4| = 16$

$|(x-4)+iy| = 16$

$\sqrt{(x-4)^2+y^2} = 16$

squaring on both sides

$(x-4)^2+y^2 = 256$

$x^2+16-8x+y^2 = 256$

$x^2+y^2-8x+16-256 = 0$

$x^2+y^2-8x-240 = 0$.

(ii) $|z-4|^2-|z-1|^2 = 16$

Let $z = x+iy$

$|z-4|^2-|z-1|^2 = 16$

$|x+iy-4|^2-|x+iy-1|^2 = 16$

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$|(x-4)+iy|^2-|(x-1)+iy|^2 = 16$.

$\Rightarrow (x-4)^2+y^2-[(x-1)^2+y^2] = 16$

$(x-4)^2+y^2-(x-1)^2-y^2 = 16$

$(x-4)^2-(x-1)^2 = 16$

$x^2+16-8x-(x^2+1-2x) = 16$

$x^2+16-8x-x^2-1+2x = 16$.

$-6x-1 = 0$

$6x+1 = 0$

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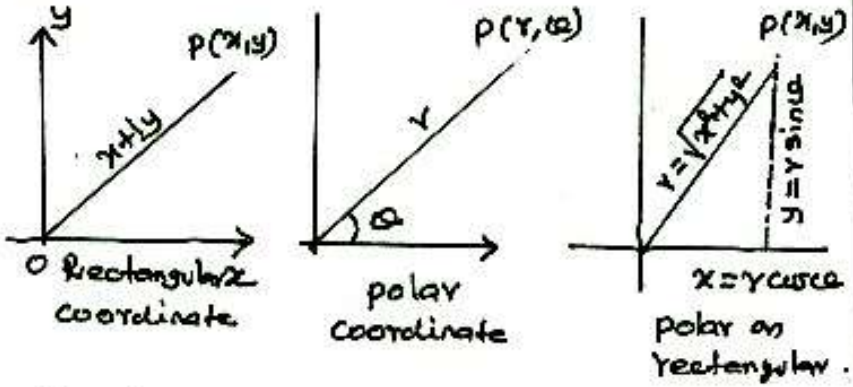
polar form of a complex no.



Any non-zero complex number

$z = x+iy = r\cos\theta + i\, r\sin\theta$.

Principal Argument of complex number.

1) If the complex no is in I st quadrant, then the principal value is $\theta = \alpha$

2) In 2nd Quadrant, the principal value is $\theta = \pi - \alpha$.

3) In 3rd Quadrant, the principal value is $\theta = \alpha - \pi$

4) In 4th Quadrant, the principal value is $\theta = -\alpha$.

properties of arguments

a) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

b) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

c) $\arg(z^n) = n \arg z$.

d) The alternate form of

$\cos\theta + i\sin\theta = \cos(2k\pi+\theta) + i\sin(2k\pi+\theta)$

---

principal argument and argument of $1, i, -1, -i$ are

z	1	i	-1	$-i$
arg(z)	0	$\pi/2$	π	$-\pi/2$
arg z	$2n\pi$	$2n\pi+\frac{\pi}{2}$	$2n\pi+\pi$	$2n\pi-\pi/2$

Euler's form of complex number

$e^{i\theta} = \cos\theta + i\sin\theta$.

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Note: To find the principal angle ω, compute $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

Ex 2.22 Find the modulus and principal argument of the following complex nos:

(i) $\sqrt{3}+i$

modulus $= |\sqrt{3}+i|$

$= \sqrt{(\sqrt{3})^2+1^2}$

$= \sqrt{3+1} = \sqrt{4} = \boxed{2}$

principal argument $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

$\alpha = \tan^{-1}\frac{1}{\sqrt{3}}$

$\alpha = 30^\circ = \boxed{\pi/6}$.

The given complex no $\sqrt{3}+i$ lies in the 1st Quadrant.

$\therefore$ The pricipal value $\omega = \alpha = \boxed{\pi/6}$

(ii) $-\sqrt{3}+i$

modulus $= |-\sqrt{3}+i|$

$= \sqrt{(-\sqrt{3})^2+1} = \sqrt{3+1} = \sqrt{4}$

$= \boxed{2}$.

principal argument $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

$= \tan^{-1}\frac{1}{\sqrt{3}}$

$= \boxed{\pi/6}$

The complex no $-\sqrt{3}+i$ lies in IInd Quadrant.

$\therefore$ The principal value $\omega = \pi - \alpha$

$= \pi - \pi/6$

$= \boxed{\frac{5\pi}{6}}$

(iii) $-\sqrt{3}-i$

modulus $= |-\sqrt{3}-i|$

$= \sqrt{(-\sqrt{3})^2+(-1)^2}$

$= \sqrt{3+1} = \sqrt{4}$

$= \boxed{2}$

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principal argument $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

$= \tan^{-1}\frac{1}{\sqrt{3}}$

$\therefore \alpha = \pi/6$

The complex no $-\sqrt{3}-i$ lies in IVth quadrant

$\therefore$ principal value $\theta = -\alpha = -\pi/6$.

Ex 2.23 Represent the complex number in polar form.

(i) $-1-i$

$-1-i = r(\cos\theta + i\sin\theta)$

we have $r = \sqrt{x^2+y^2} = \sqrt{1+1}$

$= \sqrt{2}$

$\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

$\alpha = \tan^{-1}(1) = \pi/4$.

The complex no $-1-i$ lies in 3rd quadrant.

$\therefore \theta = \alpha - \pi = \pi/4 - \pi$

$\theta = -3\pi/4$

$\therefore -1-i = \sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\frac{3\pi}{4}\right]$

$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$= \sqrt{2}\left(\cos\left(\frac{3\pi}{4}+2k\pi\right) + i\sin\left(\frac{3\pi}{4}+2k\pi\right)\right)$.

(ii) $1+i\sqrt{3}$

$1+i\sqrt{3} = r(\cos\theta + i\sin\theta)$

$\therefore r = \sqrt{x^2+y^2} = \sqrt{1+3} = \sqrt{4}$

$= \boxed{2}$

$\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

$= \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$

$= \tan^{-1}\sqrt{3} = \pi/3$.

The complex no $1+i\sqrt{3}$ lies in 1st quadrant

$\therefore \theta = \alpha = \pi/3$.

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$\therefore 1+i\sqrt{3} = 2\left[\cos \pi/3 + i \sin \pi/3\right]$

$= 2\left[\cos\left(\frac{\pi}{3}+2k\pi\right) + i \sin\left(\frac{\pi}{3}+2k\pi\right)\right]$

Ex. 2.24 Find the principal arg z, when $z = \frac{-2}{1+i\sqrt{3}}$

$z = \frac{-2}{1+i\sqrt{3}}$

$\arg z = \arg\left(\frac{-2}{1+i\sqrt{3}}\right)$

$= \arg(-2) - \arg(1+i\sqrt{3})$

$= \pi - \pi/3$

$= 2\pi/3$

$\therefore$ principal argument of z is $2\pi/3$

arg (-2)

$\alpha = \tan^{-1}\left(\frac{0}{-2}\right)$

$= 0$ (2nd Q)

$\therefore \omega = \pi - \alpha$

$= \pi$

arg (1+i√3)

$\alpha = \tan^{-1}(\sqrt{3}) = \pi/3$

1+i√3 lies in 1st Q

$\therefore \omega = \alpha$.

---

properties of polar form:

1) If $z = r(\cos\omega + i\sin\omega)$ then

$z^{-1} = \frac{1}{r}(\cos\omega - i\sin\omega)$

2) If $z_1 = r_1(\cos\omega_1 + i\sin\omega_1)$

$z_2 = r_2(\cos\omega_2 + i\sin\omega_2)$

Then $z_1 z_2 = r_1 r_2\left[\cos(\omega_1+\omega_2) + i\sin(\omega_1+\omega_2)\right]$

3) If $z_1 = r_1(\cos\omega_1 + i\sin\omega_1)$

$z_2 = r_2(\cos\omega_2 + i\sin\omega_2)$

Then $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\omega_1-\omega_2) + i\sin(\omega_1-\omega_2)\right]$

Note: $\arg\left(\frac{z_1}{z_2}\right) = \omega_1 - \omega_2$

$= \arg(z_1) - \arg(z_2)$

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Ex 2.25

Find the product $\frac{3}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\cdot 6\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$ in rectangular form.

The product $\frac{3}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\cdot 6\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

$= \frac{3}{2}\times 6\left[\cos\left(\frac{\pi}{3}+\frac{5\pi}{6}\right)+i\sin\left(\frac{\pi}{3}+\frac{5\pi}{6}\right)\right]$

$= 9\left[\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right]$

$= 9\left[\cos(\pi+\pi/6)+i\sin(\pi+\pi/6)\right]$

$= 9\left[-\cos\pi/6-i\sin\pi/6\right]$

$= 9\left[-\cos 30-i\sin 30\right]$

$= 9\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$ which is in rectangular form.

Ex 2.26

Find the quotient $\frac{2\left(\cos\frac{9\pi}{4}+i\sin\frac{9\pi}{4}\right)}{4\left[\cos\left(-\frac{3\pi}{2}\right)+i\sin\left(-\frac{3\pi}{2}\right)\right]}$ in rectangular form.

$\frac{2\left(\cos\frac{9\pi}{4}+i\sin\frac{9\pi}{4}\right)}{4\left[\cos\left(-\frac{3\pi}{2}\right)+i\sin\left(-\frac{3\pi}{2}\right)\right]}$

$= \frac{1}{2}\left[\cos\left(\frac{9\pi}{4}-\left(-\frac{3\pi}{2}\right)\right)+i\sin\left(\frac{9\pi}{4}-\left(-\frac{3\pi}{2}\right)\right)\right]$

$= \frac{1}{2}\left[\cos\left(\frac{9\pi}{4}+\frac{3\pi}{2}\right)+i\sin\left(\frac{9\pi}{4}+\frac{3\pi}{2}\right)\right]$

$= \frac{1}{2}\left[\cos\left(\frac{9\pi+6\pi}{4}\right)+i\sin\left(\frac{9\pi+6\pi}{4}\right)\right]$

$= \frac{1}{2}\left[\cos\frac{15\pi}{4}+i\sin\frac{15\pi}{4}\right]$

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$= \frac{1}{2}\left[\cos\left(4\pi - \frac{\pi}{4}\right) + i\sin\left(4\pi - \frac{\pi}{4}\right)\right]$ (4 quadrant)

$= \frac{1}{2}\left[\cos \pi/4 - i\sin \pi/4\right]$

$= \frac{1}{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$ which is in rectangular form.

Ex 2.27

If $z = x+iy$ and $\arg\left(\frac{z-1}{z+1}\right) = \pi/2$ Show that $x^2+y^2 = 1$.

$z = x+iy$

$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy}$

$= \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$

$= \frac{(x+1)(x-1) - iy(x-1) + iy(x+1) - i^2y^2}{(x+1)^2+y^2}$

$= \frac{((x+1)(x-1)+y^2) + i(y(x+1) - y(x-1))}{(x+1)^2+y^2}$ (XII | CH 2 | 2.7 | pb)

$= \frac{(x^2+y^2-1) + i(y(x+1-x+1))}{(x+1)^2+y^2}$

$= \frac{(x^2+y^2-1) + i(2y)}{(x+1)^2+y^2}$

$\arg\left(\frac{z-1}{z+1}\right) = \pi/2 \Rightarrow \tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right) = \pi/2$

$\Rightarrow \frac{2y}{x^2+y^2-1} = \tan \pi/2$

$\Rightarrow \frac{2y}{x^2+y^2-1} = \infty = \frac{1}{0}$

$\Rightarrow x^2+y^2-1 = 0$

i.e. $\boxed{x^2+y^2 = 1}$

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EXERCISE 2.7

1. Write in polar form of complex number.

(i) $2+i2\sqrt{3}$

$2+i2\sqrt{3} = r(\cos\theta + i\sin\theta)$

$\therefore r = \sqrt{2^2+(2\sqrt{3})^2}$

$= \sqrt{4+12} = \sqrt{16} = 4$.

This complex no lies in 1st Q.

Now $\alpha = \tan^{-1}\frac{2\sqrt{3}}{2} = \tan^{-1}\sqrt{3}$

$= \pi/3$

$\therefore \theta = \alpha = \pi/3$

$\therefore 2+i2\sqrt{3} = 4\left(\cos\pi/3 + i\sin\pi/3\right)$

(ii) $3-i\sqrt{3}$

$3-i\sqrt{3} = r(\cos\theta + i\sin\theta)$

$r = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$

$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi/6$

The complex no lies in 4th Q.

$\therefore \theta = -\alpha = -\pi/6$

$\therefore 3-i\sqrt{3} = 2\sqrt{3}\left(\cos(-\pi/6) + i\sin(-\pi/6)\right)$

(iii) $-2-i2$

$-2-i2 = r(\cos\theta + i\sin\theta)$

$r = \sqrt{(-2)^2+(-2)^2} = \sqrt{4+4}$

$= \sqrt{2\times4}$

$= 2\sqrt{2}$

$\alpha = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) = \pi/4$

The complex no lies in 3rd Q

$\therefore \theta = \alpha - \pi$

$= \pi/4 - \pi = -\frac{3\pi}{4}$

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$\therefore -2-i2 = 2\sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i \sin\left(-\frac{3\pi}{4}\right)\right]$

(iv) $\dfrac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$

$-1+i = r(\cos\alpha + i\sin\alpha)$

$r = \sqrt{1+1} = \sqrt{2}$

$\alpha = \tan^{-1}(1) = \pi/4$

This complex no lies in 2nd Q.

$\therefore \alpha = \pi - \alpha = \pi - \pi/4 = \frac{3\pi}{4}$

$-1+i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$\therefore \dfrac{-1+i}{\cos\pi/3 + i\sin\pi/3} = \dfrac{\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)}{\cos\pi/3 + i\sin\pi/3}$

$= \sqrt{2}\left[\cos\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{3\pi}{4} - \frac{\pi}{3}\right)\right]$

$= \sqrt{2}\left[\cos\left(\frac{9\pi - 4\pi}{12}\right) + i\sin\left(\frac{9\pi - 4\pi}{12}\right)\right]$

$= \sqrt{2}\left[\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right]$

---

2) Find the Cartesian form of the complex nos

(i) $\left(\cos\pi/6 + i\sin\pi/6\right)\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

$= \cos\left(\frac{\pi}{6} + \frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{6} + \frac{\pi}{12}\right)$

$= \cos\frac{3\pi}{12} + i\sin\frac{3\pi}{12}$

$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

(ii) $\dfrac{\cos\pi/6 - i\sin\pi/6}{2\left(\cos\pi/3 + i\sin\pi/3\right)}$

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$$= \frac{\cos(-\pi/6) + i\sin(-\pi/6)}{2(\cos\pi/3 + i\sin\pi/3)}$$

$$= \frac{1}{2}\left[\cos\left(-\frac{\pi}{6} - \frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{6} - \frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(-\frac{3\pi}{6}\right) + i\sin\left(-\frac{3\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[\cos(-\pi/2) + i\sin(-\pi/2)\right]$$

$$= \frac{1}{2}\left[\cos\pi/2 - i\sin\pi/2\right]$$

$$= \frac{1}{2}[0 - i] = \boxed{\frac{-i}{2}}$$

---

3) If $(x_1+iy_1)(x_2+iy_2)\cdots(x_n+iy_n) = a+ib$ show that

(i) $(x_1^2+y_1^2)(x_2^2+y_2^2)\cdots(x_n^2+y_n^2) = a^2+b^2$

(ii) $\sum_{r=1}^{n} \tan^{-1}\left(\frac{y_r}{x_r}\right) = \tan^{-1}\left(\frac{b}{a}\right) + k\pi$.

(i)

$(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\cdots(x_n+iy_n) = a+ib$.

Taking modulus on both sides.

$$|(x_1+iy_1)(x_2+iy_2)\cdots(x_n+iy_n)| = |a+ib|$$

$$|x_1+iy_1||x_2+iy_2|\cdots|x_n+iy_n| = |a+ib|$$

$$|x_1+iy_1|^2|x_2+iy_2|^2\cdots|x_n+iy_n|^2 = |a+ib|^2$$

$$(x_1^2+y_1^2)(x_2^2+y_2^2)\cdots(x_n^2+y_n^2) = a^2+b^2$$

Taking argument on both sides.

$$\arg[(x_1+iy_1)(x_2+iy_2)\cdots(x_n+iy_n)] = \arg(a+ib)$$

$$\arg(x_1+iy_1) + \arg(x_2+iy_2) + \cdots + \arg(x_n+iy_n) = \arg(a+ib)$$

i.e. $\tan^{-1}\frac{y_1}{x_1} + \tan^{-1}\frac{y_2}{x_2} + \cdots + \tan^{-1}\frac{y_n}{x_n} = \tan^{-1}\left(\frac{b}{a}\right) + k\pi$.

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4) If $\frac{1+z}{1-z} = \cos 2\alpha + i \sin 2\alpha$

Show that $z = i \tan\alpha$

$$\frac{1+z}{1-z} = \cos 2\alpha + i \sin 2\alpha$$

$$\frac{1+z}{1-z} = e^{i2\alpha}$$

$$(1-z)e^{i2\alpha} = 1+z$$

$$e^{i2\alpha} - ze^{i2\alpha} = 1+z$$

$$e^{i2\alpha} - 1 = z + ze^{i2\alpha}$$

$$e^{i2\alpha} - 1 = z(1+e^{i2\alpha})$$

$$z = \frac{e^{i2\alpha} - 1}{1 + e^{i2\alpha}}$$

$$= \frac{e^{i\alpha} - e^{-i\alpha}}{e^{i\alpha} + e^{-i\alpha}}$$

$$= \frac{\cos\alpha + i\sin\alpha - (\cos\alpha - i\sin\alpha)}{\cos\alpha + i\sin\alpha + \cos\alpha - i\sin\alpha}$$

$$= \frac{2i\sin\alpha}{2\cos\alpha} = i\tan\alpha.$$

$\therefore z = i\tan\alpha$.

---

5) If $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$.

Then show that

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

Soln: Let $a = \cos\alpha + i\sin\alpha = e^{i\alpha}$

$b = \cos\beta + i\sin\beta = e^{i\beta}$

$c = \cos\gamma + i\sin\gamma = e^{i\gamma}$.

Now $a+b+c =$

$(\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 + i0 = 0$

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$\therefore a+b+c = 0$

If $a+b+c = 0$ then

$a^3+b^3+c^3 = 3abc$

i $(e^{i\alpha})^3 + (e^{i\beta})^3 + (e^{i\gamma})^3 =$

$3e^{i\alpha}e^{i\beta}e^{i\gamma}$

ii $e^{i3\alpha} + e^{i3\beta} + e^{i3\gamma} = 3e^{i(\alpha+\beta+\gamma)}$

$\Rightarrow \cos 3\alpha + i\sin 3\alpha + \cos 3\beta + i\sin 3\beta + \cos 3\gamma + i\sin 3\gamma =$

$3[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$

ie $(\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$

$= 3\cos(\alpha+\beta+\gamma) + i\,3\sin(\alpha+\beta+\gamma)$.

equating real and Imaginary part.

XII/CH2/2.7/P11

$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

---

6) If $z = x+iy$ and $\arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$, then show that $x^2+y^2+3x-3y+2=0$.

$z = x+iy$

$$\frac{z-i}{z+2} = \frac{x+iy-i}{x+iy+2} = \frac{x+i(y-1)}{(x+2)+iy}$$

$$= \frac{x+i(y-1)}{(x+2)+iy} \times \frac{(x+2)-iy}{(x+2)-iy}$$

$$= \frac{x(x+2) - ixy + i(y-1)(x+2) - i^2y(y-1)}{(x+2)^2+y^2}$$

$$= \frac{x(x+2)+y(y-1) + i\left((y-1)(x+2) - xy\right)}{(x+2)^2+y^2}$$

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Given

$$\arg\left(\frac{z-i}{z+2}\right) = \pi/4$$

$$\text{i.e. } \tan^{-1}\left[\frac{(y-1)(x+2)-xy}{x(x+2)+y(y-1)}\right] = \frac{\pi}{4}$$

$$\text{i.e. } \frac{(y-1)(x+2)-xy}{x(x+2)+y(y-1)} = \tan\pi/4 = 1$$

$$(y-1)(x+2)-xy = x(x+2)+y(y-1)$$

$$xy+2y-x-2-xy = x^2+2x+y^2-y$$

$$x^2+y^2+2x+x-y-2y+2=0$$

$$x^2+y^2+3x-3y+2=0$$



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$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Note:-

1) $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$

2) $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

3) $(\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta$

4) $(\sin\theta + i\cos\theta) = i(\cos\theta - i\sin\theta)$

Ex 2.28 If $z = \cos\theta + i\sin\theta$, show that $z^n + \frac{1}{z^n} = 2\cos n\theta$, and $z^n - \frac{1}{z^n} = 2i\sin n\theta$.

Soln: $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta$ ——— (1)

$\frac{1}{z^n} = z^{-n} = (\cos\theta + i\sin\theta)^{-n}$

$= \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$ ——— (2).

(1) + (2)

$\Rightarrow z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= \boxed{2\cos n\theta}$.

(1) − (2) $\Rightarrow z^n - \frac{1}{z^n}$

$= \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$

$= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$

$= \boxed{2i\sin n\theta}$

Ex 2.29 Simplify $\left(\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}\right)^{18}$

$\sin\frac{\pi}{6} + i\cos\frac{\pi}{6} = i\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

$\therefore \left(\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}\right)^{18} = i^{18}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^{18}$

$= i^{16} \cdot i^2 \left(\cos\frac{18\pi}{6} - i\sin\frac{18\pi}{6}\right)$

$= -\left(\cos 3\pi - i\sin 3\pi\right)$

$= -(-1) = 1 = \boxed{1 + 0i}$

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Ex 2.30:- Simplify $\left(\dfrac{1+\cos 2\alpha + i\sin 2\alpha}{1+\cos 2\alpha - i\sin 2\alpha}\right)^{30}$

Let $\cos 2\alpha + i\sin 2\alpha = z$.

WKT $|z| = |z|^2 = z\bar{z} = 1$

$\therefore \bar{z} = \dfrac{1}{z} = z^{-1}$

$= (\cos 2\alpha + i\sin 2\alpha)^{-1}$

$= \cos 2\alpha - i\sin 2\alpha$.

$\therefore \left(\dfrac{1+\cos 2\alpha + i\sin 2\alpha}{1+\cos 2\alpha - i\sin 2\alpha}\right)^{30} = \left(\dfrac{1+z}{1+\frac{1}{z}}\right)^{30}$

$= \left(\dfrac{1+z}{\frac{z+1}{z}}\right)^{30} = \left(\dfrac{(1+z)\times z}{(z+1)}\right)^{30}$

$= z^{30}$

$= (\cos 2\alpha + i\sin 2\alpha)^{30}$

$= \cos 60\alpha + i\sin 60\alpha$ //

Ex 2.31 (i) Simplify $(1+i)^{18}$

Let $1+i = r(\cos\alpha + i\sin\alpha)$

$r = \sqrt{1+1} = \sqrt{2}$.

$\alpha = \tan(1) = \pi/4$

$1+i$ lies in 1st quadrant.

$\therefore \alpha = \alpha = \pi/4$

$\therefore 1+i = \sqrt{2}\left(\cos \pi/4 + i\sin \pi/4\right)$

$(1+i)^{18} = (\sqrt{2})^{18}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{18}$

$= 2^9\left(\cos\frac{18\pi}{4} + i\sin\frac{18\pi}{4}\right)$

$= 2^9\left(\cos\frac{9\pi}{2} + i\sin\frac{9\pi}{2}\right)$

$= 2^9\left(\cos\left(4\pi+\frac{\pi}{2}\right) + i\sin\left(4\pi+\frac{\pi}{2}\right)\right)$

$= 2^9\left(\cos \pi/2 + i\sin \pi/2\right)$

$= 2^9(0+i) = 2^9 i$

$= 512i$ //

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Eg: 2.31 ii) $\left(-\sqrt{3}+3i\right)^{31}$

$-\sqrt{3}+3i = r(\cos\alpha + i\sin\alpha)$

$r = \sqrt{(\sqrt{3})^2+3^2} = \sqrt{3+9} = \sqrt{12} = \boxed{2\sqrt{3}}$

$\alpha = \tan^{-1}\left(\frac{3}{\sqrt{3}}\right) = \tan^{-1}\sqrt{3} = \boxed{\pi/3}$.

$-\sqrt{3}+3i$ lies in II quadrant.

$\therefore \omega = \pi - \alpha = \pi - \pi/3$

$= \boxed{\frac{2\pi}{3}}$

$\therefore \left(-\sqrt{3}+3i\right)^{31} = (2\sqrt{3})^{31}\left[\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right]^{31}$

$= (2\sqrt{3})^{31}\left[\cos\frac{62\pi}{3}+i\sin\frac{62\pi}{3}\right]$

$= (2\sqrt{3})^{31}\left[\cos\left(20\pi+\frac{2\pi}{3}\right)+i\sin\left(20\pi+\frac{2\pi}{3}\right)\right]$

$= (2\sqrt{3})^{31}\left[\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right]$

$= (2\sqrt{3})^{31}\left[\cos\left(\pi-\frac{\pi}{3}\right)+i\sin\left(\pi-\frac{\pi}{3}\right)\right]$

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$= (2\sqrt{3})^{31}\left[-\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right]$

$= (2\sqrt{3})^{31}\left[-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right]$

Finding n^{th} roots of a complex number

The nth roots of unity:

The soln of the eqn $z^n = 1$ are the nth root of unity.

In polar form of the eqn $z = 1$ can be written as

$z = \cos 2k\pi + i\sin 2k\pi$

$= \cos(0+2k\pi)+i\sin(0+2k\pi)$

$= e^{i2k\pi}$, $k = 0, 1, 2 \ldots\ldots$

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Using de Moivre's Th, we find n^{th} root of unity from the given eqn is

$$z^{\frac{1}{n}} = (\cos 2k\pi + i \sin 2k\pi)^{\frac{1}{n}}$$

$$= \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$$

$$= e^{\frac{i2k\pi}{n}}, \quad i = 0, 1, 2 \cdots n-1$$

---

* A complex number z is called an nth root of unity iff $z^n = 1$. If we denote the complex no ω, then

$$\omega = e^{\frac{2\pi i}{n}} = \cos \frac{2\pi i}{n} + i \sin \frac{2\pi i}{n}$$

$$\Rightarrow \omega^n = \left(e^{\frac{2\pi i}{n}}\right)^n$$

$$= e^{2\pi i} = 1$$

Note:

The sum of all n^{th} roots of unity is

$$1 + \omega + \omega^2 + \omega^3 + \cdots + \omega^{n-1} = 0.$$

The product of all n^{th} roots of unity is $1 \cdot \omega \cdot \omega^2 \cdot \omega^3 + \cdots + \omega^{n-1} = (-1)^{n-1}$.

⊛ All the nth roots of unity are in GP.

Ex 2.32 Find the cube root of unity

Let $z^3 = 1$

In polar form, the eqn $z = 1$ is

$$z = \cos(0 + 2k\pi) + i \sin(0 + 2k\pi)$$

$$= e^{i2k\pi}$$

$$\therefore z^{\frac{1}{3}} = \left(e^{i2k\pi}\right)^{\frac{1}{3}}$$

$$= e^{\frac{i2k\pi}{3}}$$

$$= \left(\cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3}\right) \quad k = 0, 1, 2.$$

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When $k=0$, $z = \cos 0 + i \sin 0 = 1$

$k=1$, $z = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$

$= \cos\left(\pi - \frac{\pi}{3}\right) + i \sin\left(\pi - \frac{\pi}{3}\right)$

$= -\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$

$= -\frac{1}{2} + i \frac{\sqrt{3}}{2}$

$k=2$, $z = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$

$= \cos\left(\pi + \frac{\pi}{3}\right) + i \sin\left(\pi + \frac{\pi}{3}\right)$

$= -\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}$

$= -\frac{1}{2} - i \frac{\sqrt{3}}{2}$

∴ The cube root of unity are

$1, \quad \frac{-1+i\sqrt{3}}{2}, \quad \frac{-1-i\sqrt{3}}{2} \quad i\frac{2\pi}{3}$

$\Rightarrow 1, \omega, \omega^2$ where $\omega = e$

$= -\frac{1}{2} + i \frac{\sqrt{3}}{2}$

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Ex 2.33 Find the fourth root of unity.

Let $z^4 = 1$

polar form of $z = 1$ is

$z = \cos(0 + 2k\pi) + i \sin(0 + 2k\pi)$

$\therefore z^{\frac{1}{4}} = (\cos 2k\pi + i \sin 2k\pi)^{\frac{1}{4}}$

$= \cos \frac{2k\pi}{4} + i \sin \frac{2k\pi}{4}$

$= \cos \frac{k\pi}{2} + i \sin \frac{k\pi}{2}$, $k = 0, 1, 2, 3$

Taking $k=0$,

$z = \cos 0 + i \sin 0 = 1$

$k=1$, $z = \cos \pi/2 + i \sin \pi/2 = 0 + i = i$

$k=2$, $z = \cos \pi + i \sin \pi = -1 + 0 = -1$

$k=3$, $z = \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}$

$= \cos(\pi + \pi/2) + i \sin(\pi + \frac{\pi}{2})$

$= -\cos \frac{\pi}{2} - i \sin \pi/2 = 0 - i$

$= \boxed{-i}$

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∴ Fourth root of unity are

$1, i, -1, -i \Rightarrow 1, w, w^2, w^3$

where $w = e^{i2\pi/4} = i$

Ex 2.34 solve the eqn $z^3 + 8i = 0$

Let $z^3 + 8i = 0$.

$z^3 = -8i = 8(-i)$

$= 8\left[\cos\left(-\frac{\pi}{2} + 2k\pi\right) + i\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right]$

$= 8\left[\cos\left(\frac{4k\pi - \pi}{2}\right) + i\sin\left(\frac{4k\pi - \pi}{2}\right)\right]$ $k = 0, 1, 2$

$\therefore z = 8^{\frac{1}{3}}\left[\cos\frac{4k\pi - \pi}{2} + i\sin\frac{4k\pi - \pi}{2}\right]^{\frac{1}{3}}$

$= 2\left[\cos\frac{4k\pi - \pi}{6} + i\sin\frac{4k\pi - \pi}{6}\right]$ $k = 0, 1, 2$.

when $k = 0$,

$z = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$

$= 2\left[\cos\pi/6 - i\sin\pi/6\right]$

$= 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$

$= \boxed{\sqrt{3} - i}$

when $k = 1$,

$z = 2\left[\cos\frac{3\pi}{6} + i\sin\frac{3\pi}{6}\right]$

$= 2\left[\cos\pi/2 + i\sin\pi/2\right]$

$= 2[0 + i(1)] = \boxed{2i}$

when $k = 2$,

$z = 2\left[\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right]$

$= 2\left[\cos\left(\pi + \frac{\pi}{6}\right) + i\sin\left(\pi + \frac{\pi}{6}\right)\right]$

$= 2\left[-\cos\pi/6 - i\sin\pi/6\right]$

$= 2\left[-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right] = \boxed{-\sqrt{3} - i}$

∴ The value of z are

$\sqrt{3} - i, 2i, -\sqrt{3} - i$

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Ex 2.35 Find all cube root of $\sqrt{3}+i$

Let $z^3 = \sqrt{3}+i = r(\cos\omega + i\sin\omega)$

$r = \sqrt{(\sqrt{3})^2+1^2} = \sqrt{3+1} = \sqrt{4} = \boxed{2}$

$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi/6$.

$\sqrt{3}+i$ lies in 1st quadrant.

$\therefore \omega = \alpha = \pi/6$.

$z^3 = 2(\cos\pi/6 + i\sin\pi/6)$

$z = 2^{\frac{1}{3}}(\cos\pi/6 + i\sin\pi/6)^{\frac{1}{3}}$

$= 2^{\frac{1}{3}}\left[\cos\left(2k\pi+\frac{\pi}{6}\right)+i\sin\left(2k\pi+\frac{\pi}{6}\right)\right]^{\frac{1}{3}}$

$= 2^{\frac{1}{3}}\left[\cos\left(\frac{12k\pi+\pi}{6}\right)+i\sin\left(\frac{12k\pi+\pi}{6}\right)\right]$

$= 2^{\frac{1}{3}}\left[\cos\left(\frac{12k\pi+\pi}{18}\right)+i\sin\left(\frac{12k\pi+\pi}{18}\right)\right]$

$k = 0, 1, 2$.

Taking $k=0$

$z = 2^{\frac{1}{3}}\left[\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right]$

$k=1,\ z = 2^{\frac{1}{3}}\left[\cos\frac{13\pi}{18}+i\sin\frac{13\pi}{18}\right]$

$k=2,\ z = 2^{\frac{1}{3}}\left[\cos\frac{25\pi}{18}+i\sin\frac{25\pi}{18}\right]$

Ex 2.36 Suppose z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in the circle $|z|=2$. If $z_1 = 1+i\sqrt{3}$, then find z_2 and z_3.

$|z|=2$ represent the circle with centre $(0,0)$ and radius 2.

z_1, z_2, z_3 are the vertices of an equilateral triangle.

The Side AB, BC, CA subtends $\frac{2\pi}{3}$ radians (120°) at the centre.

Note: The complex number $ze^{i\alpha}$ is a rotation of z by α radians in the counter clockwise

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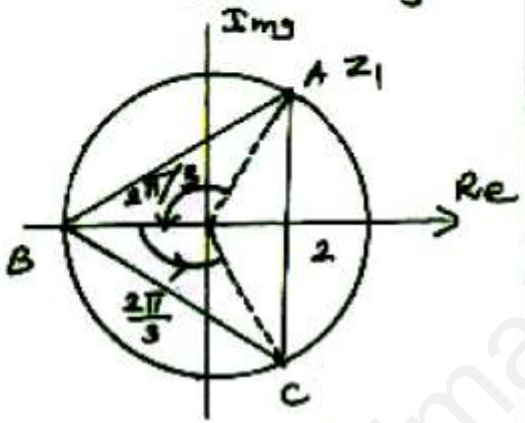
Direction about the origin.

∴ z_2 is obtain by rotation of z_1 by $\frac{2\pi}{3}$ and z_3 is obtained by rotation of z_1 by $\frac{4\pi}{3}$ or z_2 by $2\pi/3$



Given

$OA = z_1 = 1+i\sqrt{3}$

$OB = z_1 e^{i\frac{2\pi}{3}}$

$= (1+i\sqrt{3})\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$

$= (1+i\sqrt{3})\left[\cos\left(\pi-\frac{\pi}{3}\right) + i\sin\left(\pi-\frac{\pi}{3}\right)\right]$

$= (1+i\sqrt{3})\left[-\cos\pi/3 + i\sin\pi/3\right]$

$= (1+i\sqrt{3})\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= -\frac{1}{2} + i\frac{\sqrt{3}}{2} - i\frac{\sqrt{3}}{2} - \frac{3}{2}$

$= -\frac{1}{2} - \frac{3}{2} = \boxed{-2}$

$OC = z_1 e^{i\frac{4\pi}{3}}$ or $z_2 e^{i\frac{2\pi}{3}}$

$= -2\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$

$= -2\left[-\cos\pi/3 + i\sin\pi/3\right]$

$= -2\left[-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$

$= \boxed{1 - i\sqrt{3}}$

EXERCISE 2.8

1) $w \neq 1$ is a cube root of unity, Then Show that $\frac{a+bw+cw^2}{b+cw+aw^2} + \frac{a+bw+cw^2}{c+aw+bw^2} = -1$

$LHS = \left(\frac{a+bw+cw^2}{b+cw+aw^2}\right) + \left(\frac{a+bw+cw^2}{c+aw+bw^2}\right)$

$= \frac{w^2(a+bw+cw^2)}{w^2(b+cw+aw^2)} + \frac{w^2(a+bw+cw^2)}{w^2(c+aw+bw^2)}$

$= \frac{aw^2+bw^3+cw^4}{w^2(b+cw+aw^2)} + \frac{w^2(a+bw+cw^2)}{cw^2+aw^3+bw^4}$

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$$= \frac{a\omega^2 + b\omega + c}{\omega^2(b + c\omega + a\omega^2)} + \frac{\omega^2(a + b\omega + c\omega^2)}{(c\omega^2 + a + b\omega)}$$

$$= \frac{1}{\omega^2} + \frac{\omega^2}{1}$$

$$= \frac{1 + \omega^4}{\omega^2} = \frac{1+\omega}{\omega^2} \qquad \because 1 + \omega + \omega^2 = 0,\ \omega^3 = 1$$

$$= \frac{1+\omega}{-(1+\omega)} \qquad \because \omega^2 = -(1+\omega)$$

$$= -1$$

$$= \text{RHS}$$

2) Show that $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5 = -\sqrt{3}$

polar form of $\frac{\sqrt{3}}{2} + \frac{i}{2} = r(\cos\theta + i\sin\theta)$

$$r = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{\frac{4}{4}} = 1$$

$$\alpha = \tan^{-1}\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi/6.$$

$\frac{\sqrt{3}}{2} + \frac{i}{2}$ lies in 1st quadrant.

$\therefore \theta = \alpha = \pi/6$.

$$\therefore \frac{\sqrt{3}}{2} + \frac{i}{2} = \cos\pi/6 + i\sin\pi/6$$

$$\text{W.K.T } \frac{\sqrt{3}}{2} - \frac{i}{2} = \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}$$

$$\text{LHS} = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$$

$$= \left(\cos\pi/6 + i\sin\frac{\pi}{6}\right)^5 + \left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^5$$

$$= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} + \cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}$$

$$= 2\cos\frac{5\pi}{6}$$

$$= 2\cos(\pi - \pi/6)$$

$$= -2\cos(\pi/6) = -2\frac{\sqrt{3}}{2}$$

$$= \boxed{-\sqrt{3}}$$

$$= \text{RHS}.$$

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3) Find the value of $\left(\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right)^{10}$

Let $z = \sin\frac{\pi}{10} + i\cos\frac{\pi}{10}$.

$\therefore \frac{1}{z} = \sin\frac{\pi}{10} - i\cos\frac{\pi}{10}$

$\therefore \left(\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right)^{10} = \left(\frac{1+z}{1+\frac{1}{z}}\right)^{10}$

$= \left(\frac{1+z}{\frac{z+1}{z}}\right)^{10} = z^{10}$

$= \left(\sin\frac{\pi}{10} + i\cos\frac{\pi}{10}\right)^{10}$

$= i^{10}\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{10}$.

$= i^8 \cdot i^2\left(\cos\pi - i\sin\pi\right)$

$= -1\left(-1 - i(0)\right) = 1$ //

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4) If $2\cos\alpha = x + \frac{1}{x}$ and $2\cos\beta = y + \frac{1}{y}$ show that

(i) $\frac{x}{y} + \frac{y}{x} = 2\cos(\alpha-\beta)$

$2\cos\alpha = x + \frac{1}{x}$

$2\cos\alpha = \frac{x^2+1}{x}$

$x^2 + 1 = 2\cos\alpha\, x$

$x^2 - 2\cos\alpha\, x + 1 = 0$

$x = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4(1)(1)}}{2}$

$= \frac{2\cos\alpha \pm \sqrt{4(\cos^2\alpha - 1)}}{2}$

$= \frac{2\cos\alpha \pm \sqrt{-4\sin^2\alpha}}{2}$

$= \frac{2\cos\alpha \pm 2\sin^2\alpha\, i}{2}$

$= \cos\alpha \pm i\sin\alpha$

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$x = \cos\alpha + i\sin\alpha = e^{i\alpha}$

and $y = \cos\beta + i\sin\beta = e^{i\beta}$

$$\frac{x}{y} = \frac{e^{i\alpha}}{e^{i\beta}} = e^{i\alpha - i\beta} \cdot e$$

$$= e^{i(\alpha-\beta)}$$

$$= \cos(\alpha-\beta) + i\sin(\alpha-\beta) \quad \text{---} (1)$$

$$\frac{y}{x} = \frac{1}{\frac{x}{y}} = \frac{1}{\cos(\alpha-\beta) + i\sin(\alpha-\beta)}$$

$$= [\cos(\alpha-\beta) + i\sin(\alpha-\beta)]^{-1}$$

$$= \cos(\alpha-\beta) - i\sin(\alpha-\beta) \quad \text{---} (2)$$

$$\frac{x}{y} + \frac{y}{y} = 2\cos(\alpha-\beta)$$

(ii) $xy - \frac{1}{xy} = 2i\sin(\alpha+\beta)$

$xy = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$= e^{i\alpha} \cdot e^{i\beta} = e^{i(\alpha+\beta)}$

$= \cos(\alpha+\beta) + i\sin(\alpha+\beta) \quad \text{---} (1)$

$$\frac{1}{xy} = [\cos(\alpha+\beta) + i\sin(\alpha+\beta)]^{-1}$$

$$= \cos(\alpha+\beta) - i\sin(\alpha+\beta) \quad \text{---} (2)$$

$$xy - \frac{1}{xy} = \cos(\alpha+\beta) + i\sin(\alpha+\beta) - [\cos(\alpha+\beta) - i\sin(\alpha+\beta)]$$

$$= \cos(\alpha+\beta) + i\sin(\alpha+\beta) - \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$$= 2i\sin(\alpha+\beta)$$

(iii) $\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2i\sin(m\alpha - n\beta)$

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$x = \cos\alpha + i\sin\alpha = e^{i\alpha}$

$y = \cos\beta + i\sin\beta = e^{i\beta}$

$x^m = (\cos\alpha + i\sin\alpha)^m = (e^{i\alpha})^m$

$= e^{i\alpha m}$

$y^n = (e^{i\beta})^n = e^{in\beta}$.

$\frac{x^m}{y^n} = x^m \cdot y^{-n}$

$= e^{i\alpha m - in\beta} \cdot y$

$= e^{i(\alpha m - n\beta)}$

$= \cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)$. ——— (1)

$\frac{y^n}{x^m} = \frac{1}{x^m/y^n} = \left(\frac{x^m}{y^n}\right)^{-1}$

$= [\cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)]^{-1}$

$= \cos(m\alpha - n\beta) - i\sin(m\alpha - n\beta)$. ——— (2)

(1) − (2) ⇒ $\frac{x^m}{y^n} - \frac{y^n}{x^m}$

$= \cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)$

$- [\cos(m\alpha - n\beta) - i\sin(m\alpha - n\beta)]$

$= \cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)$

$- \cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)$

$= 2i\sin(m\alpha - n\beta)$

(iv) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$

$x = e^{i\alpha}$, $y = e^{i\beta}$

$x^m y^n = (e^{i\alpha})^m \cdot (e^{i\beta})^n$

$= e^{i\alpha m} \cdot e^{i\beta n} = e^{i(\alpha m + \beta n)}$

$\frac{1}{x^m y^n} = e^{-i(\alpha m + \beta n)}$.

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$x^m y^n = \cos(\alpha m + \beta n) + i \sin(\alpha m + \beta n)$

$\frac{1}{x^m y^n} = \cos(\alpha m + \beta n) - i \sin(\alpha m + \beta n)$

$x^m y^n + \frac{1}{x^m y^n} = 2\cos(\alpha m + \beta n)$.

Note:

$1 = \cos 0 + i \sin 0$

$-1 = \cos \pi + i \sin \pi$

$i = \cos \pi/2 + i \sin \pi/2$

$-i = \cos(-\pi/2) + i \sin(-\pi/2)$

---

5) Solve the eqn $z^3 + 27 = 0$

$z^3 + 27 = 0$

$\Rightarrow z^3 = -27 = 27(-1)$

$z^3 = 3^3(-1)$

$z = 3(-1)^{\frac{1}{3}}$

$= 3[\cos \pi + i \sin \pi]^{\frac{1}{3}}$

$= 3\left[\cos(2k\pi + \pi) + i \sin(2k\pi + \pi)\right]^{\frac{1}{3}}$

$= 3\left[\cos\frac{(2k\pi + \pi)}{3} + i \sin\frac{(2k\pi + \pi)}{3}\right]$

$k = 0, 1, 2$

$k = 0,\ z = 3\left[\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right] = 3\left[\frac{1}{2} + \frac{\sqrt{3}}{2}\right]$

$k = 1,\ z = 3\left[\cos\frac{3\pi}{3} + i \sin\frac{3\pi}{3}\right]$

$= 3[\cos \pi + i \sin \pi]$

$= 3(-1) = -3$

$k = 2,\ z = 3\left[\cos\left(5\frac{\pi}{3}\right) + i \sin\frac{5\pi}{3}\right]$

$= 3[\cos(2\pi - \pi/3) + i \sin(2\pi - \pi/3)]$

$= 3[\cos \pi/3 - i \sin \pi/3]$

$= 3\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$ //

---

6) If $w \neq 1$ is a cube root of unity, show that the roots of the eqn $(z-1)^3 + 8 = 0$ are $-1, 1-2w, 1-2w^2$.

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$(z-1)^3 + 8 = 0$

$(z-1)^3 = -8 = 8(-1)$

$z-1 = 8^{\frac{1}{3}}(-1)^{\frac{1}{3}}$

$= 2\left[\cos\pi + i\sin\pi\right]^{\frac{1}{3}}$

$= 2\left[\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)\right]^{\frac{1}{3}}$

$= 2\left[\cos\left(\frac{2k\pi+\pi}{3}\right) + i\sin\left(\frac{2k\pi+\pi}{3}\right)\right], k=0,1,2$

When $k=0$,

$z-1 = 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$

$= 2\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right] = (1+i\sqrt{3})$

$z-1 = 1+i\sqrt{3}$

$\boxed{z = 2+i\sqrt{3}}$

when $k=1$, $2\left[\cos\pi + i\sin\pi\right] = 2(-1) = -2$

$\therefore z-1 = -2 \Rightarrow \boxed{z=-1}$

When $k=2$,

$z-1 = 2\left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right]$

$= 2\left[\cos\left(2\pi-\frac{\pi}{3}\right) + i\sin\left(2\pi-\frac{\pi}{3}\right)\right]$

$= 2\left[\cos\pi/3 - i\sin\pi/3\right]$

$= 2\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right] = 1-i\sqrt{3}$

$z-1 = 1-i\sqrt{3}$

$\boxed{z = 2-i\sqrt{3}}$

$\therefore$ The value of z are

$2+i\sqrt{3}, -1, 2-i\sqrt{3}$

namely $-1, 1-2w, 1-2w^2$

---

7) Find the value of

$\sum_{k=1}^{8}\left(\cos\frac{2k\pi}{9} + i\sin\frac{2k\pi}{9}\right)$

Let $w = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$

Then $1, w, w^2, w^3, \ldots w^8$ are 9th root of unity

$\therefore 1+w+w^2+\ldots+w^8 = 0$

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$\omega^2 + \omega^8 = -1$

ii) $\sum_{k=1}^{8} \left(\cos\frac{2k\pi}{9} + i\sin\frac{2k\pi}{9}\right) = -1$

---

8) If $\omega \neq 1$ is a cube root of unity, show that

(i) $(1-\omega+\omega^2)^6 + (1+\omega-\omega^2)^6 = 128$

LHS $= (1-\omega+\omega^2)^6 + (1+\omega-\omega^2)^6$

$= (-\omega-\omega)^6 + (-\omega^2-\omega^2)^6$

$= (-2\omega)^6 + (-2\omega^2)^6$

$= (-2)^6\{\omega^6 + \omega^{12}\}$

$= 64\{(\omega^3)^2 + (\omega^3)^4\}$

$= 64\{1+1\} = 64(2)$

$= 128$

(ii) $(1+\omega)(1+\omega^2)(1+\omega^4)\cdots(1+\omega^{2^n}) = 1$

LHS $= (1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\cdots(1+\omega^{2^n})$

$= (1+\omega^{3^0})(1+\omega^{2^1})(1+\omega^{2^2})(1+\omega^{2^3})\cdots(1+\omega^{2^n})$

$= \{(1+\omega)(1+\omega^2)\}\{(1+\omega)(1+\omega^2)\}$ 6 times

$= \{(-\omega^2)(-\omega)\}\{(-\omega^2)(-\omega)\}$ 6 times

$= \omega^3 \cdot \omega^3 \ldots$ 6 times

$= 1\cdot 1\cdot 1 \cdots$ 6 times

$= 1$

---

9) If $z = 2-2i$, find the rotation of z by α radians in the counter clockwise direction about the origin when

a) $\alpha = \frac{\pi}{3}$

$z = 2-2i = r(\cos\theta + i\sin\theta)$

$r = \sqrt{4+4} = 2\sqrt{2}$

$2-2i$ lies 4th quadrant

$\alpha = \tan^{-1}(1) = \pi/4$

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$\therefore \quad \theta = -\alpha = -\pi/4$

$\therefore \quad z = r(\cos\theta + i\sin\theta)$

$= 2\sqrt{2}\left(\cos(-\pi/4) + i\sin(-\pi/4)\right)$

$=$

a) $\theta = \pi/3$

$z e^{i\pi/3} = (2-2i)\left(\cos\pi/3 + i\sin\pi/3\right)$

$= 2\sqrt{2}\left(\cos(-\pi/4) + i\sin(-\pi/4)\right)\left(\cos\pi/3 + i\sin\pi/3\right)$

$= 2\sqrt{2}\left[\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\right]$

$= 2\sqrt{2}\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right]$

b) $\theta = \frac{2\pi}{3}$

$z e^{i2\pi/3} = 2\sqrt{2}\left[\cos(-\pi/4) + i\sin\left(-\frac{\pi}{4}\right)\right]\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$

$= 2\sqrt{2}\left[\cos\left(\frac{2\pi}{3} - \frac{\pi}{4}\right) + i\sin\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)\right]$

$= 2\sqrt{2}\left[\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right]$

c) $\theta = \frac{3\pi}{2}$

$z e^{i\frac{3\pi}{2}} = 2\sqrt{2}\left[\cos(-\pi/4) + i\sin(-\pi/4)\right] \times \left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right]$

$= 2\sqrt{2}\left[\cos\left(\frac{3\pi}{2} - \frac{\pi}{4}\right) + i\sin\left(\frac{3\pi}{2} - \frac{\pi}{4}\right)\right]$

$= 2\sqrt{2}\left[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right]$ //

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10) Prove that the values of $\sqrt[4]{-1}$ are $\pm \frac{1}{\sqrt{2}}(1 \pm i)$.

Let $z = (-1)^{\frac{1}{4}}$

$-1 = \cos\pi + i\sin\pi$

$(-1)^{\frac{1}{4}} = [\cos\pi + i\sin\pi]^{\frac{1}{4}}$

$= [\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)]^{\frac{1}{4}}$

$= \cos\left(\frac{2k\pi+\pi}{4}\right) + i\sin\left(\frac{2k\pi+\pi}{4}\right)$ $k = 0,1,2,3$

when $k=0$, $z = \cos\pi/4 + i\sin\pi/4$

$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \boxed{\frac{1}{\sqrt{2}}(1+i)}$

when $k=1$, $z = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$= \cos(\pi - \pi/4) + i\sin(\pi - \pi/4)$

$= -\cos\pi/4 + i\sin\pi/4$

$= -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \boxed{\frac{-1}{\sqrt{2}}(1-i)}$

when $k=2$,

$z = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$

$= \cos(\pi + \frac{\pi}{4}) + i\sin(\pi + \frac{\pi}{4})$

$= -\cos\pi/4 - i\sin\pi/4$

$= -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} = \boxed{\frac{-1}{\sqrt{2}}(1+i)}$

when $k=3$,

$z = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$

$= \cos(2\pi - \frac{\pi}{4}) + i\sin(2\pi - \frac{\pi}{4})$

$= +\cos\pi/4 - i\sin\pi/4$

$= \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} = \boxed{\frac{1}{\sqrt{2}}(1-i)}$

$\therefore$ The roots are

$\frac{1}{\sqrt{2}}(1+i)$, $-\frac{1}{\sqrt{2}}(1-i)$, $-\frac{1}{\sqrt{2}}(1+i)$, $\frac{1}{\sqrt{2}}(1-i)$

ie $\pm\frac{1}{\sqrt{2}}(1 \pm i)$

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Chapter 3 – Theory of Equations

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CHAPTER - 3

Theory of Equations

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is called polynomial Equation or polynomial function.

The eqn $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$ is called polynomial eqn.

Quadratic eqn:

For a quadratic eqn $ax^2 + bx + c = 0$, $b^2 - 4ac$ is called the discriminant and is denoted by Δ.

1) $\Delta = 0 \iff$ The roots are equal
2) $\Delta > 0 \iff$ the roots are real and distinct.
3) $\Delta < 0 \iff$ The roots are imaginary or no real roots.

Let α and β be the roots of the quadratic eqn $ax^2 + bx + c = 0$

$\therefore \alpha + \beta = -\frac{b}{a}$

and $\alpha\beta = \frac{c}{a}$

* A quadratic eqn whose roots are α and β is

$x^2 - (\text{Sum of the roots})x + \text{product of the roots} = 0$.

Ex 3.1 If α and β are the roots of the eqn $17x^2 + 43x - 73 = 0$, construct a quadratic eqn whose roots are $\alpha+2$, $\beta+2$.

Given α and β are roots of $17x^2 + 43x - 73 = 0$

$\therefore \alpha + \beta = -\frac{b}{a} = -\frac{43}{17}$

$\alpha\beta = \frac{c}{a} = -\frac{73}{17}$

$\alpha+2$, $\beta+2$ are roots of the required eqn.

$\therefore$ Sum of the roots $= \alpha+2+\beta+2$

$= \alpha+\beta+4$

$= -\frac{43}{17} + 4$

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$= \frac{-43+68}{17} = \frac{25}{17}$

product of the roots $= (\alpha+2)(\beta+2)$

$= \alpha\beta + 2(\alpha+\beta) + 4$

$= \frac{-73}{17} + 2\left(\frac{-43}{17}\right) + 4$

$= \frac{-73 - 86 + 68}{17}$

$= \frac{-159+68}{17} = \frac{-91}{17}$.

∴ The required eqn is

$x^2 - Sx + P = 0$

i.e. $x^2 - \frac{25}{17}x - \frac{91}{17} = 0$

i.e. $\boxed{17x^2 - 25x - 91 = 0}$

Ex 3.2 If α and β are the roots of $2x^2 - 7x + 13 = 0$, construct a quadratic eqn whose roots are α^2 and β^2.

α and β are roots of $2x^2 - 7x + 13 = 0$

∴ $\alpha + \beta = -\frac{b}{a} = \frac{7}{2}$

$\alpha\beta = \frac{c}{a} = \frac{13}{2}$

α^2 and β^2 are roots of the required eqn.

∴ $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= \left(\frac{7}{2}\right)^2 - 2\left(\frac{13}{2}\right)$

$= \frac{49}{4} - 13 = \frac{49-52}{4}$

$= \boxed{\frac{-3}{4}}$

$\alpha^2\beta^2 = (\alpha\beta)^2$

$= \left(\frac{13}{2}\right)^2 = \frac{169}{4}$

∴ The required eqn is

$x^2 - Sx + P = 0$

i.e. $x^2 + \frac{3}{4}x + \frac{169}{4} = 0$

$\boxed{4x^2 + 3x + 169 = 0}$

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Vieta's formula for polynomial eqn of degree 3.

Consider the cubic eqn

$ax^3 + bx^2 + cx + d = 0$.

It has 3 roots namely α, β and γ.

$\therefore ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma)$

$\therefore ax^3 + bx^2 + cx + d$

$= a\{x^2 - (\alpha+\beta)x + \alpha\beta\}(x-\gamma)$

$= a\{x^3 - x^2\gamma - (\alpha+\beta)x^2 + (\alpha+\beta)\gamma x + \alpha\beta x - \alpha\beta\gamma\}$

$= a\{x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\alpha\gamma)x - \alpha\beta\gamma\}$

$= ax^3 - a(\alpha+\beta+\gamma)x^2 + a(\alpha\beta+\beta\gamma+\alpha\gamma)x - a(\alpha\beta\gamma)$

comparing the coefficient.

$\alpha+\beta+\gamma = -b/a$

$\alpha\beta+\beta\gamma+\alpha\gamma = \frac{c}{a}$

$\alpha\beta\gamma = -\frac{d}{a}$.

Vieta's formula for polynomial eqn of degree $n > 3$

$\sum_1'$ = Coefficient of $x^{n-1} = -\sum_1' \alpha_1$

$\sum_2'$ = coefficient of $x^{n-2} = \sum_2' \alpha_1\alpha_2$

$\sum_3'$ = Coefficient of $x^{n-3} = -\sum_3' \alpha_1\alpha_2\alpha_3$

$\sum_{n-1}'$ = coefficient of $x = (-1)^n \sum' \alpha_1\alpha_2\cdots\alpha_{n-1}$

coefficient of x^0 = constant term

$= (-1)^n \sum' \alpha_1\alpha_2\cdots\alpha_{n-1}$

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Ex 3.3 If α, β, γ are the roots of the eqn $x^3+px^2+qx+r=0$, find the value of $\sum \frac{1}{\beta\gamma}$ in terms of the coefficients.

α, β, γ are roots of $x^3+px^2+qx+r=0$

Here a = coeff of $x^3 = 1$

$$\Sigma_1 = \alpha+\beta+\gamma = \frac{-p}{1} = -p$$

$$\Sigma_2 = \alpha\beta+\beta\gamma+\gamma\alpha = \frac{q}{1} = q$$

$$\Sigma_3 = \alpha\beta\gamma = \frac{-r}{1} = -r$$

$$\therefore \sum \frac{1}{\beta\gamma} = \frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\alpha\gamma}$$

$$= \frac{\gamma+\alpha+\beta}{\alpha\beta\gamma}$$

$$= \frac{-p}{-r} = \boxed{\frac{p}{r}}$$

Ex 3.4

Find the sum of the squares of the roots of $ax^4+bx^3+cx^2+dx+e=0$.

Let α, β, γ and δ be the roots of $ax^4+bx^3+cx^2+dx+e=0$.

$$\Sigma_1 = \alpha+\beta+\gamma+\delta = -\frac{b}{a}$$

$$\Sigma_2 = \alpha\beta+\beta\gamma+\alpha\delta = \frac{c}{a}$$

$$\Sigma_3 = \alpha\beta\gamma+\alpha\beta\delta+\alpha\delta\gamma+\beta\gamma\delta = -\frac{d}{a}$$

$$\Sigma_4 = \alpha\beta\gamma\delta = e/a$$

To find $\alpha^2+\beta^2+\gamma^2+\delta^2$

WKT

$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2(ab+bc+cd+ad)$$

$$\therefore \alpha^2+\beta^2+\gamma^2+\delta^2 = (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta+\beta\gamma+\gamma\delta+\alpha\delta)$$

$$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{d}{a}\right)$$

$$= \frac{b^2}{a^2} - \frac{2d}{a} = \frac{b^2-2ad}{a^2} \;//$$

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3.5 Find the condition that the roots of cubic eqn $x^3+ax^2+bx+c=0$ are in the ratio $p:q:r$.

Ratio of roots of $x^3+ax^2+bx+c=0$ is $p:q:r$

$\therefore$ The roots are $p\lambda, q\lambda, r\lambda$.

Now,

$\Sigma_1' = p\lambda+q\lambda+r\lambda = -a$ — ①

$\Sigma_2' = pq\lambda^2+qr\lambda^2+pr\lambda^2 = b$ — ②

$\Sigma_3' = pqr\lambda^3 = -c$ — ③

From ① $\lambda(p+q+r) = -a$

$\lambda = \frac{-a}{p+q+r}$ — ④

From ③ $\lambda^3 = \frac{-c}{pqr}$ — ⑤

From ④ and ⑤

$\left(\frac{-a}{p+q+r}\right)^3 = \frac{-c}{pqr}$

$\frac{-a^3}{(p+q+r)^3} = -\frac{c}{pqr}$

$c(p+q+r)^3 = a^3pqr$

Ex 3.6 Form the eqn whose roots are the squares of the roots of the cubic eqn $x^3+ax^2+bx+c=0$.

Let α, β, γ be the roots of $x^3+ax^2+bx+c=0$.

$\therefore \Sigma_1' = \alpha+\beta+\gamma = -a$ — ①

$\Sigma_2' = \alpha\beta+\beta\gamma+\alpha\gamma = b$ — ②

$\Sigma_3' = \alpha\beta\gamma = -c$ — ③

Now $\Sigma_1' = \alpha^2+\beta^2+\gamma^2$

$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= (-a)^2 - 2(b)$

$= \boxed{a^2-2b}$

$\Sigma_2' = \alpha^2\beta^2+\beta^2\gamma^2+\alpha^2\gamma^2$

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$= (\alpha\beta+\beta\gamma+\alpha\gamma)^2 - 2[(\alpha\beta)(\beta\gamma)+(\beta\gamma)(\alpha\gamma)+(\alpha\beta)(\alpha\gamma)]$

$= (\alpha\beta+\beta\gamma+\alpha\gamma)^2 - 2[\alpha^2\beta\gamma + \beta^2\alpha\gamma + \gamma^2\alpha\beta]$

$= (\alpha\beta+\beta\gamma+\alpha\gamma)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$

$= b^2 - 2(-c)(a)$

$= b^2 + 2ac.$

$\Sigma'_3 = \alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = (-c)^2 = c^2$

Hence the required eqn is

$x^3 - \Sigma'_1 x^2 + \Sigma'_2 x - \Sigma'_3 = 0.$

$\therefore x^3 - (a^2-2b)x^2 + (b^2+2ac)x - c^2 = 0$

Ex 3.7 If p is real, discuss the nature of the roots of the eqn $4x^2 + 4px + p + 2 = 0$, in terms of p.

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The eqn is $4x^2 + 4px + p + 2 = 0.$

$\Delta = b^2 - 4ac$

$= (4p)^2 - 4(4)(p+2)$

$= 16p^2 - 16(p+2)$

$= 16p^2 - 16p - 32$

$= 16(p^2 - p - 2)$

$= 16(p+1)(p-2)$

$\Delta < 0 \Rightarrow (p+1)(p-2) < 0$

$\Rightarrow -1 < p < 2$

$\Delta = 0 \Rightarrow p = -1$ or $p = 2$

$\Delta > 0 \Rightarrow -\infty < p < -1$ or $2 < p < \infty.$

∴ Roots are imaginary if $-1 < p < 2$

roots are equal if $p = -1$ or $p = 2$

roots are real if $-\infty < p < -1$ or $2 < p < \infty.$

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EXERCISE 3.1

1. If the sides of a cubic box are increased by 1, 2, 3 unit respectively to form a cuboid. Then the volume of the cube increased by 52 cubic units. Find the volume of the cuboid.

Let x be the side of the cube.

$\therefore$ Volume of the cube $= x^3$.

By data $(x+1)(x+2)(x+3) = x^3+52$

$\Rightarrow (x+1)(x^2+5x+6) = x^3+52$

$x^3+5x^2+6x+x^2+5x+6 = x^3+52$

$6x^2+11x+6-52=0$

$6x^2+11x-46=0$.

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$-12 \quad +23$

$6 \quad 6$

$-2 \quad \frac{23}{6}$

$(x-2)(6x+23)=0$

$x-2=0 \mid 6x+23=0$

$x=2 \mid x=-\frac{23}{6}$

$x=-\frac{23}{6}$ is not possible.

$\therefore x=2$.

Volume of the cuboid $= x^3+52$

$= 2^3+52$

$= 8+52$

$= 60$ cubic units.

---

2) Construct a cubic eqn with roots are

(i) 1, 2, 3

Given roots are $\alpha=1, \beta=2, \gamma=3$

$\therefore \Sigma_1 = \alpha+\beta+\gamma = 1+2+3 = 6$

$\Sigma_2 = \alpha\beta+\beta\gamma+\alpha\gamma = 2+6+3 = 11$

$\Sigma_3 = \alpha\beta\gamma = 1(2)(3) = 6$.

$\therefore$ The required eqn is.

$x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$\boxed{x^3 - 6x^2 + 11x - 6 = 0.}$

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(ii) $1, 1, -2$

Given roots are $\alpha=1, \beta=1, \gamma=-2$

$\therefore \Sigma_1 = \alpha+\beta+\gamma = 1+1-2 = 0$

$\Sigma_2 = \alpha\beta+\beta\gamma+\alpha\gamma = 1-2-2 = -3$

$\Sigma_3 = \alpha\beta\gamma = 1(1)(-2) = -2$

$\therefore$ The required cubic eqn is

$x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

i.e. $x^3 - 0x^2 + (-3)x - (-2) = 0$

$\boxed{x^3 - 3x + 2 = 0}$.

(iii) $2, \frac{1}{2}, 1$

Given roots are $\alpha=2, \beta=\frac{1}{2}, \gamma=1$

$\Sigma_1 = \alpha+\beta+\gamma = 2+\frac{1}{2}+1 = \frac{7}{2}$

$\Sigma_2 = \alpha\beta+\beta\gamma+\alpha\gamma = 2(\frac{1}{2}) + \frac{1}{2}(1) + 2(1)$

$= 1+\frac{1}{2}+2 = \frac{7}{2}$

$\Sigma_3 = \alpha\beta\gamma = 2(\frac{1}{2})(1) = 1$

$\therefore$ The required cubic eqn is

$x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$x^3 - \frac{7}{2}x^2 + \frac{7}{2}x - 1 = 0$

$2x^3 - 7x^2 + 7x - 1 = 0$.

---

3) If α, β, γ are the roots of the cubic eqn $x^3+2x^2+3x+4=0$ form a cubic eqn whose roots are

(i) $2\alpha, 2\beta, 2\gamma$.

Given α, β, γ are roots of

$x^3+2x^2+3x+4=0$

$\Sigma_1 = \alpha+\beta+\gamma = -2$

$\Sigma_2 = \alpha\beta+\beta\gamma+\gamma\alpha = 3$

$\Sigma_3 = \alpha\beta\gamma = -4$

Roots of required eqn are $2\alpha, 2\beta, 2\gamma$.

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$\therefore \Sigma_1 = 2\alpha+2\beta+2\gamma = 2(\alpha+\beta+\gamma)$
$= 2(-2) = -4$

$\Sigma_2 = (2\alpha)(2\beta)+(2\beta)(2\gamma)+(2\alpha)(2\gamma)$
$= 4\alpha\beta + 4\beta\gamma + 4\alpha\gamma$
$= 4(\alpha\beta+\beta\gamma+\alpha\gamma)$
$= 4(3) = 12$

$\Sigma_3 = (2\alpha)(2\beta)(2\gamma)$
$= 8\alpha\beta\gamma = 8(-4) = -32$

$\therefore$ The required cubic eqn is
$x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

i.e. $x^3 - (-4)x^2 + 12x - (-32) = 0$

$\boxed{x^3 + 4x^2 + 12x + 32 = 0}$

(ii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

$\Sigma_1 = \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$
$= \boxed{\frac{3}{-4}}$

$\Sigma_2 = \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)+\left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right)+\left(\frac{1}{\alpha}\right)\left(\frac{1}{\gamma}\right)$
$= \frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\alpha\gamma} = \frac{\gamma+\alpha+\beta}{\alpha\beta\gamma}$
$= \frac{-2}{-4} = \boxed{\frac{1}{2}}$

$\Sigma_3 = \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right) = \frac{1}{\alpha\beta\gamma} = \boxed{-\frac{1}{4}}$

$\therefore$ The required eqn is
$x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$.

i.e. $x^3 - \left(-\frac{3}{4}\right)x^2 + \frac{1}{2}x - \left(-\frac{1}{4}\right) = 0$

$x^3 + \frac{3x^2}{4} + \frac{x}{2} + \frac{1}{4} = 0$

$\frac{4x^3+3x^2+2x+1}{4} = 0$

$\Rightarrow \boxed{4x^3+3x^2+2x+1 = 0}$

(iii) $-\alpha, -\beta, -\gamma$.

$\Sigma_1 = -\alpha-\beta-\gamma = -(\alpha+\beta+\gamma)$
$= -(-2) = 2$.

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$\Sigma_2' = \alpha\beta + \beta\gamma + \alpha\gamma = 3$

$\Sigma_3' = (-\alpha)(-\beta)(-\gamma) = -\alpha\beta\gamma$

$= -(-4) = 4$

∴ The required eqn is

$x^3 - \Sigma_1' x^2 + \Sigma_2' x - \Sigma_3' = 0$

i.e. $x^3 - 2x^2 + 3x - 4 = 0$.

---

4) Solve the eqn $3x^3 - 16x^2 + 23x - 6 = 0$ if the product of two roots is 1.

The given eqn is $3x^3 - 16x^2 + 23x - 6 = 0$.

Let the roots be $\alpha, \frac{1}{\alpha}, \beta$.

$\Sigma_1' = \alpha + \frac{1}{\alpha} + \beta = -\frac{(-16)}{3} = \frac{16}{3}$ — ①

$\Sigma_2' = \alpha\left(\frac{1}{\alpha}\right) + \frac{1}{\alpha}(\beta) + \alpha\beta = \frac{23}{3}$

$= 1 + \frac{\beta}{\alpha} + \alpha\beta = \frac{23}{3}$ — ②

$\Sigma_3' = \alpha\left(\frac{1}{\alpha}\right)(\beta) = -\frac{(-6)}{3} = 2$ — ③

i.e. $\boxed{\beta = 2}$

Sub $\beta = 2$ in ①

$\alpha + \frac{1}{\alpha} + 2 = \frac{16}{3}$

$\frac{\alpha^2+1}{\alpha} = \frac{16}{3} - 2$

$\frac{\alpha^2+1}{\alpha} = \frac{16-6}{3}$

$\frac{\alpha^2+1}{\alpha} = \frac{10}{3}$

$3\alpha^2 + 3 = 10\alpha$

$3\alpha^2 - 10\alpha + 3 = 0$

$(\alpha - 3)(3\alpha - 1) = 0$

$\alpha = 3$ or $\alpha = \frac{1}{3}$

9

$-\frac{9}{3}$ $-\frac{1}{3}$

-3 $-\frac{1}{3}$

∴ The roots are $3, \frac{1}{3}, 2$.

---

5) Find the sum of squares of roots of the eqn $2x^4 - 8x^3 + 6x^2 - 3 = 0$

The given eqn is $2x^4 - 8x^3 + 6x^2 - 3 = 0$

Let the roots be $\alpha, \beta, \gamma, \delta$.

$\Sigma_1' = \alpha + \beta + \gamma + \delta = -\frac{(-8)}{2} = 4$

$\Sigma_2' = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{6}{2} = 3$

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$\Sigma_3 = \alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta = \frac{0}{2} = 0$

$\Sigma_4 = \alpha\beta\gamma\delta = -\frac{3}{2}$

To find $\alpha^2+\beta^2+\gamma^2+\delta^2$

$= (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta+\beta\gamma+\gamma\delta+\alpha\delta+\alpha\gamma+\beta\delta)$

$= 4^2 - 2(3) = 16-6 = 10.$

---

6) Solve the eqn $x^3-9x^2+14x+24=0$. If it is given that two of its roots are in the ratio 3:2.

The given eqn is $x^3-9x^2+14x+24=0$.

ratio of two roots are 3:2.

∴ The roots are $3\lambda, 2\lambda, \alpha$.

$\Sigma_1 = 3\lambda+2\lambda+\alpha = \frac{-(-9)}{1} = 9$ — ①

$\Sigma_2 = (3\lambda)(2\lambda)+(2\lambda)\alpha+\alpha(3\lambda) = \frac{14}{1} = 14$ — ②

$\Sigma_3 = 3\lambda(2\lambda)(\alpha) = 6\lambda\alpha = -24.$ — ③

From ① $5\lambda+\alpha = 9$ — ④.

From ③ $6\lambda^2\alpha = -24$ — ⑤.

From ⑤ $\alpha = \frac{-24}{6\lambda^2} = -\frac{4}{\lambda^2}$

∴ ④ ⇒ $5\lambda + \left(-\frac{4}{\lambda^2}\right) = 9$

$\frac{5\lambda^3-4}{\lambda^2} = 9$

$5\lambda^3-4 = 9\lambda^2$

$5\lambda^3-9\lambda^2-4 = 0.$

Let $P(\lambda) = 5\lambda^3-9\lambda^2-4$

$P(1) = 5-9-4 = -8 \neq 0$

∴ $\lambda = 1$ is not a factor

$P(2) = 5(8)-9(4)-4$

$= 40-36-4 = 40-40 = 0$

∴ $\lambda = 2$ is a factor.

∴

2	5	-9	0	-4
	0	10	2	4
	5	1	2	0

∴ The other root is

$5\lambda^2+\lambda+2=0$ has imaginary roots ∵ $\Delta = b^2-4ac$

$= 1-4(5)(2) < 0$

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$\therefore \lambda = 2$

From ④ $5(2) + \alpha = 9$

$10 + \alpha = 9$

$\alpha = 9 - 10$

$\boxed{\alpha = -1}$

$\therefore$ The roots are $3\lambda, 2\lambda, \alpha$

i.e. $3(2), 2(2), -1$

i.e. $\underline{6, 4, -1}$.

7) If α, β, γ are the roots of the polynomial eqn $ax^3 + bx^2 + cx + d = 0$, find the value of $\sum \frac{\alpha}{\beta\gamma}$ interms of the coefficients.

The given eqn is $ax^3 + bx^2 + cx + d = 0$.

$(\div a)$ $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$.

Given that α, β, γ are roots.

$\therefore \Sigma_1 = \alpha + \beta + \gamma = -b/a$

$\Sigma_2 = \alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$

$\Sigma_3 = \alpha\beta\gamma = -\frac{d}{a}$.

$$\sum \frac{\alpha}{\beta\gamma} = \frac{\alpha}{\beta\gamma} + \frac{\beta}{\gamma\alpha} + \frac{\gamma}{\alpha\beta}$$

$$= \frac{\alpha^2 + \beta^2 + \gamma^2}{\alpha\beta\gamma}$$

$$= \frac{(\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)}{\alpha\beta\gamma}.$$

$$= \frac{\left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)}{-d/a}$$

$$= \left(\frac{b^2}{a^2} - \frac{2c}{a}\right)\left(\frac{-a}{d}\right)$$

$$= \left(\frac{b^2 - 2ac}{a^2}\right)\left(-\frac{a}{d}\right)$$

$$= \frac{b^2 - 2ac}{a}\left(-\frac{1}{d}\right)$$

$$= \frac{2ac - b^2}{ad} \quad //$$

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8) If α, β, γ and δ are the roots of the polynomial eqn $2x^4+5x^3-7x^2+8=0$ find a quadratic eqn with integer coefficients whose roots are $\alpha+\beta+\gamma+\delta$ and $\alpha\beta\gamma\delta$.

The given eqn is $2x^4+5x^3-7x^2+8=0$

$\div 2$, $x^4+\frac{5}{2}x^3-\frac{7}{2}x^2+\frac{0}{2}x+4=0$

Let α, β, γ and δ be the root of the given eqn

$\therefore \alpha+\beta+\gamma+\delta = \Sigma_1 = -5/2$

$\alpha\beta\gamma\delta : \Sigma_4 = 2$

Roots of the required eqn are $\alpha+\beta+\gamma+\delta$ and $\alpha\beta\gamma\delta$.

$\therefore$ S.O.R $= \alpha+\beta+\gamma+\delta+\alpha\beta\gamma\delta$

$= -\frac{5}{2}+4 = \boxed{\frac{3}{2}}$

P.O.R $= (\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma\delta)$

$= -\frac{5}{2}\times 4 = \boxed{-10}$

$\therefore$ The required eqn is

$x^2 - S.O.R\,x + P.O.R = 0$

$x^2-\left(-\frac{3}{2}\right)x-10=0$

$2x^2+3x-20=0$

---

9) If p and q are roots of the eqn $lx^2+nx+n=0$, show that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0$.

The given eqn is $lx^2+nx+n=0$.

p and q are roots of the given eqn.

$\therefore p+q = -\frac{n}{l}$, $pq = \frac{n}{l}$

$\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}} = \frac{\sqrt{p}}{\sqrt{q}}+\frac{\sqrt{q}}{\sqrt{p}}+\frac{\sqrt{n}}{\sqrt{l}}$

$= \frac{p+q}{\sqrt{pq}}+\frac{\sqrt{n}}{\sqrt{l}}$

$= \frac{-n/l}{\sqrt{\frac{n}{l}}}+\sqrt{\frac{n}{l}} \Rightarrow$

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$$= \frac{-\frac{n}{l}+\frac{n}{l}}{\sqrt{\frac{n}{l}}} = \frac{0}{Dr} = 0.$$

---

10) If the eqns $x^2+px+q=0$ and $x^2+p'x+q'=0$ have a common roots, show that it must be equal to $\frac{pq'-p'q}{q-q'}$ or $\frac{q-q'}{p'-p}$.

Let α be the common roots of the given eqns.

$\therefore$ $\alpha^2+p\alpha+q=0$ — ①

$\alpha^2+p'\alpha+q'=0$ — ②

Solving ① and ② we get.

α^2	α	1
p	q	1 p
p'	q'	1 p'

$$\frac{\alpha^2}{pq'-p'q} = \frac{\alpha}{q-q'} = \frac{1}{p'-p}$$

$$\frac{\alpha^2}{pq'-p'q} = \frac{1}{p'-p} \Rightarrow \alpha^2 = \frac{pq'-p'q}{p'-p} \quad -①$$

$$\frac{\alpha}{q-q'} = \frac{1}{p'-p} \Rightarrow \alpha = \frac{q-q'}{p'-p} \quad -②$$

$$\frac{\alpha^2}{\alpha} = \frac{\frac{pq'-p'q}{p'-p}}{\frac{q-q'}{p'-p}} \Rightarrow \alpha = \frac{pq'-p'q}{p'-p} \times \frac{p'-p}{q-q'}$$

$$\alpha = \frac{pq'-p'q}{q-q'}$$

$\therefore$ The common root is

$$\alpha = \frac{pq'-p'q}{q-q'} \text{ or } \frac{q-q'}{p'-p}$$

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Imaginary roots:-

If a complex no z_0 is a root of a polynomial eqn with real coefficients, Then its complex conjugate $\bar{z}_0$ is also a root.

If $\sqrt{p}+\sqrt{q}$ is a root of a polynomial eqn with rational coeff, then $\sqrt{p}-\sqrt{q}$, $-\sqrt{p}+\sqrt{q}$, $-\sqrt{p}-\sqrt{q}$ are also roots of the same polynomial eqn.

Ex 3.8 Find the monic polynomial eqn of minimum degree with real coefficients having $2-\sqrt{3}i$ as a root.

$2-\sqrt{3}i$ is a root of the required polynomial.

$\therefore$ $2+\sqrt{3}i$ is also a root of the same polynomial.

Sum of the roots $= 4$

product of the roots $= (2+\sqrt{3}i)(2-\sqrt{3}i)$

$= 4+3 = 7$.

$\therefore$ The required eqn is

$x^2 - S.O.R\,x + P.O.R = 0$

$x^2 - 4x + 7 = 0$.

Ex 3.9 Find a polynomial eqn of minimum degree with rational coefficients, having $2-\sqrt{3}$ as a root.

Given $2-\sqrt{3}$ is a root

$\therefore$ $2+\sqrt{3}$ is also a root.

$S.O.R = 4$, $P.O.R = (2-\sqrt{3})(2+\sqrt{3})$

$= 4-3 = 1$

$\therefore$ The required eqn is

$x^2 - S.O.R\,x + P.O.R = 0$

i.e. $x^2 - 4x + 1 = 0$.

Ex 3.10 Form a polynomial eqn with integer coefficients with $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ as a root.

Since $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ is a root $\Rightarrow x - \sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ is a factor.

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$\therefore$ $x + \sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ is also a factor.

$\therefore$ Their product $= \left(x+\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}\right)\left(x-\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}\right)$

$= x^2 - \frac{\sqrt{2}}{\sqrt{3}}$ is a factor.

$\therefore$ $x^2 + \frac{\sqrt{2}}{\sqrt{3}}$ is also a factor.

$\therefore$ Their product $= \left(x^2-\frac{\sqrt{2}}{\sqrt{3}}\right)\left(x^2+\frac{\sqrt{2}}{\sqrt{3}}\right)$

$= x^4 - \frac{2}{3}$

$= 3x^4 - 2 = 0.$

is the required polynomial eqn.

Ex 3.11 Show that the eqn $2x^2 - 6x + 7 = 0$ cannot be satisfied by any real values of x.

Given eqn is $2x^2 - 6x + 7 = 0$

$\Delta = b^2 - 4ac = (-6)^2 - 4(2)(7)$

$= 36 - 56$

$= -20 < 0.$

$\therefore$ This eqn has imaginary roots.

Ex 3.12 If $x^2 + 2(k+2)x + 9k = 0$ has equal roots, find k.

Given eqn is $x^2 + 2(k+2)x + 9k = 0.$

$\Delta = b^2 - 4ac = [2(k+2)]^2 - 4(1)(9k)$

$= 4(k+2)^2 - 36k.$

$= 4(k^2+4+4k) - 36k$

$= 4[k^2+4+4k-9k]$

$= 4[k^2-5k+4]$

Given that roots are equal.

$\therefore$ $\Delta = 0 \Rightarrow 4[k^2-5k+4] = 0$

$\Rightarrow k^2 - 5k + 4 = 0$

$\Rightarrow (k-4)(k-1) = 0$

$\Rightarrow k = 4$ or $k = 1$.

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Ex 3.13 Show that, if p, q, r are rational, the roots of the eqn $x^2 - 2px + p^2 - q^2 + 2qr - r^2 = 0$ are rational.

The given eqn is

$$x^2 - 2px + p^2 - q^2 + 2qr - r^2 = 0$$

$$\Delta = b^2 - 4ac$$

$$= (-2p)^2 - 4(1)(p^2 - q^2 + 2qr - r^2)$$

$$= 4p^2 - 4p^2 + 4q^2 + 8qr + 4r^2$$

$$= 4(q^2 - 2qr + r^2)$$

$$= 4(q-r)^2$$

which is a perfect square.
Hence the roots are rational.

Ex 3.14 prove that a line cannot intersect a circle at more than two points.

Eqn of the circle is $x^2 + y^2 = r^2$ — (1)

Eqn of the line is $y = mx + c$ — (2)

Sub (2) in (1)

$$x^2 + (mx+c)^2 = r^2$$

$$x^2 + m^2x^2 + c^2 + 2mcx - r^2 = 0$$

$$(1+m^2)x^2 + 2mcx + c^2 - r^2 = 0$$

This is a quadratic eqn, which has only two roots.

∴ The line and a circle cannot intersect at more than two points.

EXERCISE 3.2

1) If k is real, discuss the nature of the roots of the polynomial eqn $2x^2 + kx + k = 0$ interms of k.

The given eqn is $2x^2 + kx + k = 0$

Here $a = 2, b = k, c = k$.

$$\Delta = b^2 - 4ac$$

$$= k^2 - 4(2)(k)$$

$$= k^2 - 8k$$

$$= k(k-8)$$

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a) If the roots are equal

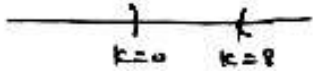
$\Delta = 0 \Rightarrow k(k-8) = 0$

$\Rightarrow k = 0$ or $k = 8$.

b) If the roots are real.

$\Delta > 0 \Rightarrow k(k-8) > 0$.

$\Rightarrow k \in (-\infty, 0)$ or $k \in (8, \infty)$



c) If the roots are imaginary

$\Delta < 0 \Rightarrow k(k-8) < 0$.

i.e. $k^2 - 8k < 0$

$k^2 < 8k$

$k \in (0, 8)$

---

2) Find a polynomial eqn of minimum degree with rational coefficients, having $2+\sqrt{3}i$ as a root.

$2+\sqrt{3}i$ is a root.

$\therefore$ $2-\sqrt{3}i$ is also a root.

S.O.R = 4, P.O.R = 4 + 3 = 7

$\therefore$ The required eqn is

$x^2 - S.O.R\,x + P.O.R = 0$

$x^2 - 4x + 7 = 0$.

---

3) Find a polynomial eqn of minimum degree with rational coefficients, having $2i+3$ as a root.

Given root is $3+2i$

$\therefore$ The other root is $3-2i$

S.O.R = 6

P.O.R = $(3+2i)(3-2i)$

= $9+4 = 13$

$\therefore$ The required eqn is

$x^2 - S.O.R\,x + P.O.R = 0$

$\boxed{x^2 - 6x + 13 = 0}$

---

4) Find a polynomial eqn of minimum degree with rational coefficients, having $\sqrt{5}-\sqrt{3}$ as a root.

one root of the eqn is $\sqrt{5}-\sqrt{3}$

$\therefore$ The other roots are

$\sqrt{5}+\sqrt{3}$, $-\sqrt{5}+\sqrt{3}$, $-\sqrt{5}-\sqrt{3}$

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The roots connected to zero are

$\sqrt{5}-\sqrt{3}$ and $\sqrt{5}+\sqrt{3}$

S.o.R $= 2\sqrt{5}$ P.o.R $= 5-3 = 2$

∴ The eqn is $x^2 - S.o.R\,x + P.o.R = 0$

i.e. $x^2 - 2\sqrt{5}x + 2 = 0$.

The roots connected to zero are

$-\sqrt{5}+\sqrt{3}$, $-\sqrt{5}-\sqrt{3}$

S.o.R $= -2\sqrt{5}$, P.o.R $= 5-3 = 2$

∴ The eqn is $x^2 - S.o.R\,x + P.o.R = 0$

i.e. $x^2 + 2\sqrt{5}x + 2 = 0$

To rationalize the coefficients with minimum degree

$(x^2 - 2\sqrt{5}x + 2)(x^2 + 2\sqrt{5}x + 2) = 0$.

$((x^2+2) - 2\sqrt{5}x)((x^2+2) + 2\sqrt{5}x) = 0$

i.e. $(x^2+2)^2 - (2\sqrt{5}x)^2 = 0$.

$x^4 + 4x^2 + 4 - 4(5)x^2 = 0$

$x^4 + 4x^2 + 4 - 20x^2 = 0$

$x^4 - 16x^2 + 4 = 0$.

5) prove that a straight line and parabola cannot interseet at more than two points.

The eqn of parabola $y^2 = 4ax$ — ①

eqn of the line $y = mx + c$ — ②.

Solving ① and ②

$(mx+c)^2 = 4ax$

$m^2x^2 + c^2 + 2mcx - 4ax = 0$

$m^2x^2 + (2mc - 4a)x + c^2 = 0$

This is a quadratic eqn, which has two solns.

Hence the line and parabola cannot interseet at more than two pts.

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Ex 3.15 If $2+i$ and $3-\sqrt{2}$ are roots of the eqn $x^6-13x^5+62x^4-126x^3+65x^2+127x-140=0$ find all the roots.

Given roots are $2+i$ and $3-\sqrt{2}$

The other roots are $2-i$ and $3+\sqrt{2}$

∴ The factors are

$$\{(x-(2+i))(x-(2-i))\}\{(x-(3-\sqrt{2}))(x-(3+\sqrt{2}))$$

$$= \{x^2-x(2-i)-x(2+i)+(2+i)(2-i)\} \times \{x^2-x(3+\sqrt{2})-x(3-\sqrt{2})+(3-\sqrt{2})(3+\sqrt{2})\}$$

$$= \{x^2-x(2-i+2+i)+4-i^2\} \times \{x^2-x(3+\sqrt{2}+3-\sqrt{2})+9-2\}$$

$$= \{x^2-x(4)+4+1\}\{x^2-6x+7\}$$

$$= (x^2-4x+5)(x^2-6x+7)$$

$$= x^4-6x^3+7x^2-4x^3+24x^2-28x+5x^2-30x+35$$

$$= x^4-10x^3+36x^2-58x+35 \quad \text{— (1)}$$

To find other roots divide the given eqn by (1).

$$x^4-10x^3+36x^2-58x+35 \;\big)\; x^6-13x^5+62x^4-126x^3+65x^2+127x-140 \;\big(\; x^2-3x-4$$

$$x^6-10x^5+36x^4-58x^3+35x^2$$

$$-3x^5+26x^4-68x^3+30x^2+127x-140$$

$$-3x^5+30x^4-108x^2+174x-105x$$

$$-4x^4+40x^2-144x^2+232x-140$$

$$-4x^4+40x^2-144x^2+232x-140$$

$$0$$

∴ The other factor is $x^2-3x-4=0$

$$(x-4)(x+1)=0$$

$$x=4 \text{ or } x=-1$$

∴ The roots are $2+i, 2-i, 3-\sqrt{2}, 3+\sqrt{2}, -1$, and 4.

Ex 3.16 Solve the eqn $x^4-9x^2+20=0$

Let $x^2=y$

∴ $x^4-9x^2+20=0 \Rightarrow y^2-9y+20=0$

$$(y-4)(y-5)=0$$

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$y-4=0$ or $y-5=0$

$y=4$ or $y=5$

i.e. $x^2=4$ | $x^2=5$

$\therefore x=\pm 2$ | $x=\pm\sqrt{5}$

∴ The required roots are

$2, -2, \sqrt{5}, -\sqrt{5}$

Ex 3.17 Solve the eqn $x^3-3x^2-33x+35=0$

The given eqn is $x^3-3x^2-33x+35=0$.

Sum of the coeff $= 1-3-33+35$

$= 36-36$

$= 0$

∴ 1 is root of the polynomial.

i.e. $x-1$ is a factor of the polynomial.

Divide the eqn by $x-1$

$$x-1 \;\overline{)\; x^3-3x^2-33x+35 \;(\; x^2-2x-35}$$

x^3-x^2

$-2x^2-33x$

$-2x^2+2x$

$-35x+35$

$-35x+35$

0

The remaining two roots are obtained by solving $x^2-2x-35=0$

$(x-7)(x+5)=0$

$x=7, -5$

∴ The roots are $1, -5, 7$.

3.18 Solve the eqn $2x^3+11x^2-9x-18=0$

The given eqn is $2x^3+11x^2-9x-18=0$

Note that,

Sum of the coeff of odd power

$= 2-9=-7$

Sum of the coeff of even power

$= 11-18=-7$

Sum of coeff of odd = Sum of coeff of even power

∴ -1 is a root of the given eqn

i.e. $x+1=0$ is a factor.

Divide the given eqn by $x+1$.

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$$x+1 \;)\; 2x^3+11x^2-9x-18 \;(\; 2x^2+9x-18$$

$2x^3+2x^2$

$9x^2-9x-18$

$9x^2+9x$

$-18x-18$

$-18x-18$

0

The other two roots are obtained by solving $2x^2+9x-18=0$

-36

$12, -3$

$\frac{12}{2}, \frac{-3}{2}$

$6, -\frac{3}{2}$

$(x+6)(2x-3)=0$

$x=-6$ or $x=\frac{3}{2}$

∴ All the roots of the given polynomial are $-1, \frac{3}{2}, -6$

Ex 3.19 obtain the condition that the roots of $x^3+px^2+qx+r=0$ are in AP.

The given polynomial is

$x^3+px^2+qx+r=0$ ——— ①

Assume that roots of the polynomial are in AP.

Let the roots be $a-d, a, a+d$.

Sum of the roots

$a-d+a+a+d = -\frac{p}{1} = -p$

$3a = -p \Rightarrow a = -\frac{p}{3}$

$\therefore a$ is the root of the polynomial

$\therefore \left(-\frac{p}{3}\right)^3 + p\left(-\frac{p}{3}\right)^2 + q\left(-\frac{p}{3}\right) + r = 0$

$-\frac{p^3}{27} + \frac{p^3}{9} - \frac{pq}{3} + r = 0$

$\frac{-p^3+3p^3-9qp+27r}{27} = 0$

$2p^3-9qp+27r=0.$

$\therefore 9pq = 2p^3+27r$

which is the required condition.

———

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Ex 3.20 Find the condition that the roots of $ax^3+bx^2+cx+d=0$ are in GP. Assume $a,b,c,d \neq 0$.

The given eqn is $ax^3+bx^2+cx+d=0$.

Assume that the roots are in GP.

Let the roots be $\frac{\alpha}{\lambda}, \alpha, \alpha\lambda$.

Sum of the roots $= \frac{\alpha}{\lambda}+\alpha+\alpha\lambda = -\frac{b}{a}$ — ①

$\Sigma 1_2 = \frac{\alpha}{\lambda}\times\alpha + \frac{\alpha}{\lambda}\times\alpha\lambda + \alpha\times\alpha\lambda = \frac{c}{a}$

i.e. $\frac{\alpha^2}{\lambda}+\alpha^2+\alpha^2\lambda = \frac{c}{a}$ — ②.

$\frac{②}{①} \Rightarrow \frac{\alpha^2\left(\frac{1}{\lambda}+1+\lambda\right)}{\alpha\left(\frac{1}{\lambda}+1+\lambda\right)} = \frac{c/a}{-b/a}$

$\Rightarrow \alpha = \frac{c}{a}\left(-\frac{a}{b}\right) = -\frac{c}{b}$

α is root of the eqn.

$\therefore a\left(-\frac{c}{b}\right)^3+b\left(-\frac{c}{b}\right)^2+c\left(-\frac{c}{b}\right)+d=0$.

$-\frac{ac^3}{b^3}+\frac{bc^2}{b^2}-\frac{c^2}{b}+d=0$.

$\frac{-ac^3+b^2c^2-b^2c^2+b^3d}{b^3}=0$

$-ac^3+b^3d=0$

$\therefore ac^3=b^3d$.

Ex 3.21 If the roots of $x^3+px^2+qx+r=0$ are in H.P. prove that $9pqr = 27r^2+2q^3$.

Given eqn is $x^3+px^2+qx+r=0$ — ①.

Assume that the roots are in HP.

i.e. their reciprocals are in A.P

$\therefore \left(\frac{1}{x}\right)^3+p\left(\frac{1}{x}\right)^2+q\left(\frac{1}{x}\right)+r=0$

$\frac{1}{x^3}+\frac{p}{x^2}+\frac{q}{x}+r=0$

$\frac{1+px+qx^2+rx^3}{x^3}=0$

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$rx^3+qx^2+px+1=0$ ——— (2)

∴ The roots of the eqn (2) are in AP.

Let the roots be $a-d, a, a+d$.

Sum of the roots $= 3a = -\frac{q}{r}$

$$a = -\frac{q}{3r}$$

a is the root of (2).

$$\therefore r\left(-\frac{q}{3r}\right)^3 + q\left(-\frac{q}{3r}\right)^2 + p\left(-\frac{q}{3r}\right) + 1 = 0.$$

$$-\frac{q^3 r}{27r^3} + \frac{q^3}{9r^2} - \frac{pq}{3r} + 1 = 0.$$

$$-\frac{q^3}{27r^2} + \frac{q^3}{9r^2} - \frac{pq}{3r} + 1 = 0$$

$$\frac{-q^3 + 3q^3 - 9pqr + 27r^2}{27r^2} = 0$$

$$2q^3 - 9pqr + 27r^2 = 0.$$

$$9pqr = 2q^3 + 27r^2$$

———— × ———— × ————

XII | CH3 | 3.3 | P5

3.22 It is known that the roots of the eqn $x^3-6x^2-4x+24=0$ are in AP. Find its roots.

The given eqn is $x^3-6x^2-4x+24=0$. ——— (1)

Given that roots of (1) are in A.P

Let the roots be $a-d, a, a+d$.

Sum of the roots $a-d+a+a+d = -(-6)$

$= 6$

$3a = 6$

$\Rightarrow \boxed{a=2}$

product of the roots $(a-d)\,a\,(a+d) = -24$

$$a(a^2-d^2) = -24$$

$$a^3 - ad^2 = -24$$

$$8 - 2d^2 = -24$$

$$2d^2 = 32$$

$$d^2 = 16 \Rightarrow d = \pm 4$$

when $d=4$, the roots are

$2-4, 2, 2+4$

i.e. $\boxed{-2, 2, 6}$

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when $d=-4$, the roots are

$2-(-4),\ 2,\ 2+(-4)$

$\boxed{6, 2, -2}$

EXERCISE 3.3

1. Solve the cubic eqn $2x^3-x^2-18x+9=0$ if sum of two of its roots vanishes.

Given eqn is $2x^3-x^2-18x+9=0$

$\div 2,\ x^3-\frac{x^2}{2}-9x+\frac{9}{2}=0.$

Let the roots be $\alpha, -\alpha, \beta$

Sum of the roots $\alpha-\alpha+\beta=-(-\frac{1}{2})$

$\Rightarrow \boxed{\beta=\frac{1}{2}}$

product of the roots $-\alpha^2\beta=\frac{-9}{2}$

$-\alpha^2(\frac{1}{2})=\frac{-9}{2}$

$\alpha^2=9$

$\alpha=\pm 3$

∴ The roots are $-3, 3, \frac{1}{2}$. //

2) Solve the eqn $9x^3-36x^2+44x-16=0$ if the roots form an AP.

Given eqn is $9x^3-36x^2+44x-16=0$

$\div 9,\ x^3-\frac{36}{9}x^2+\frac{44}{9}x-\frac{16}{9}=0.$

Given that the roots are in AP.

Let the roots be $a-d, a, a+d$.

Their sum $a-d+a+a+d=-(-\frac{36}{9})$

$=4$

$3a=4 \Rightarrow \boxed{a=\frac{4}{3}}$

Their product $a(a-d)(a+d)=\frac{16}{9}$

$a(a^2-d^2)=\frac{16}{9}$

$\frac{4}{3}\left[\frac{16}{9}-d^2\right]=\frac{16}{9}$

$\frac{16}{9}-d^2=\frac{16}{9}\times\frac{3}{4}$

$=\frac{4}{3}$

$\frac{16}{9}-\frac{4}{3}=d^2$

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$\frac{16-12}{9} = d^2 \Rightarrow d^2 = \frac{4}{3}$

$d = \pm \frac{2}{3}$

(i) when $a = \frac{4}{3}$, $d = \frac{2}{3}$

The roots are $\frac{4}{3} - \frac{2}{3}, \frac{4}{3}, \frac{4}{3} + \frac{2}{3}$

i.e. $\frac{2}{3}, \frac{4}{3}, 2$.

(ii) when $a = \frac{4}{3}$, $d = -\frac{2}{3}$

The roots are $\frac{4}{3} + \frac{2}{3}, \frac{4}{3}, \frac{4}{3} - \frac{2}{3}$

$2, \frac{4}{3}, \frac{2}{3}$

---

3) Solve the eqn $3x^3 - 26x^2 + 52x - 24 = 0$. If its roots form a G.P.

The given eqn is $3x^3 - 26x^2 + 52x - 24 = 0$ — (1)

$\div 3$, $x^3 - \frac{26}{3}x^2 + \frac{52}{3}x - \frac{24}{3} = 0$

i.e. $x^3 - \frac{26}{3}x^2 + \frac{52}{3}x - 8 = 0$.

Given that the roots are in G.P

Let the roots be $\frac{a}{\lambda}, a, a\lambda$.

Sum of the roots

$\frac{a}{\lambda} + a + a\lambda = -\left(-\frac{26}{3}\right)$

$a\left(\frac{1}{\lambda} + 1 + \lambda\right) = \frac{26}{3}$ — (2).

Their product is $\frac{a}{\lambda} \times a \times a\lambda = -(-8)$

i.e. $a^3 = 8 \Rightarrow \boxed{a = 2}$

Sub $a = 2$ in (2)

$2\left(\frac{1}{\lambda} + 1 + \lambda\right) = \frac{26}{3}$

$3\left(\frac{1+\lambda+\lambda^2}{\lambda}\right) = 13$

$3 + 3\lambda + 3\lambda^2 = 13\lambda$

$3\lambda^2 - 10\lambda + 3 = 0$

$(\lambda - 3)(3\lambda - 1) = 0$

$\lambda = 3$ or $\lambda = \frac{1}{3}$

9 → $-\frac{9}{3}$, $-\frac{1}{3}$ → $-3, -\frac{1}{3}$

when $a = 2$, $\lambda = 3$ the roots are

$\frac{2}{3}, 2, 6$.

when $a = 2$, $\lambda = \frac{1}{3}$ the roots are

$6, 2, \frac{2}{3}$ //

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4) Determine k and Solve the eqn $2x^3-6x^2+3x+k=0$ if one of the roots is twice the sum of the other two roots.

Given eqn is $2x^3-6x^2+3x+k=0$ — (1)

$\div 2$, $x^3-3x^2+\frac{3}{2}x+\frac{k}{2}=0$.

Let the roots be α, β, γ.

By data, $\alpha = 2(\beta+\gamma)$

$\beta+\gamma = \alpha/2$.

$\Sigma_1 = \alpha+\beta+\gamma = -(-3)$

$= \alpha + \frac{\alpha}{2} = 3$

$\frac{3\alpha}{2} = 3 \Rightarrow \boxed{\alpha = 2}$ — (2)

$\therefore \beta+\gamma = \frac{2}{2} = 1$

$\boxed{\beta+\gamma = 1}$ — (3)

$\Sigma_2 = \alpha\beta+\beta\gamma+\alpha\gamma = \frac{3}{2}$

$= \alpha(\beta+\gamma)+\beta\gamma = \frac{3}{2}$

$2(1)+\beta\gamma = \frac{3}{2}$

$\beta\gamma = \frac{3}{2}-2 = -\frac{1}{2}$

$\Sigma_3 = \alpha\beta\gamma = -\frac{k}{2}$ — (4).

$2\left(-\frac{1}{2}\right) = -\frac{k}{2} \Rightarrow 1 = \frac{k}{2}$

$\Rightarrow \boxed{k=2}$

from (3) $\beta+\gamma = 1 \Rightarrow \beta = 1-\gamma$

$\gamma = 1-\beta$.

from (4) $2(1-\gamma)\gamma = -1$

$2(\gamma-\gamma^2) = -1$

$2\gamma-2\gamma^2+1=0$

$2\gamma^2-2\gamma-1=0$.

$\therefore \gamma = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2 \pm \sqrt{4-4(2)(-1)}}{2(2)}$

$= \frac{2 \pm \sqrt{4+8}}{4} = \frac{2 \pm 2\sqrt{3}}{4}$

$= \frac{2(1 \pm \sqrt{3})}{4}$

$= \frac{1 \pm \sqrt{3}}{2}$.

$\therefore$ The roots are

$2, \frac{1 \pm \sqrt{3}}{2}, \quad k=2$. //

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5) find all zeros of the polynomial $x^6-3x^5-5x^4+22x^3-39x^2-39x+135$, if it is known that $1+2i$ and $\sqrt{3}$ are two of zeros.

The given eqn is $x^6-3x^5-5x^4+22x^3-39x^2-39x+135=0$ — (1)

The given roots are $1+2i$, $\sqrt{3}$.

∴ The other roots are $1-2i$, $-\sqrt{3}$

∴ The factors are

$\{(x-(1+2i))(x-(1-2i))\}\{(x-\sqrt{3})(x+\sqrt{3})\}$

$=\{x^2-x(1-2i)-x(1+2i)+(1+2i)(1-2i)\}\times\{x^2-(\sqrt{3})^2\}$

$=\{x^2-x(1-2i+1+2i)+(1+4)\}\{x^2-3\}$

$=(x^2-x(2)+5)(x^2-3)$

$=x^4-3x^2-2x^3+6x+5x^2-15$

$=x^4-2x^3+2x^2+6x-15$ — (2)

To find the other root Divide (1) by (2)

$$x^4-2x^3+2x^2+6x-15 \;\overline{)\; x^6-3x^5-5x^4+22x^3-39x^2-39x+135 \;}\; x^2-x-9$$

$x^6-2x^5+2x^4+6x^3-15x^2+135$

$-x^5-7x^4+16x^3-24x^2-39x$

$-x^5+2x^4-2x^3-6x^2+15x$

$-9x^4+18x^3-18x^2-54x+135$

$-9x^4+18x^3-18x^2-54x+135$

0

∴ The other factor is $x^2-x-9=0$.

$$\therefore x=\frac{1\pm\sqrt{1-4(1)(-9)}}{2}$$

$$=\frac{1\pm\sqrt{1+36}}{2}=\frac{1\pm\sqrt{37}}{2}$$

∴ The roots are $1\pm2i$, $\pm\sqrt{3}$, $\frac{1\pm\sqrt{37}}{2}$.

x x

6) Solve the cubic eqn

(i) $2x^3-9x^2+10x=3$

The given eqn is $2x^3-9x^2+10x-3=0$

Sum of the coefficients $=2-9+10-3=0$

∴ 1 is a root of this eqn.

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$\therefore (x-1)$ is a factor.

1	2	−9	10	−3
	0	2	−7	3
	2	−7	3	0

The other factor is $2x^2-7x+3$

$2x^2-7x+3=0$

i $(x-3)(2x-1)=0$

$\Rightarrow x=3, \; x=\frac{1}{2}$

6 → $-\frac{6}{2}, -\frac{1}{2}$ → $-3, -\frac{1}{2}$

$\therefore$ The roots are $1, 3, \frac{1}{2}$

(ii) The given eqn is $8x^3-2x^2-7x+3=0$

Sum of odd coeff = Sum of even coeff = 1

$\therefore -1$ is a root of this eqn

$\therefore x+1$ is a factor

−1	8	−2	−7	3
	0	−8	10	−3
	8	−10	3	0

$\therefore$ The other factor is $8x^2-10x+3$

$8x^2-10x+3=0$

$(4x-3)(2x-1)=0$

24 → $-\frac{6}{8}, -\frac{4}{8}$

$x=\frac{3}{4}, \; x=\frac{1}{2}$. $\left(-\frac{3}{4}, -\frac{1}{2}\right)$

$\therefore$ The roots are $-1, \frac{1}{2}, \frac{3}{4}$.

---

7) Solve the eqn $x^4-14x^2+45=0$.

The given eqn is $x^4-14x^2+45=0$

put $y=x^2$

$\therefore (x^2)^2-14x^2+45=0$

$\Rightarrow y^2-14y+45=0$

$(y-9)(y-5)=0$

$\Rightarrow y=5, 9$

i $x^2=5$ | $x^2=9$

$\Rightarrow x=\pm\sqrt{5}$ | $x=\sqrt{9}=\pm 3$.

$\therefore$ The roots are $\pm 3, \pm\sqrt{5}$.

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partly factored polynomials:-

Quadratic polynomial eqns of the form $(ax+b)(cx+d)(px+q)(rx+s)+k=0$ which can be rewritten in the form $(\alpha x^2+\beta x+\gamma)(\alpha x^2+\rho x+\mu)+k=0$.

Ex 3.23 Solve $(x-2)(x-7)(x-3)(x+2)+19=0$

Rewrite the eqn as

$(x-2)(x-3)(x-7)(x+2)+19=0$

$(x^2-5x+6)(x^2-5x-14)+19=0$

put $y = x^2-5x$

$\therefore (y+6)(y-14)+19=0$

$y^2-8y-84+19=0$

$y^2-8y-65=0$

$(y-13)(y+5)=0$

$\Rightarrow y-13=0$ and $y+5=0$

i.e. $x^2-5x-13=0$ and $x^2-5x+5=0$

$x^2-5x-13=0$

$x = \frac{5 \pm \sqrt{25-4(1)(-13)}}{2}$

$x = \frac{5 \pm \sqrt{25+52}}{2}$

$= \frac{5 \pm \sqrt{77}}{2}$

$x^2-5x+5=0$

$x = \frac{5 \pm \sqrt{25-4(1)(5)}}{2}$

$x = \frac{5 \pm \sqrt{25-20}}{2}$

$x = \frac{5 \pm \sqrt{5}}{2}$

$\therefore$ The roots are $\frac{5 \pm \sqrt{77}}{2}$, $\frac{5 \pm \sqrt{5}}{2}$.

EXERCISE 3.4

1) Solve $(x-5)(x-7)(x+6)(x+4)=504$

$(x-5)(x+4)(x-7)(x+6)=504$

$(x^2-x-20)(x^2-x-42)-504=0$

let $y = x^2-x$

$\therefore (y-20)(y-42)-504=0$

$y^2-42y-20y+840-504=0$

$y^2-62y+336=0$

$(y-56)(y-6)=0$

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$y-56=0$ and $y-6=0$.

$y-56=0$

$x^2-x-56=0$

$(x-8)(x+7)=0$

$x=-7, 8$

$y-6=0$

$x^2-x-6=0$

$(x-3)(x+2)=0$

$x=-2, 3$

∴ The roots are $-7, -2, 3, 8$

---

(11) $(x-4)(x-7)(x-2)(x+1)=16$

$(x-4)(x-2)(x-7)(x+1)-16=0$

$(x^2-6x+8)(x^2-6x-7)-16=0$

Let $x^2-6x=y$

$(y+8)(y-7)-16=0$

$y^2+y-56-16=0$

$y^2+y-72=0$

$(y+9)(y-8)=0$

$y+9=0$

$x^2-6x+9=0$

$(x-3)(x-3)=0$

$x=3, 3$

$y-8=0$

$x^2-6x-8=0$

$x=\frac{6\pm\sqrt{(-6)^2-4(1)(-8)}}{2}$

$=\frac{6\pm\sqrt{36+32}}{2}$

$=\frac{6\pm\sqrt{68}}{2}=\frac{6\pm\sqrt{4\times17}}{2}$

$=\frac{6\pm2\sqrt{17}}{2}=3\pm\sqrt{17}$.

∴ The roots are $3, 3, 3\pm\sqrt{17}$

---

2) Solve $(2x-1)(x+3)(x-2)(2x+3)+20=0$

$(2x-1)(2x+3)(x+3)(x-2)+20=0$

$(4x^2+6x-2x-3)(x^2+x-6)+20=0$

$(4x^2+4x-3)(x^2+x-6)+20=0$

$[4(x^2+x)-3][x^2+x-6]+20=0$

Let $x^2+x=y$

∴ $(4y-3)(y-6)+20=0$

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$4y^2 - 24y - 3y + 18 + 20 = 0$

$4y^2 - 27y + 38 = 0$

$(4y - 19)(y - 2) = 0$

$4y - 19 = 0$, $y - 2 = 0$

$4(x^2 + x) - 19 = 0$

$4x^2 + 4x - 19 = 0$

$$x = \frac{-4 \pm \sqrt{16 - 4(4)(-19)}}{2(4)}$$

$$x = \frac{-4 \pm \sqrt{16 + 304}}{8}$$

$$= \frac{-4 \pm \sqrt{320}}{8}$$

$$= \frac{-4 \pm \sqrt{16 \times 4 \times 5}}{8}$$

$$= \frac{-4 \pm 4 \times 2\sqrt{5}}{8}$$

$$= \frac{-4 \pm 8\sqrt{5}}{8}$$

$$= \frac{4(-2 \pm 2\sqrt{5})}{8}$$

$$= \frac{-2 \pm 2\sqrt{5}}{2}$$

$+152$

-19 , -8

4 , 4

$\frac{-19}{4}, -2$

$y - 2 = 0$

(ii) $x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x = -2$ or $x = 1$

∴ The roots are

$-2, 1, \frac{-2 \pm 2\sqrt{5}}{2}$.

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Rational Roots Theorem

Let $a_n x^n + \cdots + a_1 x + a_0$ with $a_n \neq 0$ and $a_0 \neq 0$, be a polynomial with integer coefficients. If $\frac{p}{q}$ with $(p,q) = 1$ is a root of the polynomial then p is a factor of a_0 and q is a factor of a_n.

Ex 3.26 Find the roots of $2x^3 + 3x^2 + 2x + 3$.

The given polynomial is

$2x^3 + 3x^2 + 2x + 3 = 0$

Here $a_n = 2$ and $a_0 = 3$.

If p/q is root of the polynomial where $(p,q) = 1$ then p must divide 3 and q must divide 2

∴ The possible values of p are $1, -1, 3, -3$

The possible values of q are $1, -1, 2, -2$.

∴ The possible fractions $\left(\frac{p}{q}\right)$ are $\pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}$.

Let $p(x) = 2x^3 + 3x^2 + 2x + 3$

$p(1) = 2+3+2+3 = 10 \neq 0$

$p(-1) = -2+3-2+3 = 6-4 = 2 \neq 0$

$p(3) = 2(27) + 3(9) + 2(3) + 3$

$= 54 + 27 + 6 + 3$

$= 90 \neq 0$

$p(-3) = 2(-27) + 3(9) - 2(3) + 3$

$= -54 + 27 - 6 + 3$

$= -27 - 6 + 3$

$= -30 \neq 0$

$p\left(\frac{3}{2}\right) = 2\left(\frac{27}{8}\right) + 3\left(\frac{9}{4}\right) + 2\left(\frac{3}{2}\right) + 3$

$= \frac{27}{4} + \frac{27}{4} + 3 + 3 \neq 0.$

$p\left(-\frac{3}{2}\right) = 2\left(-\frac{27}{8}\right) + 3\left(\frac{9}{4}\right) + 2\left(-\frac{3}{2}\right) + 3$

$= -\frac{27}{4} + \frac{27}{4} - 3 + 3 = 0$

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$\therefore -\frac{3}{2}$ is the only rational root of $p(x)$

$\therefore x+\frac{3}{2}$ is a factor of $p(x)$

i.e. $(2x+3)$ is a factor of $p(x)$

To find the other root divide ① by $2x+3$

$$\begin{array}{r|l} & x^2+1 \\ \hline 2x+3 & 2x^3+3x^2+2x+3 \\ & 2x^3+3x^2 \\ \hline & 2x+3 \\ & 2x+3 \\ \hline & 0 \end{array}$$

For other roots, solve $x^2+1=0$

$x^2=-1 \Rightarrow x^2=i^2$

$x=\pm i$

$\therefore$ The roots of $p(x)$ are

$-\frac{3}{2}, i, -i$ //

Theorem 3.6 A polynomial eqn

$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_2x^2 + a_1x + a_0 = 0$

is a reciprocal eqn iff one of the following two statements is true.

(i) $a_n = a_0$, $a_{n-1} = a_1$, $a_{n-2} = a_2$, ...

(ii) $a_n = -a_0$, $a_{n-1} = -a_1$, $a_{n-2} = -a_2$, ...

Note:- Type I

Reciprocal equations of type I corresponds to those in which the coefficients from the beginning are equal to the coefficients from the end.

Type II Reciprocal equations in which the coefficients from the begining are equal in magnitude but opposite in sign.

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* for an odd degree reciprocal eqn of Type I, $x = -1$ must be a soln.

* for an odd degree reciprocal eqn of Type II, $x = 1$ must be a soln.

* for an even degree reciprocal eqn of Type II, the middle term must be zero. further $x = 1$ and $x = -1$ are solutions.

* for an even degree reciprocal eqn, by taking $x + \frac{1}{x}$ or $x - \frac{1}{x}$ as y.

Ex 3.27: Solve the eqn

$7x^3 - 43x^2 = 43x - 7$.

The eqn of the polynomial is

$7x^3 - 43x^2 - 43x + 7 = 0$ ——— ①

In this eqn coeff from the beginning are equal to the coefficient from the end.

eqn ① is odd degree reciprocal.

∴ -1 is a soln of ①

∴ $x+1$ is a factor

To find the other factor Divide ① by $x+1$

$$x+1 \overline{)\ 7x^3 - 43x^2 - 43x + 7\ (\ 7x^2 - 50x + 7}$$

$7x^3 + 7x^2$

$-50x^2 - 43x$

$-50x^2 - 50x$

$7x + 7$

$7x + 7$

0

To get the other roots solve

$7x^2 - 50x + 7 = 0$

$(x-7)(7x-1) = 0$

$x = 7, \quad x = \frac{1}{7}$

∴ Roots of ① are $-1, \frac{1}{7}, 7$.

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$-49 \quad -1$

$7 \quad 7$

$-7, -\frac{1}{7}$

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Ex 3.28: Solve $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$

The eqn of the polynomial is

$x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$ ——①

In this eqn coeff of beginning are equal to coeff of from end.

Eqn ① is even degree reciprocal eqn.

$\therefore$ ① $\Rightarrow (x^4+1) + (-10x^3 - 10x) + 26x^2 = 0$

$x^2\left(x^2 + \frac{1}{x^2}\right) - 10x^2\left(x + \frac{1}{x}\right) + 26x^2 = 0$

$x^2\left[\left(x^2 + \frac{1}{x^2}\right) - 10\left(x + \frac{1}{x}\right) + 26\right] = 0.$

$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) - 10\left(x + \frac{1}{x}\right) + 26 = 0$ —— ②

put $x + \frac{1}{x} = y$

$\therefore x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$

$= y^2 - 2$

$\therefore$ ② $\Rightarrow y^2 - 2 - 10y + 26 = 0$

$y^2 - 10y + 24 = 0$

$(y-4)(y-6) = 0$

$y - 4 = 0$

$\Rightarrow y = 4$

$\Rightarrow x + \frac{1}{x} = 4$

$x^2 + 1 = 4x$

$x^2 - 4x + 1 = 0$

$x = \frac{4 \pm \sqrt{16 - 4(1)(1)}}{2}$

$x = \frac{4 \pm \sqrt{12}}{2}$

$= \frac{4 \pm 2\sqrt{3}}{2}$

$= 2 \pm \sqrt{3}$

$y - 6 = 0$

i.e. $x + \frac{1}{x} - 6 = 0$

$x^2 + 1 - 6x = 0.$

$x^2 - 6x + 1 = 0$

$x = \frac{6 \pm \sqrt{36 - 4(1)(1)}}{2}$

$= \frac{6 \pm \sqrt{32}}{2}$

$= \frac{6 \pm 4\sqrt{2}}{2}$

$= \frac{2(3 \pm 2\sqrt{2})}{2}$

$= 3 \pm 2\sqrt{2}$

$\therefore$ Solution of ① are

$2+\sqrt{3},\ 2-\sqrt{3},\ 3+2\sqrt{2},\ 3-2\sqrt{2}.$

———— ∝ ————

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Ex 3.29 Find the soln if any of the eqn $2\cos^2 x - 9\cos x + 4 = 0$.

Given eqn is $2\cos^2 x - 9\cos x + 4 = 0$

put $x = \cos x$

$\therefore p(x) = 2x^2 - 9x + 4 = 0$

$(x-4)(2x-1) = 0$

$x = 4$ or $x = \frac{1}{2}$

8 : $-\frac{8}{2}$, $-\frac{1}{2}$; $-4, -\frac{1}{2}$

$x = 4 \Rightarrow \cos x = 4$ is not possible since $\cos x$ lies betn -1 and 1.

$\therefore x = \frac{1}{2} \Rightarrow \cos x = \frac{1}{2}$

$\Rightarrow x = 2n\pi \pm \pi/3$

$\therefore x = 2n\pi \pm \pi/3$ are soln of (1)

EXERCISE 3.5

1) Solve $\sin^2 x - 5\sin x + 4 = 0$

Given eqn is $\sin^2 x - 5\sin x + 4 = 0$

put $y = \sin x$.

$\therefore y^2 - 5y + 4 = 0$

$(y-1)(y-4) = 0$

$y - 1 = 0$ i.e. $\sin x = 1$

$\Rightarrow x = \pi/2$

$x = \pi/2 + 2n\pi$

$y - 4 = 0 \Rightarrow y = 4$ i.e. $\sin x = 4$

It is not possible $\because \sin x$ lies betn -1 and 1.

$\therefore$ The soln are $x = 2n\pi + \pi/2$.

(ii) Solve $12x^3 + 8x = 29x^2 - 4$

The given eqn is $12x^3 - 29x^2 + 8x + 4 = 0$.

let $p(x) = 12x^3 - 29x^2 + 8x + 4$

$p(1) = 12 - 29 + 8 + 4 = 24 - 29 = -5 \neq 0$

$p(-1) = -12 - 29 - 8 + 4 = -49 + 4 = -45 \neq 0$

$p(2) = 96 - 116 + 16 + 4$

$= 116 - 116 = 0$

$\therefore$ 2 is a root of $p(x)$

i.e. $x - 2$ is a factor of $p(x)$

$\therefore$ Divide the eqn (1) by $x - 2$ to get other factors.

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$$x-2 \overline{)\, 12x^3 - 29x^2 + 8x + 4} \quad (12x^2 - 5x - 2)$$

$12x^3 - 24x^2$

$-5x^2 + 8x$

$-5x^2 + 10x$

$-2x + 4$

$-2x + 4$

0

∴ To get the other factors solve $12x^2 - 5x - 2 = 0$

$(3x-2)(4x+1) = 0.$

$x = 2/3, \quad x = -1/4$

∴ The roots of ① are

$-\frac{1}{4}, \frac{2}{3}, 2$

-24

$-8, 3$

$12, 12$

$-\frac{2}{3}, \frac{1}{4}$

---

2) Examine for the rational roots of (1) $2x^3 - x^2 - 1 = 0.$

Given eqn is $2x^3 - x^2 - 1 = 0$ — ①

Here $a_n = 2, \quad a_0 = -1$

XII|CH 3|3.5|p6

If $\frac{p}{q}$ is a root of $p(x)$ where $(p, q) = 1$.

The p must divide -1 and q must divide 2.

∴ The possible values of p are $-1, 1$

The possible values of q are $-2, 2, 1, -1$

∴ possible fractions are $\pm\frac{1}{1} \quad \pm\frac{1}{2}$.

Let $p(x) = 2x^3 - x^2 - 1$

$p(1) = 2 - 1 - 1 = 0.$

∴ 1 is a root of $p(x)$

$p(-1) = 2(-1) - 1 - 1 = -4 \neq 0.$

$p\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^2 - 1$

$= \frac{2}{8} - \frac{1}{4} - 1 = \frac{2-2-8}{8} = -1 \neq 0$

$p\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^2 - 1$

$= 2\left(-\frac{1}{8}\right) - \frac{1}{4} - 1$

$= -\frac{1}{4} - \frac{1}{4} - 1 = \frac{-1-1-4}{4} = -\frac{6}{4}$

$= \boxed{-\frac{3}{2}} \neq 0$

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Hence $x=1$ is a root of $p(x)$

$\therefore$ $x-1$ is a factor of $p(x)$

To find the other factor divide ① by $x-1$

$$\begin{array}{r} 2x^2+x+1 \\ x-1 \overline{)\, 2x^3-x^2+0x-1} \\ \underline{2x^3-2x^2} \\ x^2+0x \\ \underline{x^2-x} \\ x-1 \\ \underline{x-1} \\ 0 \end{array}$$

To get the other factors solve

$2x^2+x+1=0$

$$x = \frac{-1\pm\sqrt{1-4(2)(1)}}{2(2)}$$

$$= \frac{-1\pm\sqrt{-7}}{4} = \frac{-1\pm i\sqrt{7}}{4}$$

Remaining roots are imaginary.

$\therefore$ The only rational roots of $p(x)$ is 1.

(ii) $x^8-3x+1=0$

Let $p(x) = x^8-3x+1$

$a_n=1,\ a_0=1.$

If $\frac{p}{q}$ is a root of the polynomial where $(p,q)=1$. Then p must divide 1 and q must divide 1

$\therefore$ The possible values of p are ± 1 and the possible values of q are ± 1

$\therefore$ The possible fractions are $\pm\frac{1}{1}$

$p(1) = 1-3+1 \neq 0$

$p(-1) = 1+3+1 = 4 \neq 0.$

Hence $p(x)$ has no rational roots.

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3) Solve :- $8x^{\frac{3}{2n}} - 8x^{-\frac{3}{2n}} = 63$

The given eqn is

$$8x^{\frac{3}{2n}} - 8x^{-\frac{3}{2n}} - 63 = 0.$$

$$8\left[x^{\frac{3}{2n}} - \frac{1}{x^{\frac{3}{2n}}}\right] - 63 = 0$$

$$8\left[\frac{\left(x^{\frac{3}{2n}}\right)^2 - 1}{x^{\frac{3}{2n}}}\right] = 63$$

$$8\left[\left(x^{\frac{3}{2n}}\right)^2 - 1\right] = 63x^{\frac{3}{2n}}$$

$$8\left(x^{\frac{3}{2n}}\right)^2 - 63x^{\frac{3}{2n}} - 8 = 0 \quad \text{—— (1)}$$

put $x^{\frac{3}{2n}} = y$

$\therefore$ (1) $\Rightarrow$ $8y^2 - 63y - 8 = 0$

-64

$-64 \times \frac{1}{8}$ $\quad -\frac{64}{8}$ $\quad \frac{1}{8}$

$(y-8)(8y+1) = 0$

$y - 8 = 0$

$y = 8$

$x^{\frac{3}{2n}} = 2^3$

$\left(x^{\frac{1}{2n}}\right)^3 = 2^3$

$\Rightarrow x^{\frac{1}{2n}} = 2$

$\Rightarrow x = 2^{2n}$

$x = (2^2)^n$

$\boxed{x = 4^n}$

$8y + 1 = 0$

$8y = -1$

$y = -\frac{1}{8} = -\left(\frac{1}{2}\right)^3$

$x^{\frac{3}{2n}} = -\left(\frac{1}{2}\right)^3$

$\left(x^{\frac{1}{2n}}\right)^3 = -\left(\frac{1}{2}\right)^3$

$x^{\frac{1}{2n}} = -\frac{1}{2}$

$x = \left(-\frac{1}{2}\right)^{2n} = \left[\left(\frac{1}{2}\right)^2\right]^n$

$x = \left(\frac{1}{4}\right)^n = \boxed{\frac{1}{4^n}}$

-64

$-8 \quad \frac{1}{8}$

$\therefore$ The required soln of the given eqn is $x = 4^n$ (or) $x = \frac{1}{4^n}$ //

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4) $2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}$

$$2\sqrt{\frac{x}{a}} + \frac{3}{\sqrt{\frac{x}{a}}} = \frac{1}{\frac{a}{b}} + 6\frac{a}{b} \quad \text{—①}$$

put $\sqrt{\frac{x}{a}} = y$, $\frac{a}{b} = m$

$\therefore$ ① $\Rightarrow 2y + \frac{3}{y} = \frac{1}{m} + 6m$

$$\frac{2y^2+3}{y} = \frac{1+6m^2}{m}$$

$$(2y^2+3)m = y(1+6m^2)$$

$$2y^2m + 3m = y + 6ym^2$$

$$2y^2m - 6ym^2 - y + 3m = 0$$

$$2ym(y-3m) - (y-3m) = 0$$

$$(y-3m)(2ym-1) = 0$$

$y - 3m = 0$ | $2ym - 1 = 0$

$y = 3m$ | $y = \frac{1}{2m}$

(i) $y = 3m$

$\sqrt{\frac{x}{a}} = \frac{3a}{b}$

squaring

$\frac{x}{a} = \frac{9a^2}{b^2}$

$x = \boxed{\frac{9a^3}{b^2}}$

$y = \frac{1}{2m}$

$\sqrt{\frac{x}{a}} = \frac{1}{\frac{2a}{b}}$

$\sqrt{\frac{x}{a}} = \frac{b}{2a}$

squaring, $\frac{x}{a} = \frac{b^2}{4a^2}$

$x = \frac{ab^2}{4a^2} = \boxed{\frac{b^2}{4a}}$

$\therefore$ The roots of ① are

$x = \frac{9a^3}{b^2}, \frac{b^2}{4a}$.

× × ×

5) Solve:

(i) $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$ —①

The given eqn is an even degree reciprocal eqn of Type I.

$$x^2\left[6x^2 - 35x + 62 - \frac{35}{x} + \frac{6}{x^2}\right] = 0$$

$$x^2\left[6\left(x^2 + \frac{1}{x^2}\right) - 35\left(x + \frac{1}{x}\right)\right] + 62 = 0 \quad \text{—②}$$

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put $x+\frac{1}{x}=y$

$\therefore x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2$

$= y^2-2$

Sub in (2)

$6(y^2-2)-35y+62=0$

$6y^2-12-35y+62=0$

$6y^2-35y+50=0$

$(3y-10)(2y-5)=0$

300

$-\frac{20}{6}$ $-\frac{15}{6}$

$-\frac{10}{3}$ $-\frac{5}{2}$

$3y-10=0$

$3y=10$

$y=\frac{10}{3}$

i.e. $x+\frac{1}{x}=\frac{10}{3}$

$\frac{x^2+1}{x}=\frac{10}{3}$

$3x^2+3=10x$

$3x^2-10x+3=0$

9

$-\frac{9}{3}$ $-\frac{1}{3}$

$-3, -\frac{1}{3}$

$2y-5=0$

$2y=5$

$y=5/2$

$x+\frac{1}{x}=5/2$

$\frac{x^2+1}{x}=5/2$

$2(x^2+1)=5x$

$2x^2+2-5x=0$

$2x^2-5x+2=0$

$(x-3)(3x-1)=0$

$x=3, \frac{1}{3}$

$(x-2)(2x-1)=0$

$x=2, x=\frac{1}{2}$

4

$-\frac{4}{2}$ $-\frac{1}{2}$

$-2, -\frac{1}{2}$

$\therefore$ The required roots are.

$\frac{1}{2}, 2, 3, \frac{1}{3}$

(ii) Solve $x^4+3x^3-3x-1=0$.

Given eqn is $x^4+3x^3-3x-1=0$. —(1)

The given eqn is even degree reciprocal polynomial of Type II.

$\therefore$ 1 and -1 are roots of this eqn.

Hence $(x-1)$ and $(x+1)$ are factors.

Hence $(x-1)(x+1)$ also a factor

To find the other roots divide (1) by x^2-1

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$$
\begin{array}{r|l}
 & x^2+3x+1 \\
\hline
x^2-1 & x^4+3x^3+0x^2-3x-1 \\
 & x^4+0x^3-x^2 \\
\hline
 & 3x^3+x^2-3x \\
 & 3x^3 \qquad -3x \\
\hline
 & x^2+0-1 \\
 & x^2 \qquad -1 \\
\hline
 & 0
\end{array}
$$

To get the other roots solve

$x^2+3x+1=0$

$$x = \frac{-3 \pm \sqrt{9-4(1)(1)}}{2}$$

$$= \frac{-3 \pm \sqrt{5}}{2}$$

$\therefore$ The required roots of the given polynomial is

$x-1,\ x+1,\ \frac{-3+\sqrt{5}}{2},\ \frac{-3-\sqrt{5}}{2}$

XII/CH3/3.5/P11

6) Find all real numbers satisfying

$4^x - 3(2^{x+2}) + 2^5 = 0$

$(2^x)^2 - 3(2^x \cdot 2^2) + 2^5 = 0$

$(2^x)^2 - 12(2^x) + 32 = 0. \quad \text{—} (1)$

put $2^x = y$

$y^2 - 12y + 32 = 0.$

$(y-4)(y-8) = 0$

$\Rightarrow y-4=0$	$y-8=0$
$y=4$	$y=8$
$2^x=4$	$2^x=8$
$2^x=2^2$	$2^x=2^3$
$\Rightarrow \boxed{x=2}$	$\Rightarrow \boxed{x=3}$

$\therefore$ The required roots of the polynomial are $2, 3$

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7) Solve the eqn $6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0$ ... ① if it is known that $\frac{1}{3}$ is a soln.

Given $\frac{1}{3}$ is a soln for the reciprocal eqn of even degree.

Hence 3 is also a soln.

$\therefore (3x-1)(x-3) = 3x^2 - 9x - x + 3$

$= 3x^2 - 10x + 3$

is a factor of ①.

Divide ① by $3x^2 - 10x + 3$

$$3x^2 - 10x + 3 \;\overline{)}\; 6x^4 - 5x^3 - 38x^2 - 5x + 6 \;\big|\; 2x^2 + 5x + 2$$

$6x^4 - 20x^3 + 6x^2$

$15x^3 - 44x^2 - 5x$

$15x^3 - 50x^2 + 15x$

$6x^2 - 20x + 6$

$6x^2 - 20x + 6$

0

To get the other roots solve

$2x^2 + 5x + 2 = 0$

$(x+2)(2x+1) = 0$

$x = -2, \; x = -\frac{1}{2}$

4

$\frac{4}{2}$ $\frac{1}{2}$

2, $\frac{1}{2}$

Hence the roots are

$-\frac{1}{2}, -2, 3, \frac{1}{3}$.

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Discartes Rule

The sign of the coefficients using the symbols + and −

+, −, −, +, +, −, +

× ——— ×

EXERCISE 3.6

1. Discuss the maximum possible number of positive and negative roots of the polynomial eqn

$9x^9 - 4x^8 + 4x^7 - 3x^6 + 2x^5 + x^3 + 7x^2 + 7x + 2 = 0$

The given eqn is $p(x)$.

No of sign changes in $p(x) = 4$

∴ The maximum number of positive roots is 4.

$p(-x) = -9x^9 - 4x^8 - 4x^7 - 3x^6 - 2x^5 - x^3 + 7x^2 - 7x + 2$

No of sign changes in $p(x) = 2$

∴ maximum number of negative roots of $p(x) = 2$.

2) Discuss the maximum possible number of positive and negative roots of the polynomials $x^2 - 5x + 6$ and $x^2 - 5x + 16$

The given eqn is $p(x) = x^2 - 5x + 6$

No of sign changes in $p(x) = 2$.

∴ The maximum number of positive roots is 2.

$p(-x) = x^2 + 5x + 6$.

number of sign changes in $p(-x) = 0$.

∴ $p(x)$ has no negative roots.

× ——— ×

(ii) The given polynomial $q(x) = x^2 - 5x + 16$.

Number of sign changes in $q(x) = 2$

∴ The maximum number of positive roots is 2.

$q(-x) = x^2 + 5x + 16$

Number of sign changes in $q(-x) = 0$.

∴ $q(x)$ has no negative roots.

———— × ————

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3) Show that the eqn $x^9 - 5x^5 + 4x^4 + 2x^2 + 1 = 0$ has atleast 6 imaginary solutions.

The given eqn is

$p(x) = x^9 - 5x^5 + 4x^4 + 2x^2 + 1 = 0$.

Number of sign changes in $p(x) = 2$.

∴ The maximum no of positive roots $= 2$

Number of sign changes in

$p(-x) = -x^9 + 5x^5 + 4x^4 + 2x^2 + 1 = 0$ is 1

∴ The maximum no of negative roots $= 1$

0 does not satisfy $p(x)$

∴ 0 is not a root of $p(x)$

∴ $p(x)$ has maximum $2+3 = 3$ real roots.

Total no of roots of $p(x) = 9$

Hence $p(x)$ has $9 - 3 = 6$ imaginary roots.

———— ∝ ————

4) Determine the number of positive and negative roots of the eqn $x^9 - 5x^8 - 14x^7 = 0$.

Let $p(x) = x^9 - 5x^8 - 14x^7 = 0$

$x^7(x^2 - 5x - 14) = 0$

$\Rightarrow x^2 - 5x - 14 = 0$

$(x-7)(x+2) = 0$

$x = 7, \; x = -2$

∴ This eqn has one positive and one negative root.

———— ∝ ———— ∝ ————

5) Find the exactly number of real zeros and imaginary of the polynomial $x^9 + 9x^7 + 7x^5 + 5x^3 + 3x$

The given polynomial $p(x)$

$= x^9 + 9x^7 + 7x^5 + 5x^3 + 3x$

Number of sign changes in $p(x) = 0$.

∴ $p(x)$ has no positive roots.

$p(-x) = -x^9 - 9x^7 - 7x^5 - 5x^3 - 3x$

Number of changes in $p(-x) = 0$

∴ $p(x)$ has no negative roots.

But 0 is a roots of $p(x)$.

∴ $p(x)$ has 8 imaginary roots.

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Ex 3.30 Show that the polynomial $9x^9 + 2x^5 - x^4 - 7x^2 + 2$ has atleast 6 imaginary roots.

Let $p(x) = 9x^9 + 2x^5 - x^4 - 7x^2 + 2$

Number of sign changes in $p(x) = 2$

∴ The max number of positive roots is 2.

$p(-x) = -9x^9 - 2x^5 - x^4 - 7x^2 + 2$

Number of sign changes in $p(-x) = 1$

∴ The max number of negative roots is 1.

Hence $p(x)$ has max number of real roots $= 2+1 = 3$.

Total no of roots of $p(x) = 9$

∴ $p(x)$ has $9-3 = 6$ imaginary roots.

3.31 Discuss the nature of roots of

(i) $x^{2018} + 1947x^{1950} + 15x^8 + 26x^6 + 2019$

Let $p(x) = x^{2018} + 1947x^{1950} + 15x^8 + 26x^6 + 2019$

No of sign changes in $p(x) = 0$.

∴ $p(x)$ has no positive real roots.

$p(-x) = x^{2018} + 1947x^{1950} + 15x^8 + 26x^6 + 2019$

No of sign changes in $p(-x) = 0$.

∴ $p(+x)$ has negative real roots.

0 is not a root of $p(x)$.

∴ $p(x)$ has no real roots.

Hence all the roots of $p(x)$ are imaginary.

(ii) $x^5 - 19x^4 + 2x^3 + 5x^2 + 11$

Let $p(x) = x^5 - 19x^4 + 2x^3 + 5x^2 + 11$.

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Let $p(x) = x^5 - 19x^4 + 2x^3 + 5x^2 + 11$

Number of sign changes in $p(x) = 2$

∴ Maximum no of positive roots of $p(x) = 2$.

$p(-x) = -x^5 - 19x^4 - 2x^3 + 5x^2 + 11$

Number of sign changes in $p(-x) = 1$

∴ Max no of negative roots = 1.

⊗ Let the number of sign changes in $p(x) = n$

The no of positive roots = p

Then By Descrates Th

$n - p$ must be an even no

For positive roots:-

$n = 2$, $p = 2$ (p must be 0, or 1, or 2)

∴ $n = 2$, $p = 0$, then $n - p = 2 - 0 = 2$ (an even no)

∴ Number of positive roots of $p(x)$ is equal 0 is possible.

$n = 2$, $p = 1$

Then $n - p = 2 - 1 = 1$ (not an even no)

∴ positive roots of $p(x) = 1$ is not possible.

$n = 2$, $p = 2$

Then $n - p = 2 - 2 = 0$ (even no)

∴ positive roots of $p(x) = 0$ is possible.

∴ positive roots of $p(x) = 2$ (nos) i.e either 0 or 2.

For negative roots:

$n = 4$, $s = 1$ (p must 0 or 1)

∴ $n = 4$, $s = 0$ then $n - s = 4 - 0 = 1$ (not an even no).

∴ negative root of $p(x) = 1$ is not possible.

$n = 1$, $s = 1$, Then $n - s = 1 - 1 = 0$ (even no)

∴ negative root of $p(x) = 0$ is possible.

∴ No of negative roots of $p(x) = 1$.

∴ $p(x)$ has $(2 + 1) = 3$ real roots.

∴ The other two roots are imaginary.

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Chapter 4 – Inverse Trigonometric Functions

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Trigonometric Ratios of some angles

θ	0°	30°	45°	60°	90°	120°	135°	150	180°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	ND	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
$\cot\theta$	ND	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	ND
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	ND	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1
$\operatorname{cosec}\theta$	ND	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	ND

α equal	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$	$\sec\alpha$	$\operatorname{cosec}\alpha$
$-\theta$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$	$-\cot\theta$	$\sec\theta$	$-\operatorname{cosec}\theta$
$90-\theta$	$\cos\theta$	$\sin\theta$	$\cot\theta$	$\tan\theta$	$\operatorname{cosec}\theta$	$\sec\theta$
$90+\theta$	$\cos\theta$	$-\sin\theta$	$-\cot\theta$	$-\tan\theta$	$-\operatorname{cosec}\theta$	$\sec\theta$
$180-\theta$	$\sin\theta$	$-\cos\theta$	$-\tan\theta$	$-\cot\theta$	$-\sec\theta$	$\operatorname{cosec}\theta$
$180+\theta$	$-\sin\theta$	$-\cos\theta$	$\tan\theta$	$\cot\theta$	$-\sec\theta$	$-\operatorname{cosec}\theta$
$360-\theta$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$	$-\cot\theta$	$\sec\theta$	$-\operatorname{cosec}\theta$
$360+\theta$	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$

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CH-4 Inverse Trigonometric functions:-

Domain and range of Inverse Trigonometric functions:-

Function	Domain	Range (principal value)
1. $y = \sin^{-1}x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
2. $y = \cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$
3. $y = \tan^{-1}x$	$(-\infty, \infty)$	$-\pi/2 < y < \pi/2$
4. $y = \operatorname{cosec}^{-1}x$	$x \ge 1$ or $x \le -1$	$-\pi/2 \le y \le \pi/2$, $y \ne 0$.
5. $y = \sec^{-1}x$	$x \ge 1$ or $x \le -1$	$0 \le y \le \pi$, $y \ne \pi/2$
6. $y = \cot^{-1}x$	$(-\infty, \infty)$	$0 < y < \pi/2$

Some important formula:-

1) a) $\sin^{-1}(\sin\theta) = \theta, \quad -\pi/2 \le \theta \le \pi/2$

b) $\cos^{-1}(\cos\theta) = \theta, \quad 0 \le \theta \le \pi$

c) $\tan^{-1}(\tan\theta) = \theta, \quad -\pi/2 < \theta < \pi/2$

d) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta, \quad -\pi/2 \le \theta \le \pi/2$

e) $\sec^{-1}(\sec\theta) = \theta, \quad 0 \le \theta \le \pi$

f) $\cot^{-1}(\cot\theta) = \theta, \quad 0 < \theta < \pi$

2(a) $\sin(\sin^{-1}x) = x, \quad -1 \le x \le 1$

b) $\cos(\cos^{-1}x) = x, \quad -1 \le x \le 1$

c) $\tan(\tan^{-1}x) = x, \quad -\infty < x < \infty$

d) $\cot(\cot^{-1}x) = x, \quad -\infty < x < \infty$

e) $\sec(\sec^{-1}x) = x, \quad x \le -1$ or $x \ge 1$

f) $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x, \quad x \le -1$ or $x \ge 1$

3(a) $\sin^{-1}(-x) = -\sin^{-1}x$

b) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

c) $\tan^{-1}(-x) = -\tan^{-1}(x)$

d) $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$

e) $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$

f) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}(x)$

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4(a) $\operatorname{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$, $x \le -1$ or $x \ge 1$

b) $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$, $x \le -1$ or $x \ge 1$

c) $\cot^{-1} x = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right), & x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right), & x < 0 \end{cases}$

5(a) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, $-1 \le x \le 1$

b) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \pi/2$, $x \le -1$ or $x \ge 1$

c) $\tan^{-1} x + \cot^{-1} x = \pi/2$, $-\infty < x < \infty$

---

6) a) If $xy < 1$,

$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left[\frac{x+y}{1-xy}\right]$

b) If $xy > 1$,

$\tan^{-1} x + \tan^{-1} y = \begin{cases} \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) & x > 0, \ y > 0 \\ -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) & x < 0 \ y < 0 \end{cases}$

c) If $xy = 1$,

$\tan^{-1} x + \tan^{-1} y = \begin{cases} \pi/2, & x > 0, y > 0 \\ -\pi/2, & x < 0, y < 0 \end{cases}$

d) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

---

7) a) $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$

If $x, y \ge 0$, $x^2 + y^2 \le 1$

b) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left(xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right)$

---

8)

a) $\sin^{-1} x = \begin{cases} = \cos^{-1}\sqrt{1-x^2} \\ = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) \\ = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) \\ = \sec^{-1}\frac{1}{\sqrt{1-x^2}} \\ = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right) \end{cases}$

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b) $\cos^{-1} x$

$= \sin^{-1}\sqrt{1-x^2}$

$= \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$= \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$= \sec^{-1}\left(\frac{1}{x}\right)$

$= \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

c) $\tan^{-1} x$

$= \sin^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$= \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

$= \sec^{-1}\left(\sqrt{1+x^2}\right)$

$= \operatorname{cosec}^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$= \cot^{-1}\left(\frac{1}{x}\right)$

9 (a) $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

b) $2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

c) $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

---

10) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z$

$= \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-xz}\right)$

---

11) a) $3\sin^{-1} x = \sin^{-1}(3x-4x^3)$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

b) $3\cos^{-1} x = \cos^{-1}(4x^3-3x)$, $x \in \left[\frac{1}{2}, 1\right]$

A function f is even iff $f(-x) = f(x)$

A function f is odd iff $f(-x) = -f(x)$.

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Ex 4.1 Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ (in radians and degrees)

Let $\sin^{-1}\left(-\frac{1}{2}\right) = y$

$\therefore \sin y = -\frac{1}{2}$

$\sin y = -\sin \pi/6 = \sin(-\pi/6)$,

$\therefore y = -\pi/6$ $\quad -\pi/6 \in [-\pi/2, \pi/2]$

∴ principle value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is $-\pi/6$

Ex 4.2 Find the principal value of $\sin^{-1}(2)$. if it exists.

Domain of $\sin^{-1}(x)$ is $[-1, 1]$

$2 \notin [-1, 1]$

∴ $\sin^{-1}(2)$ does not exist.

4.3 Find the principal value of

a) $\sin^{-1}\left(\frac{1}{2}\right)$

Let $\sin^{-1}\left(\frac{1}{2}\right) = y$

$\sin y = \frac{1}{2}$

$\sin y = \sin \pi/6$

$\Rightarrow y = \pi/6$

∴ principal value of $\sin^{-1}\left(\frac{1}{2}\right)$ is $\pi/6$

b) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = y$

$\Rightarrow \sin y = \frac{1}{\sqrt{2}}$

i.e. $\sin y = \sin \pi/4$

$\Rightarrow y = \pi/4$

∴ principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \pi/4$

(iii) $\sin^{-1}\left(\sin(-\pi/3)\right)$

Let $\sin^{-1}\left(\sin(-\pi/3)\right) = y$

$\sin(-\pi/3) = \sin y$

i.e. $\Rightarrow y = -\pi/3 \in [-\pi/2, \pi/2]$

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∴ The principal value of $\sin^{-1}\left[\sin\left(-\frac{11}{3}\right)\right]$ is $-\frac{11}{3}$.

(iv) $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

Let $\sin^{-1}\left[\sin\left(\frac{5\pi}{6}\right)\right] = y$

$$\sin\frac{5\pi}{6} = \sin y$$

i.e. $\sin\left[\pi - \frac{11}{6}\right] = \sin y \Rightarrow \sin\frac{11}{6} = \sin y$

$$\Rightarrow y = \frac{11}{6} \in \left[-\frac{11}{2}, \frac{11}{2}\right]$$

∴ The principal value of $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$ is $\frac{11}{6}$

Ex 4.4 Find the domain of $\sin^{-1}(2-3x^2)$

WKT the domain of $\sin^{-1}(x)$ is $[-1,1]$

$$\therefore -1 \le 2-3x^2 \le 1$$

$$-1 \le 2-3x^2$$
$$-3 \le -3x^2$$
$$1 \ge x^2$$
$$x^2 \le 1 \quad \text{—— (1)}$$

$$2-3x^2 \le 1$$
$$-3x^2 \le 1-2$$
$$-3x^2 \le -1$$
$$3x^2 \ge 1$$
$$x^2 \ge \frac{1}{3} \quad \text{—— (2)}$$

From (1) and (2)

$$\frac{1}{3} \le x^2 \le 1 \Rightarrow \frac{1}{\sqrt{3}} \le |x| \le 1$$

$$\Rightarrow x \in \left[-1, -\frac{1}{\sqrt{3}}\right] \cup \left[\frac{1}{\sqrt{3}}, 1\right]$$

Note: $a \le |x| \le b \Rightarrow x \in [-b,-a] \cup [a,b]$

Amplitude:- The Amplitude is the maximum distance of the graph from x-axis. i.e the amplitude of the fun is the height from the x-axis to its maximum or minimum.

Period:- The period is the distance required for the fun to complete one full cycle.

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EXERCISE 4.1

1. Find all the values of x such that (i) $-10\pi \le x \le 10\pi$ and $\sin x = 0$

$\sin x = 0 \Rightarrow x = n\pi$

when $n = \pm 0, 1, 2 \cdots \pm 10$.

(ii) $-8\pi \le x \le 8\pi$ and $\sin x = -1$

$\sin x = -1 = -\sin \pi/2$

$= \sin(-\pi/2)$

$= \sin(2n\pi - \pi/2)$

$= \sin\left(\frac{4n\pi - \pi}{2}\right)$

$= \sin\left[(4n-1)\frac{\pi}{2}\right]$

$\therefore x = (4n-1)\pi/2$

$n = 0, \quad x = -\pi/2 \in [-8\pi, 8\pi]$

$n = 1, \quad x = 3\pi/2 \in [-8\pi, 8\pi]$

$n = -1, \quad x = -\frac{5\pi}{2} \in [-8\pi, 8\pi]$

$n = 2, \quad x = \frac{7\pi}{2} \in [-8\pi, 8\pi]$

$n = -2, \quad x = -9\pi/2 \in [-8\pi, 8\pi]$

$n = 3, \quad x = 11\frac{\pi}{2} \in [-8\pi, 8\pi]$

$n = -3, \quad x = -13\pi/2 \in [-8\pi, 8\pi]$

$n = 4, \quad x = 15\pi/2 \in [-8\pi, 8\pi]$

$n = -4, \quad x = -\frac{17\pi}{2} \notin [-8\pi, 8\pi]$

$\therefore$ The values of n are $0, \pm 1, \pm 2, \pm 3, 4$.

Note: 1) period of $\sin x$, $\cos x$, $\operatorname{cosec} x$, $\sec x$ is 2π.

period of $\tan x$ and $\cot x$ is π.

2) If $y = a\sin bx$ then the amplitude and period are $|a|$ and $\frac{2\pi}{|b|}$.

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2) Find the period and amplitude of

(i) $y = \sin 7x$. ——— (1)

If $y = a \sin bx$ then

Amplitude $= |a|$ and period is $\frac{2\pi}{|b|}$.

Now $y = \sin 7x$ ——— (2)

Compare (1) and (2).

Amplitude $|a| = |1| = 1$

period $= \frac{2\pi}{7}$.

(ii) $y = -\sin\left(\frac{1}{3}x\right)$

$y = \sin\left(-\frac{1}{3}x\right)$ ——— (1)

Compare this with $y = a \sin bx$

$\therefore$ Amplitude $|a| = 1$

period $= \frac{2\pi}{\frac{1}{3}} = 6\pi$.

(iii) $y = 4 \sin(-2x)$

Compare this with $y = a \sin bx$.

amplitude $= |a|$ and period is $\frac{2\pi}{|b|}$

$\therefore$ Amplitude $|a| = |4| = 4$ $\frac{2\pi}{|b|}$

period $= \frac{2\pi}{2} = \pi$.

---

3) Sketch the graph of $y = \sin\left(\frac{1}{3}x\right)$ for $0 \le x \le 6\pi$.

Given $y = \sin\left(\frac{1}{3}x\right)$ $0 \le x \le 6\pi$

when $x = 0$, $y = \sin\left(\frac{0}{3}\right) = \sin 0 = \boxed{0}$

when $x = \pi/2$, $y = \sin(\pi/6) = \boxed{\frac{1}{2}}$

$x = \pi$, $y = \sin \pi/3 = \boxed{\frac{\sqrt{3}}{2}}$

$x = \frac{3\pi}{2}$, $y = \sin\left(\frac{3\pi}{2\times 3}\right) = \sin \pi/2 = \boxed{1}$

$x = 2\pi$, $y = \sin\left(\frac{2\pi}{3}\right) = \sin(\pi - \pi/3)$

$= \sin \pi/3 = \boxed{\frac{\sqrt{3}}{2}}$

$x = \frac{5\pi}{2}$, $y = \sin\left(\frac{5\pi}{2\times 3}\right) = \sin\frac{5\pi}{6}$

$= \sin(\pi - \pi/6) = \sin \pi/6 = \boxed{\frac{1}{2}}$

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When $x = 3\pi$, $y = \sin\frac{3\pi}{3} = \sin\pi = \boxed{0}$.

$x = 3\pi + \pi/2 = \frac{7\pi}{2}$, $y = \sin\left(\frac{7\pi}{2\times3}\right) = \sin\frac{7\pi}{6}$

$= \sin(\pi + \pi/6) = -\sin\pi/6$

$= \boxed{-\frac{1}{2}}$

$x = 7\pi/2 + \pi/2 = \frac{8\pi}{2} = 4\pi$

$\therefore y = \sin\frac{4\pi}{3} = \sin(\pi + \pi/3)$

$= -\sin\pi/3 = \boxed{-\frac{\sqrt{3}}{2}}$

$x = 4\pi + \frac{\pi}{2} = \frac{9\pi}{2}$

$\therefore y = \sin\left(\frac{9\pi}{2\times3}\right) = \sin\frac{3\pi}{2} = \sin(\pi + \pi/2)$

$= -\sin\pi/2 = \boxed{-1}$

$x = \frac{9\pi}{2} + \pi/2 = \frac{10\pi}{2} = 5\pi$

$\therefore y = \sin\left(\frac{5\pi}{3}\right) = \sin(2\pi - \pi/3)$

$= -\sin\pi/3 = \boxed{-\frac{\sqrt{3}}{2}}$

$x = \frac{5\pi}{3} + \pi/2$

$= \frac{11\pi}{6}$

$\therefore y = \sin\left(\frac{11\pi}{2\times3}\right) = \sin\left(\frac{11\pi}{6}\right)$

$= \sin(2\pi - \pi/6) = -\sin\pi/6$

$= \boxed{-\frac{1}{2}}$

$x = 11\pi/2 + \pi/2 = \frac{12\pi}{2} = 6\pi$.

$\therefore y = \sin\left(\frac{6\pi}{3}\right) = \sin 2\pi = 0$.

x	0	$\pi/2$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π	$\frac{9\pi}{2}$
y	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1

5π	$\frac{11\pi}{2}$	6π
$-\frac{\sqrt{3}}{2}$	$-1/2$	0

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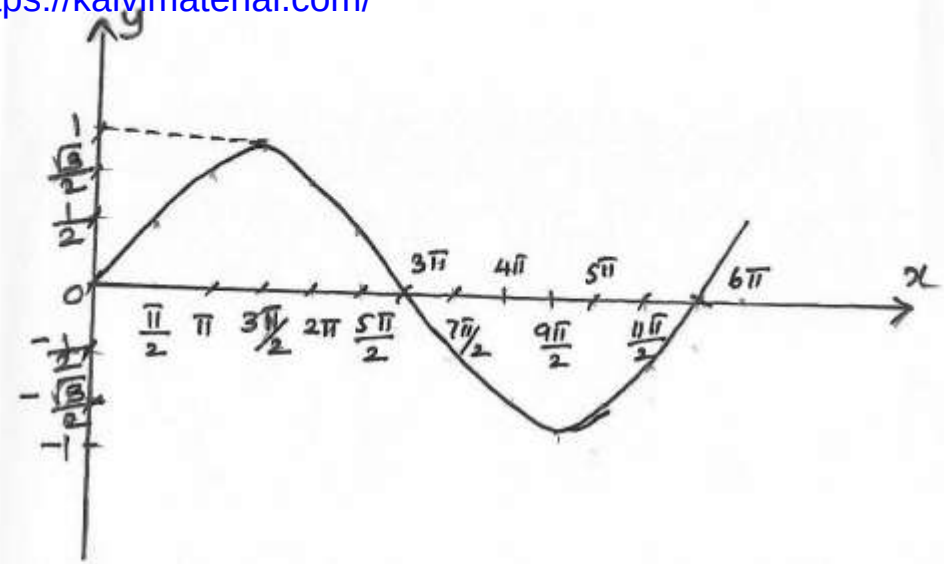

4) Find the values of

(i) $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ $\because \frac{2\pi}{3} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Let $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = y$

$\sin\frac{2\pi}{3} = \sin y$

$\sin\left(\pi - \frac{\pi}{3}\right) = \sin y$

$\sin\frac{\pi}{3} = \sin y \Rightarrow y = \frac{\pi}{3}$

$\therefore \sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$

(ii) $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$

Let $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right) = y$

$\sin\frac{5\pi}{4} = \sin y \Rightarrow \sin\left(\pi + \frac{\pi}{4}\right) = \sin y$

$\because \frac{5\pi}{4} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$-\sin\frac{\pi}{4} = \sin y$

$\sin\left(-\frac{\pi}{4}\right) = \sin y$

$\Rightarrow y = -\frac{\pi}{4}$

$\therefore \sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right) = -\frac{\pi}{4}$

---

5) For what value of x does $\sin x = \sin^{-1} x$?

$y = \sin x$ and $y = \sin^{-1} x$ both the curves are intersect at the origin.

$\therefore x = 0$.

---

6) Find the domain of the following

(i) $f(x) = \sin^{-1}\left(\frac{x^2+1}{2x}\right)$

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Given $f(x) = \sin^{-1}\left(\frac{x^2+1}{2x}\right)$

WKT The Domain of $\sin^{-1}x$ is -1 to 1

$\therefore \quad -1 \le \frac{x^2+1}{2x} \le 1$

Consider $-1 \le \frac{x^2+1}{2x}$

$-2x \le x^2+1$

$x^2+1+2x \ge 0$

$(x+1)^2 \ge 0$

$\Rightarrow x \ge 0$ or $x = -1$ — (1)

Consider $\frac{x^2+1}{2x} \le 1$

$x^2+1 \le 2x$

$x^2-2x+1 \le 0$

$(x-1)^2 \le 0$

$\Rightarrow x \le 0$ or $x = 1$ — (2)

From (1) and (2)

Domain is $\{-1, 1\}$

(ii) $g(x) = 2\sin^{-1}(2x-1) - \pi/4$

WKT The Domain of $\sin^{-1}x$ is -1 to 1

$\therefore \quad -1 \le 2x-1 \le 1$ (consider only argument (x-value) of the function $2x-1$)

$-1 \le 2x-1$

$0 \le 2x$

$2x \ge 0$

$x \ge 0$

$2x-1 \le 1$

$2x \le 2$

$x \le 1$

$\therefore$ Domain $= [0, 1]$

---

7) Find the value of

$\sin^{-1}\left(\sin\frac{5\pi}{9}\cos\pi/9 + \cos\frac{5\pi}{9}\sin\frac{\pi}{9}\right)$

$\sin^{-1}\left(\sin\frac{5\pi}{9}\cos\pi/9 + \cos\frac{5\pi}{9}\sin\pi/9\right)$

$= \sin^{-1}\left[\sin\left(\frac{5\pi}{9}+\frac{\pi}{9}\right)\right] = \sin^{-1}\left[\sin\frac{6\pi}{9}\right]$

$= \sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}(\pi - \pi)$

$= \sin^{-1}\left[\sin\left(\pi - \pi/3\right)\right] = \sin^{-1}\left[\sin\pi/3\right]$

$= \pi/3 \in [-\pi/2, \pi/2]$

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The cosine and Inverse of cosine function:-

The cosine function is a function with domain is R and its range is $[-1,1]$.

The period of cosine function is 2π.

The cosine function is continuous, there is no break or discontinuous.

The cosine function is even, the graph is symmetric about y-axis.

Ex 4.5 Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$

$\cos y = \frac{\sqrt{3}}{2} \Rightarrow y = \pi/6 \in [0,\pi]$

$\therefore$ The principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\pi/6$.

The range of the principal value of $y = \cos^{-1}x$ is $[0,\pi]$

Ex 4.6 Find (i) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Let $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = y$

$\cos y = -\frac{1}{\sqrt{2}} \Rightarrow \cos y = -\cos \pi/4$

$\Rightarrow \cos y = \cos(\pi - \pi/4)$

$= \cos 3\pi/4$

$\Rightarrow \boxed{y = 3\pi/4}$

(ii) $\cos^{-1}\left(\cos(-\pi/3)\right)$

$\cos^{-1}\left(\cos(-\pi/3)\right)$

$= \cos^{-1}\left(\cos \pi/3\right)$

$= \pi/3 \in [0,\pi]$

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(iii) $\cos^{-1}\left[\cos\left(\frac{7\pi}{6}\right)\right]$

$\cos^{-1}\left[\cos\left(\frac{7\pi}{6}\right)\right] = \cos^{-1}\left[\cos\left(\pi+\pi/6\right)\right]$

$= \cos^{-1}\left[-\cos \pi/6\right]$

$= \cos^{-1}\left[\cos\left(\pi-\pi/6\right)\right]$

$= \cos^{-1}\left[\cos 5\pi/6\right]$

$= 5\pi/6 \in [0,\pi]$

Ex 4.7 Find the domain of $\cos^{-1}\left(\frac{2+\sin x}{3}\right)$

The domain of $\cos^{-1}x$ is $[-1,1]$

$-1 \le \frac{2+\sin x}{3} \le 1$

Consider $-1 \le \frac{2+\sin x}{3}$

$-3 \le 2+\sin x$

$-5 \le \sin x$ ——— (1)

$\frac{2+\sin x}{3} \le 1 \Rightarrow 2+\sin x \le 3$

$\sin x \le 3-2$

$\sin x \le 1$ ——— (2)

From (1) and (2)

$-5 \le \sin x \le 1 \Rightarrow -1 \le \sin x \le 1$

$\Rightarrow -\sin^{-1}(1) \le x \le \sin^{-1}(1)$

$\Rightarrow -\pi/2 \le x \le \pi/2$

$\therefore \cos^{-1}\left[\frac{2+\sin x}{3}\right]$ is $[-\pi/2, \pi/2]$

EXERCISE 4.2

1) Find all the values of x such that (i) $-6\pi \le x \le 6\pi$ and $\cos x = 0$

$\cos x = 0 \Rightarrow \cos(2\pi + \pi/2) = 0$

$\cos\left(\frac{2\pi n + \pi}{2}\right) = 0$

$\cos(2n+1)\pi/2 = 0$

$\Rightarrow x = (2n+1)\pi/2$

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$n = 0, \pm1, \pm2, \pm3, \pm4, \pm5, -6$

$n=0,\ x = \pi/2 \in [-6\pi, 6\pi]$

$n=1,\ x = 3\pi/2 \in [-6\pi, 6\pi]$

$n=2,\ x = \frac{5\pi}{2} \in [-6\pi, 6\pi]$

$n=3,\ x = \frac{7\pi}{2} \in [-6\pi, 6\pi]$

$n=4,\ x = \frac{9\pi}{2} \in [-6\pi, 6\pi]$

$n=5,\ x = \frac{11\pi}{2} \in [-6\pi, 6\pi]$

$n=6,\ x = \frac{13\pi}{2} \notin [-6\pi, 6\pi]$

iii)

$n=-1,$	$x = -\pi/2$
$n=-2$	$x = -\frac{3\pi}{2}$
$n=-3,$	$x = -\frac{5\pi}{2}$
$n=-4$	$x = -\frac{7\pi}{2}$
$x=-5,$	$x = -\frac{9\pi}{2}$
$x=-6,$	$x = -\frac{11\pi}{2}$

---

2) State the reason for $\cos^{-1}[\cos(-\pi/6)] \neq -\pi/6$

The range of $\cos^{-1}x$ is $[0, \pi]$

$-\pi/6 \notin [0, \pi]$

$\therefore \cos^{-1}[\cos(-\pi/6)] \neq -\pi/6$

---

3) Is $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$ is true. Justify your answer.

$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

$\cos(\pi - \cos^{-1}x) = -x$

i.e. $\cos\pi \cdot \cos(\cos^{-1}x) + \sin\pi \sin(\cos^{-1}x) = -x$

$(-1)(x) + 0 \cdot \sin(\cos^{-1}x) = -x$

$-x + 0 = -x$

$-x = -x$

$\therefore \cos^{-1}(-x) = \pi - \cos^{-1}(x)$ is true.

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4) Find the principal value of $\cos^{-1}\left(\frac{1}{2}\right)$.

Let $\cos^{-1}\left(\frac{1}{2}\right) = y \Rightarrow \cos y = \frac{1}{2}$

$\cos y = \cos \pi/3$

$\Rightarrow y = \pi/3 \in [0, \pi]$

∴ The principal value of $\cos^{-1}\left(\frac{1}{2}\right) = \pi/3$

---

5 Find the value of a) $2\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$

$2\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$

$= 2 \times \pi/3 + \pi/6 \Rightarrow \frac{2\pi}{3} + \frac{\pi}{6}$

$= \frac{4\pi + \pi}{6} = \boxed{\frac{5\pi}{6}}$

(ii) $\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}(-1)$

$= \pi/3 + (-\sin^{-1}(1))$

$= \pi/3 - \pi/2$

$\therefore \frac{2\pi - 3\pi}{6} = \boxed{-\pi/6}$

(iii) $\cos^{-1}\left[\cos \pi/7 \cos \frac{\pi}{17} - \sin \frac{\pi}{7} \sin \frac{\pi}{17}\right]$

$= \cos^{-1}\left[\cos\left(\frac{\pi}{7} + \frac{\pi}{17}\right)\right]$

$= \cos^{-1}\left[\cos\left(\frac{17\pi + 7\pi}{119}\right)\right]$

$= \cos^{-1}\left[\cos\left(\frac{24\pi}{119}\right)\right] = \boxed{\frac{24\pi}{119}}$

---

6) Find the domain of

(i) $f(x) = \sin^{-1}\left(\frac{|x| - 2}{3}\right) + \cos^{-1}\left(\frac{1 - |x|}{4}\right)$

The domain of $\sin^{-1}x$ is $[-1, 1]$

The domain of $\cos^{-1}x$ is $[-1, 1]$.

Consider $\sin^{-1}\left(\frac{|x| - 2}{3}\right)$

$-1 \le \frac{|x| - 2}{3} \le 1$

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$-3 \le |x| - 2 \le 3$

$-3+2 \le |x| \le 3+2$

$-1 \le |x| \le 5$ —— (1).

Consider $\cos^{-1}\left(\frac{1-|x|}{4}\right)$

$-1 \le \frac{1-|x|}{4} \le 1$

$-4 \le 1-|x| \le 4$

$-5 \le -|x| \le 4-1$

$-5 \le -|x| \le 3$

$5 \ge |x| \ge -3$

i.e. $-3 \le |x| \le 5$ — (2)

From (1) and (2) $|x| \le 5$

i.e. $-5 \le x \le 5$

∴ Domain of $\sin^{-1}\left(\frac{|x|-2}{3}\right) + \cos^{-1}\left(\frac{1-|x|}{4}\right)$ is $[-5, 5]$

(ii) $y(x) = \sin^{-1}x + \cos^{-1}x$

The domain of $\sin^{-1}x$ is $[-1, 1]$

The domain of $\cos^{-1}x$ is $[-1, 1]$

∴ The domain of $\sin^{-1}x + \cos^{-1}x$ is $[-1, 1]$.

---

7) For what value of x, the inequality $\pi/2 < \cos^{-1}(3x-1) < \pi$ holds?

$\pi/2 < \cos^{-1}(3x-1) < \pi$

$\cos\pi/2 > 3x-1 > \cos\pi$

$0 > 3x-1 > -1$

$1 > 3x > 0$

$0 < 3x < 1$

$0 < x < \frac{1}{3}$

$\cos x$ is decreasing function ∴ The inequality get rerersed.

∴ The given inequality holds if $0 < x < \frac{1}{3}$.

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8) Find the value of

a) $\cos\left[\cos^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right)\right]$

WKT $\cos^{-1}x + \sin^{-1}x = \pi/2$

$\therefore \cos\left[\cos^{-1}\frac{4}{5} + \sin^{-1}\frac{4}{5}\right]$

$= \cos\left[\pi/2\right] = 0$

b) $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right) + \cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$

$= \cos^{-1}\left(\cos\left(\pi + \pi/3\right)\right) + \cos^{-1}\left(\cos\left(\pi + \pi/4\right)\right)$

$= \cos^{-1}\left(-\cos\pi/3\right) + \cos^{-1}\left(-\cos\pi/4\right)$

$= \cos^{-1}\left(\cos\left(\pi - \pi/3\right)\right) + \cos^{-1}\left(\cos\left(\pi - \pi/4\right)\right)$

$= \cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right) + \cos^{-1}\left(\cos\left(3\pi/4\right)\right)$

$= \frac{2\pi}{3} + \frac{3\pi}{4} = \frac{8\pi + 9\pi}{12}$

$= \frac{17\pi}{12}$ //.

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The tangent function and the inverse tangent function:-

1) The tangent function is not defined at $x = \dots -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$

2) The domain of tangent function is $\{x : x \in R, x \neq \frac{\pi}{2} + n\pi, n \in Z\}$

3) The range of tangent function is $(-\infty, \infty)$.

4) The period of $y = \tan x$ is π.

properties:-

1) The graph $y = \tan x$ is not continuous and has discontinuous at $x = (2n+1)\pi/2$.

2) The partial graph $y = \tan x$ is symmetric about the origin for $-\pi/2 < x < \pi/2$.

3) It has infinitely many vertical asymptotes $x = (2n+1)\pi/2$.

Inverse tangent function:-

1) $y = \tan^{-1} x$ is odd function.

2) $\tan(\tan^{-1} x) = x$

3) $\tan^{-1}(\tan x) = x$ if $-\pi/2 < x < \pi/2$.

4) $\tan^{-1}(\tan \pi) = 0$ and not π.

Ex 4.8 Find the principal value of $\tan^{-1}(\sqrt{3})$.

Let $y = \tan^{-1}(\sqrt{3})$

$\Rightarrow \tan y = \sqrt{3}$

$\tan y = \tan \pi/3 \Rightarrow y = \pi/3 \in (-\pi/2, \pi/2)$

Hence the principal value of $\tan^{-1}(\sqrt{3}) = \pi/3$

Ex 4.9 Find (i) $\tan^{-1}(-\sqrt{3})$

Let $y = \tan^{-1}(-\sqrt{3})$

$\tan y = -\sqrt{3}$

$\tan y = -\tan \pi/3$

$\tan y = \tan(-\pi/3)$

$\Rightarrow y = -\pi/3 \in (-\pi/2, \pi/2)$

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(ii) $\tan^{-1}\left(\tan\frac{3\pi}{5}\right)$

Let $y = \tan^{-1}\left(\tan\frac{3\pi}{5}\right)$

$\tan y = \tan\frac{3\pi}{5}$

$= \tan\left(\pi - \frac{2\pi}{5}\right)$

$= -\tan\frac{2\pi}{5}$

$= \tan\left(-\frac{2\pi}{5}\right)$

$\therefore y = -\frac{2\pi}{5} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iii) $\tan\left(\tan^{-1}(2019)\right)$

WKT $\tan(\tan^{-1}(x)) = x$

$\therefore \tan(\tan^{-1}(2019)) = 2019$.

4.10 Find the value of

$\tan^{-1}(-1) + \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

Consider $\tan^{-1}(-1)$

Let $y = \tan^{-1}(-1)$

$\tan y = -1$

$\tan y = -\tan\pi/4$

$\tan y = \tan(-\pi/4)$

$\Rightarrow y = -\pi/4 \in \left(-\pi/2, \pi/2\right)$.

$\cos^{-1}\left(\frac{1}{2}\right)$

Let $y = \cos^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow \cos y = \frac{1}{2}$

$\cos y = \cos\pi/3$

$\Rightarrow y = \pi/3 \in [0, \pi]$

$\sin^{-1}\left(-\frac{1}{2}\right)$

Let $y = \sin^{-1}\left(-\frac{1}{2}\right)$

$\sin y = -\frac{1}{2}$

$\sin y = -\sin\pi/6$

$\sin y = \sin(-\pi/6)$

$\Rightarrow y = -\pi/6 \in \left[-\pi/2, \pi/2\right]$

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$\tan^{-1}(-1) + \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

$= -\pi/4 + \pi/3 - \pi/6$

$= \frac{-3\pi + 4\pi - 2\pi}{12} = -\pi/12$

Ex 4.11 (X)

prove that $\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}, \quad -1 < x < 1$

$\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}$

when $x = 0$, $\tan(\sin^{-1}(0)) = \frac{0}{Dr} = 0$

$\tan(0) = 0$

$0 = 0$ ——— (1).

$\therefore$ when $x = 0$, satisfies the eqn.

Assume that $0 < x < 1$

Let $\theta = \sin^{-1}x \Rightarrow \sin\theta = x$

$0 < x < 1$

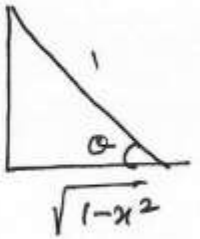
Taking $\sin^{-1}$ throughout

$\sin^{-1}(0) < \sin^{-1}(x) < \sin^{-1}(1)$

$0 < \sin^{-1}x < \pi/2$

ie $0 < \theta < \pi/2$.

In Δl,



$\sin\theta = x$

$\tan\theta = \frac{x}{\sqrt{1-x^2}}$

$\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}, \quad 0 < x < 1$ ——— (2)

Assume that $-1 < x < 0$.

taking $\sin^{-1}$ throughout

$\sin^{-1}(-1) < \sin^{-1}x < \sin^{-1}(0)$

$-\sin^{-1}(1) < \theta < 0$

$-\pi/2 < \theta < 0$

$\sin\theta = x$

$\tan\theta = \frac{x}{\sqrt{1-x^2}}$

$\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}, \quad -1 < x < 0$ ——— (3)

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From ①, ②, ③

$$\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}, \quad -1<x<1$$

EXERCISE 4.3

1) Find the domain of

a) $\tan^{-1}(\sqrt{9-x^2})$

WKT the domain of $\tan^{-1}x$ is $(-\infty,\infty)$

$\tan^{-1}(\sqrt{9-x^2})$ is defined only for

$9-x^2 \geq 0$

$9 \geq x^2 \Rightarrow x^2 \leq 9$

$\Rightarrow x \leq \pm 3$

∴ The domain of $\tan^{-1}(\sqrt{9-x^2})$ is $[-3,3]$.

b) $\frac{1}{2}\tan^{-1}(1-x^2) - \pi/4$.

WKT the domain of $\tan^{-1}(x)$ is $(-\infty\ \infty)$

$\therefore -\infty < 1-x^2 < \infty$

$-\infty - 1 < -x^2 \leq \infty - 1$

$-\infty < -x^2 < \infty$

$\infty > x^2 > -\infty$

$\therefore -\infty < x^2 < \infty$

i.e. $-\infty < x < \infty$

Hence the domain of $\tan^{-1}(1-x^2)$ is $(-\infty,\infty)$

Hence the domain of $\frac{1}{2}\tan^{-1}(1-x^2) - \pi/4$ is $(-\infty,\infty)$. i.e R.

2) Find the value of

a) $\tan^{-1}(\tan\frac{5\pi}{4})$

$\tan^{-1}(\tan(\frac{5\pi}{4})) \neq \frac{5\pi}{4}$ ∵ $\frac{5\pi}{4} \notin (-\pi/2, \pi/2)$

$\therefore \tan^{-1}(\tan(\frac{5\pi}{4}))$

$= \tan^{-1}(\tan(\pi + \pi/4))$

$= \tan^{-1}(+\tan\pi/4)$

$= \tan^{-1}(\tan(+\pi/4))$

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$= +\pi/4 \in (-\pi/2, \pi/2)$.

Hence $\tan^{-1}(\tan(\frac{5\pi}{4})) = \pi/4$.

(ii) $\tan^{-1}(\tan(-\pi/6))$

$\tan^{-1}(\tan(-\pi/6)) = -\pi/6 \in (-\pi/2, \pi/2)$.

---

3) Find the value of

a) $\tan(\tan^{-1}(\frac{7\pi}{4}))$

W.K.T $\tan(\tan^{-1}x) = x$

$\therefore \tan(\tan^{-1}(\frac{7\pi}{4})) = \frac{7\pi}{4}$

b) $\tan(\tan^{-1}(1947))$

W.K.T $\tan(\tan^{-1}(x)) = x$

$\therefore \tan(\tan^{-1}(1947)) = 1947$.

(iii) $\tan(\tan^{-1}(-0.2021))$

W.K.T $\tan(\tan^{-1}x) = x$

$\therefore \tan(\tan^{-1}(-0.2021))$

$= -0.2021$.

---

4) Find the value of

(i) $\tan[\cos^{-1}(\frac{1}{2}) - \sin^{-1}(-\frac{1}{2})]$

Consider $\cos^{-1}(\frac{1}{2})$

Let $y = \cos^{-1}(\frac{1}{2})$

$\cos y = \frac{1}{2} \Rightarrow \cos y = \cos \pi/3$

$\Rightarrow y = \pi/3$

i $\boxed{\cos^{-1}(\frac{1}{2}) = \pi/3}$

Consider $\sin^{-1}(-\frac{1}{2})$

Let $y = \sin^{-1}(-\frac{1}{2})$

$\sin y = -\frac{1}{2}$

$\sin y = -\sin \pi/6$

$\sin y = \sin(-\pi/6) \Rightarrow y = -\pi/6$

$\therefore \boxed{\sin^{-1}(-\frac{1}{2}) = -\pi/6}$

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Hence $\tan\left[\cos^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right]$

$= \tan\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right]$

$= \tan\left[\frac{\pi}{3} + \frac{\pi}{6}\right]$

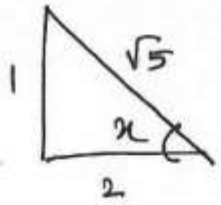
$= \tan\left[\frac{2\pi + \pi}{6}\right] = \tan\frac{3\pi}{6}$

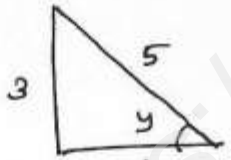
$= \tan\frac{\pi}{2} = \boxed{\infty}$

(ii) $\sin\left[\tan^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{4}{5}\right)\right]$

Let $x = \tan^{-1}\left(\frac{1}{2}\right)$ and $y = \cos^{-1}\left(\frac{4}{5}\right)$

$\tan x = \frac{1}{2}$ and $\cos y = \frac{4}{5}$





Now $\sin\left[\tan^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{4}{5}\right)\right]$

$= \sin\left[x - y\right]$

$= \sin x \cos y - \cos x \sin y$

$= \frac{1}{\sqrt{5}} \times \frac{4}{5} - \frac{2}{\sqrt{5}} \times \frac{3}{5}$

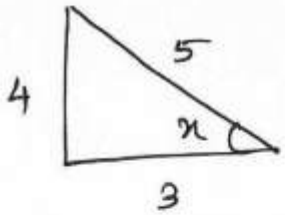
$\therefore \quad \frac{4}{5\sqrt{5}} - \frac{6}{5\sqrt{5}} = -\frac{2}{5\sqrt{5}}$

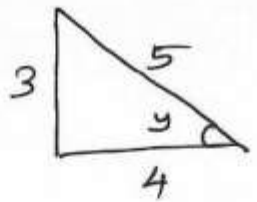
$= -\frac{2}{5\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \boxed{-\frac{2\sqrt{5}}{25}}$

(iii) $\cos\left[\sin^{-1}\left(\frac{4}{5}\right) - \tan^{-1}\left(\frac{3}{4}\right)\right]$

Let $x = \sin^{-1}\left(\frac{4}{5}\right)$ and $y = \tan^{-1}\left(\frac{3}{4}\right)$

$\sin x = \frac{4}{5}$ and $\tan y = \frac{3}{4}$





Now $\cos\left[\sin^{-1}\left(\frac{4}{5}\right) - \tan^{-1}\left(\frac{3}{4}\right)\right]$

$= \cos\left[x - y\right]$

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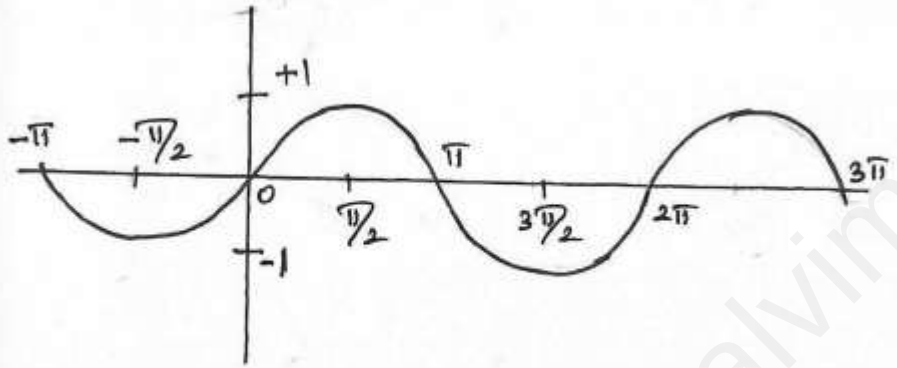
$\cos x \cos y + \sin x \sin y$

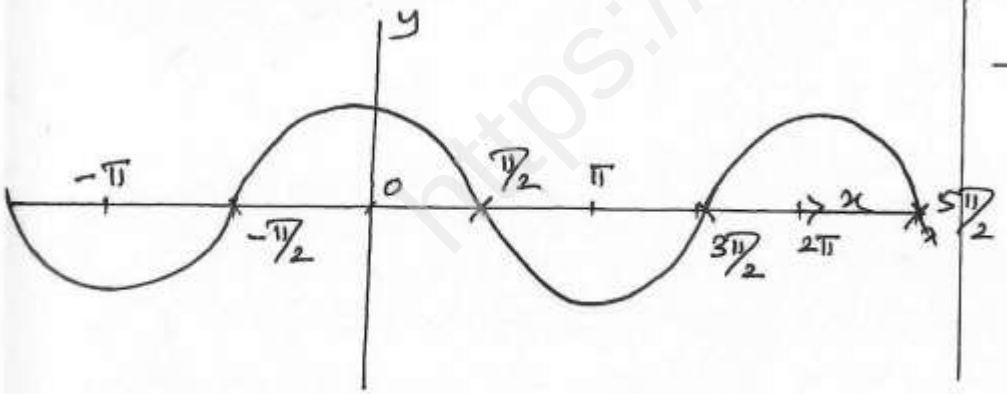
$= \frac{3}{5} \times \frac{4}{5} + \frac{4}{5} \times \frac{3}{5}$

$= \frac{12}{25} + \frac{12}{25} = \boxed{\frac{24}{25}}$

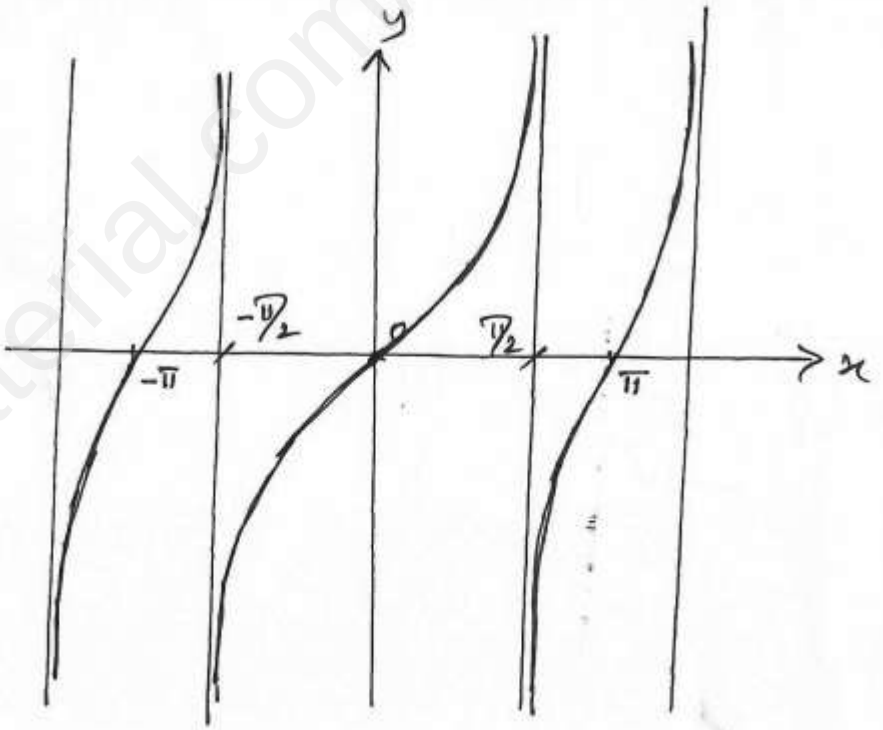
Sine - Graph



Cosine - Graph



Tangent - graph



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The cosecant function and the inverse coseeant function.

* The domain of cosecant function is $R \setminus \{ n\pi : n \in Z \}$.

* Since $-1 \le \sin x \le 1$, $y = \text{cosec}\, x$ does not take any value in betn -1 and 1.

* ∴ The range of coseeant function is $(-\infty, -1] \cup [1, \infty)$

* The inverse cosecant function $\text{cosec}^{-1} x : (-\infty, -1] \cup [1, \infty) \to [-\pi/2, 0) \cup (0, \pi/2]$

The Secant function and Inverse of seeant function.

* The domain of $y = \sec x$ is $R - \{ (2n+1)\pi/2 : n \in Z \}$

* $y = \sec x$ does not take any value betn -1 and 1.

* The range of seeant function is $(-\infty, -1] \cup [1, \infty)$

The inverse of Secant function $\sec^{-1} : R \setminus (-1, 1) \to [0, \pi] - \{\pi/2\}$ is defined by $\sec^{-1}(x) = y$ and $y \in [0, \pi] \setminus \{\pi/2\}$.

The cotangent function and the inverse of cotangent function.

* The domain of cotangent function is $R \setminus \{ n\pi : n \in Z \}$ and its range is $(-\infty, \infty)$.

* tangent and cotangent function is odd function. and the period is π.

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Inverse of cotangent function :-

The inverse of cotangent function $\cot^{-1} : (-\infty, \infty) \to (0, \pi)$ is defined by $\cot^{-1}(x) = y$ iff $\cot y = x$ and $y \in (0, \pi)$.

Ex 4.12

Find the principal value of

a) $\operatorname{cosec}^{-1}(-1)$

Let $\operatorname{cosec}^{-1}(-1) = y$

$\therefore \operatorname{cosec} y = -1$

$\frac{1}{\sin y} = -1 \Rightarrow \sin y = -1$

$\Rightarrow \sin y = -\sin \pi/2$

$\sin y = \sin(-\pi/2)$

$\Rightarrow y = -\pi/2 \in [-\pi/2, \pi/2] - \{0\}$

Thus the principal value of $\operatorname{cosec}^{-1}(-1)$ is $-\pi/2$

(ii) $\sec^{-1}(-2)$

Let $\sec^{-1}(-2) = y$

$\therefore \sec y = -2$

$\frac{1}{\cos y} = -2 \Rightarrow \cos y = -\frac{1}{2}$

i.e. $\cos y = -\cos \pi/3$

$\cos y = \cos(\pi - \pi/3)$

$\cos y = \cos 2\pi/3$

$\Rightarrow y = 2\pi/3 \in [0, \pi] - \{\pi/2\}$.

$\therefore$ The principal value of $\sec^{-1}(-2) = 2\pi/3$

Ex 4.13 Find the value of $\sec^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$

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Let $\sec^{-1}\left(-\frac{2\sqrt{3}}{3}\right) = y$

$\Rightarrow \sec y = -\frac{2\sqrt{3}}{3}$

$\sec y = -\frac{2}{\sqrt{3}}$

$\frac{1}{\cos y} = -\frac{2}{\sqrt{3}} \Rightarrow \cos y = -\frac{\sqrt{3}}{2}$

$\cos y = -\frac{\sqrt{3}}{2}$

i.e. $\cos y = -\cos \pi/6$

$\cos y = \cos(\pi - \pi/6)$

$= \cos\left(\frac{5\pi}{6}\right)$

$\therefore y = 5\pi/6 \in [0, \pi] - \{\pi/2\}$.

EX 4.14 If $\cot^{-1}\left(\frac{1}{7}\right) = \alpha$ find the value of $\cos\alpha$.

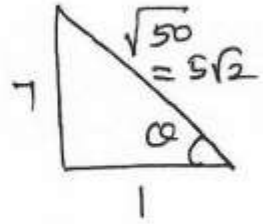
$\cot^{-1}\left(\frac{1}{7}\right) = \alpha \Rightarrow \cot\alpha = \frac{1}{7}$

$\tan\alpha = 7$

i.e. $\tan\alpha = \frac{7}{1}$

From the triangle

$\cos\alpha = \frac{1}{5\sqrt{2}}$



EX 4.15 Show that $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \sec^{-1}x$; $|x| > 1$

Let $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \alpha$ — ①

$\therefore \cot\alpha = \frac{1}{\sqrt{x^2-1}}$

From the given data, construct a right angled triangle.

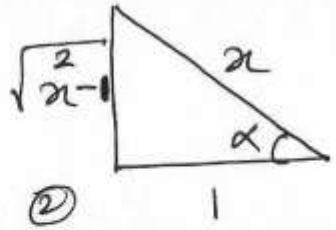
From the triangle.

$\sec\alpha = x$

$\therefore \alpha = \sec^{-1}x$ — ②

From ① and ②

$\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \sec^{-1}x$ //.



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EXERCISE 4.4

1. Find the principal value of

a) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y \Rightarrow \sec y = \frac{2}{\sqrt{3}}$

i.e. $\cos y = \frac{\sqrt{3}}{2}$

$\cos y = \cos \pi/6 \Rightarrow y = \pi/6$

∴ The principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is $\pi/6$.

(ii) $\cot^{-1}(\sqrt{3})$

Let $\cot^{-1}(\sqrt{3}) = y \Rightarrow \cot y = \sqrt{3}$

i.e. $\tan y = \frac{1}{\sqrt{3}}$

$\tan y = \tan \pi/6$

$\Rightarrow y = \pi/6$.

∴ The principal value of $\cot^{-1}(\sqrt{3})$ is $\pi/6$.

(iii) $\operatorname{cosec}^{-1}(-\sqrt{2})$

Let $\operatorname{cosec}^{-1}(-\sqrt{2}) = y$

$\Rightarrow \operatorname{cosec} y = -\sqrt{2}$

i.e. $\frac{1}{\sin y} = -\sqrt{2}$

$\Rightarrow \sin y = -\frac{1}{\sqrt{2}} = -\sin \pi/4$

$= \sin(-\pi/4)$

∴ $y = -\pi/4$ is the principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$.

---

2) Find the value of

a) $\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$

Consider $\tan^{-1}(\sqrt{3}) = y$

$\tan y = \sqrt{3}$

$\tan y = \tan \pi/3$

$\Rightarrow \boxed{y = \pi/3}$

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Let $\sec^{-1}(-2) = x$

$\sec x = -2$

$\cos x = -\frac{1}{2}$

$\cos x = -\cos \pi/3$

$\cos x = \cos(\pi - \pi/3)$

$= \cos\left(\frac{2\pi}{3}\right)$

$\therefore x = \frac{2\pi}{3}$.

Hence $\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$

$= \pi/3 - \left(\frac{2\pi}{3}\right)$

$= \frac{\pi - 2\pi}{3} = \boxed{-\pi/3}$

(ii) $\sin^{-1}(-1) + \cos^{-1}\left(\frac{1}{2}\right) + \cot^{-1}(2)$

Let $\sin^{-1}(-1) = x$

$\sin x = -1$

$\sin x = -\sin \pi/2$

$\sin x = \sin(-\pi/2)$

$\Rightarrow x = -\pi/2$

Let $\cos^{-1}\left(\frac{1}{2}\right) = y \Rightarrow \cos y = \frac{1}{2}$

$\cos y = \cos \pi/3$

$\Rightarrow y = \pi/3$

Let $\cot^{-1}(2) = z$.

Hence $\sin^{-1}(-1) + \cos^{-1}\left(\frac{1}{2}\right) + \cot^{-1}(2)$

$= -\pi/2 + \pi/3 + \cot^{-1}(2)$

$= \frac{-3\pi + 2\pi}{6} + \cot^{-1}(2)$

$= -\pi/6 + \cot^{-1}(2)$

(iii) $\cot^{-1}(1) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sec^{-1}(-\sqrt{2})$

Let $\cot^{-1}(1) = x$

$\cot x = 1$

$\cot x = \cot \pi/4$

$\Rightarrow x = \pi/4$

Let $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = y$

$\sin y = -\frac{\sqrt{3}}{2}$

$\sin y = -\sin \pi/3$

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$\sin y = \sin\left(-\frac{\pi}{3}\right)$

$\Rightarrow y = -\frac{\pi}{3}$

$\sec^{-1}(-\sqrt{2}) = z$

$\sec z = -\sqrt{2}$

$\cos z = -\frac{1}{\sqrt{2}}$

$\cos z = -\cos\frac{\pi}{4}$

$\cos z = \cos\left(\pi - \frac{\pi}{4}\right)$

$\cos z = \cos\frac{3\pi}{4}$

$\Rightarrow z = \frac{3\pi}{4}$

Hence $\cot^{-1}(1) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sec^{-1}(-\sqrt{2})$

$= \frac{\pi}{4} + \left(-\frac{\pi}{3}\right) - \frac{3\pi}{4}$

$= \frac{\pi}{4} - \frac{\pi}{3} - \frac{3\pi}{4} = \frac{3\pi - 4\pi - 9\pi}{12}$

$= -\frac{10\pi}{12}$

$= -\frac{5\pi}{6}$ //

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4.16 prove that $\frac{\pi}{2} \le \sin^{-1}x + 2\cos^{-1}x \le \frac{3\pi}{2}$

$\sin^{-1}x + 2\cos^{-1}x$

$= \sin^{-1}x + \cos^{-1}x + \cos^{-1}x$

$= \frac{\pi}{2} + \cos^{-1}x$

WKT, $0 \le \cos^{-1}x \le \pi$

$0 + \pi/2 \le \cos^{-1}x + \frac{\pi}{2} \le \pi + \frac{\pi}{2}$

$\pi/2 \le \sin^{-1}x + 2\cos^{-1}x \le \frac{3\pi}{2}$

Hence the proof.

Ex 4.17 Simplify

a) $\cos^{-1}\left[\cos\left(\frac{13\pi}{3}\right)\right]$

$\cos^{-1}\left[\cos\left(\frac{13\pi}{3}\right)\right] = \frac{13\pi}{3} \notin [0, \pi]$

Since the range of $\cos^{-1}x$ is $[0, \pi]$

$\therefore \cos^{-1}\left[\cos\left(\frac{13\pi}{3}\right)\right]$

$= \cos^{-1}\left[\cos\left(4\pi + \frac{\pi}{3}\right)\right]$

$= \cos^{-1}\left[\cos\frac{\pi}{3}\right]$

$= \pi/3 \in [0, \pi]$

$\therefore \cos^{-1}\left[\cos\left(\frac{13\pi}{3}\right)\right] = \pi/3$

(ii) $\tan^{-1}\left[\tan\left(\frac{3\pi}{4}\right)\right]$

$\tan^{-1}\left[\tan\left(\frac{3\pi}{4}\right)\right] = \frac{3\pi}{4} \notin [-\pi/2, \pi/2]$

Since the range of $\tan^{-1}x$ is $[-\pi/2, \pi/2]$

$\therefore \tan^{-1}\left[\tan\left(\frac{3\pi}{4}\right)\right]$

$= \tan^{-1}\left[\tan\left(\pi - \pi/4\right)\right]$

$= \tan^{-1}\left[-\tan\pi/4\right]$

$= \tan^{-1}\left[\tan(-\pi/4)\right]$

$= -\pi/4 \in [-\pi/2, \pi/2]$

$\therefore \tan^{-1}\left[\tan\frac{3\pi}{4}\right] = -\pi/4$ //.

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(iii) $\sec^{-1}\left[\sec\left(\frac{5\pi}{3}\right)\right]$

$\sec^{-1}\left[\sec\left(\frac{5\pi}{3}\right)\right] = \frac{5\pi}{3} \notin [0,\pi] \setminus \{\frac{\pi}{2}\}$

Since the ~~pri~~ range of $\sec^{-1}x$ is $[0,\pi] \setminus \{\pi/2\}$.

$\therefore \sec^{-1}\left[\sec\left(\frac{5\pi}{3}\right)\right] = \sec^{-1}\left[\sec\left(2\pi - \pi/3\right)\right]$

$= \sec^{-1}\left[\sec \pi/3\right]$

$= \pi/3 \in [0,\pi] \setminus \{\pi/2\}$

$\therefore \sec^{-1}\left[\sec\left(\frac{5\pi}{3}\right)\right] = \pi/3$

Ex 4.18 Find the value of

a) $\sin\left[\pi/3 - \sin^{-1}(-\frac{1}{2})\right]$

$\sin\left[\pi/3 - \sin^{-1}\left(-\sin \pi/6\right)\right]$

$\sin\left[\frac{\pi}{3} - \sin^{-1}\left(\sin(-\pi/6)\right)\right]$

$= \sin\left[\pi/3 - (-\pi/6)\right]$

$= \sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right]$

$= \sin\left[\frac{2\pi+\pi}{6}\right] = \sin\frac{3\pi}{6}$

$= \sin \pi/2$

$= 1$

$\therefore \sin\left[\pi/3 - \sin^{-1}(-\frac{1}{2})\right] = 1$

(ii) $\cos\left[\frac{1}{2}\cos^{-1}(\frac{1}{8})\right]$

Let $\cos^{-1}\frac{1}{8} = \alpha \Rightarrow \cos\alpha = \frac{1}{8}$

i.e. $2\cos^2\frac{\alpha}{2} - 1 = \frac{1}{8}$

$2\cos^2\frac{\alpha}{2} = \frac{1}{8} + 1$

$2\cos^2\frac{\alpha}{2} = \frac{9}{8}$

$\cos^2\frac{\alpha}{2} = \frac{9}{16}$

$\cos\frac{\alpha}{2} = \frac{3}{4}$

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Now, $\cos\left[\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right]$

$= \cos\left[\frac{1}{2}\theta\right]$

$= \cos\left[\frac{\theta}{2}\right] = 3/4$

$\therefore \cos\left[\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right] = \frac{3}{4}$

(iii) $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$

Let $a = \tan\theta$

$\therefore \tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$

$= \tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)\right]$

$= \tan\left[\frac{1}{2}\sin^{-1}(\sin 2\theta)+\frac{1}{2}\cos^{-1}(\cos 2\theta)\right]$

$= \tan\left[\frac{1}{2}(2\theta)+\frac{1}{2}(2\theta)\right]$

$= \tan[\theta+\theta] = \tan 2\theta$

$= \frac{2\tan\theta}{1-\tan^2\theta}$

$= \frac{2a}{1-a^2}$ //.

EX 4.19

prove that $\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}, |x|<1$

Let $\sin^{-1}x = \theta \Rightarrow \sin\theta = x$.

and $-1 \le x \le 1$.

$\tan(\sin^{-1}x) = \tan\theta = \frac{\sin\theta}{\cos\theta}$

$= \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$

$= \frac{x}{\sqrt{1-x^2}}$

Hence the proof:

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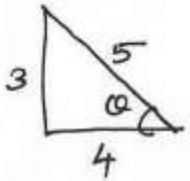
Ex 4.20:-

Evaluate: $\sin\left[\sin^{-1}\left(\frac{3}{5}\right)+\sec^{-1}\left(\frac{5}{4}\right)\right]$

$\sec^{-1}\frac{5}{4}=\alpha \Rightarrow \sec\alpha=\frac{5}{4}$

$\therefore \cos\alpha=\frac{4}{5}$



From the right angled Δ

$\sin\alpha=3/5$

$\Rightarrow \alpha=\sin^{-1}\left(\frac{3}{5}\right)$.

But $\alpha=\sec^{-1}(5/4)$

$\therefore \sin^{-1}\left(\frac{3}{5}\right)=\sec^{-1}(5/4)$

Now $\sin\left[\sin^{-1}\left(\frac{3}{5}\right)+\sec^{-1}\left(\frac{5}{4}\right)\right]$

$=\sin\left[\sin^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{3}{5}\right)\right]$

$=\sin\left[2\sin^{-1}\left(\frac{3}{5}\right)\right]$ ——— (1)

WKT, $2\sin^{-1}x=\sin^{-1}\left[2x\sqrt{1-x^2}\right]$

Now, $2\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left[2\times\frac{3}{5}\sqrt{1-\left(\frac{3}{5}\right)^2}\right]$

$=\sin^{-1}\left[\frac{6}{5}\sqrt{1-\frac{9}{25}}\right]$

$=\sin^{-1}\left[\frac{6}{5}\sqrt{\frac{16}{25}}\right]$

$=\sin^{-1}\left[\frac{6}{5}\times\frac{4}{5}\right]$

$=\sin^{-1}\left[\frac{24}{25}\right]$

$\therefore$ From (1).

$\sin\left[\sin^{-1}\left(\frac{24}{25}\right)\right]=\frac{24}{25}\in[-1,1]$

Ex 4.21 prove that

a) $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\pi/4$

WKT $\tan^{-1}(x)+\tan^{-1}(y)=\tan^{-1}\left[\frac{x+y}{1-xy}\right]$

$\therefore \tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\tan^{-1}\left[\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\left(\frac{1}{3}\right)}\right]$

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$$= \tan^{-1}\left[\frac{\frac{3+2}{6}}{1-\frac{1}{6}}\right]$$

$$= \tan^{-1}\left[\frac{5/6}{5/6}\right] = \tan^{-1}(1)$$

$$= \pi/4$$

Hence proved.

(ii) $2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{31}{17}\right)$

WKT $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

i.e. $2\tan^{-1}x = \tan^{-1}x + \tan^{-1}x$

$$= \tan^{-1}\left[\frac{x+x}{1-x\cdot x}\right]$$

$$= \tan^{-1}\left[\frac{2x}{1-x^2}\right]$$

$\therefore$ $2\tan^{-1}\left(\frac{1}{2}\right) = \tan^{-1}\left[\frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2}\right]$

$$= \tan^{-1}\left[\frac{1}{1-1/4}\right]$$

$$= \tan^{-1}\left[\frac{1}{3/4}\right] = \tan^{-1}\left(\frac{4}{3}\right)$$

Now $2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$$= \tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left[\frac{\frac{4}{3}+\frac{1}{7}}{1-\left(\frac{4}{3}\right)\left(\frac{1}{7}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\frac{28+3}{21}}{1-\frac{4}{21}}\right]$$

$$= \tan^{-1}\left[\frac{31}{21}\times\frac{21}{17}\right]$$

$$= \tan^{-1}\left[\frac{31}{17}\right] //$$

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Ex 4.22 If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ and $0 < x, y, z < 1$ show that $x^2 + y^2 + z^2 + 2xyz = 1$

Given: $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ and $0 < x, y, z < 1$.

Let $\cos^{-1}x = \alpha$, $\cos^{-1}y = \beta$, $\cos^{-1}z = \gamma$

$$\alpha + \beta + \cos^{-1}z = \pi$$

$$\alpha + \beta = \pi - \cos^{-1}z \quad \text{—— (1)}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \cos\alpha\cos\beta - \sqrt{1-\cos^2\alpha}\sqrt{1-\cos^2\beta}$$

$$\therefore \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sqrt{1-x^2}\sqrt{1-y^2} \quad \text{—— (2)}$$

sub (1) in (2)

$$\cos\left[\pi - \cos^{-1}z\right] = xy - \sqrt{1-x^2}\sqrt{1-y^2}$$

$$-\cos(\cos^{-1}z) = xy - \sqrt{1-x^2}\sqrt{1-y^2}$$

$$= -z = xy - \sqrt{1-x^2}\sqrt{1-y^2}$$

$$= (-xy - z)^2 = \left(-\sqrt{1-x^2}\sqrt{1-y^2}\right)^2$$

$$x^2y^2 + z^2 + 2xyz = (1-x^2)(1-y^2)$$

$$x^2y^2 + z^2 + 2xyz = 1 - y^2 - x^2 + x^2y^2$$

$$\boxed{x^2 + y^2 + z^2 + 2xyz = 1}$$

Ex 4.23 If $a_1, a_2, \ldots a_n$ is an arithmetic progression with common difference d, prove that

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \cdots + \tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right] = \frac{a_n - a_1}{1+a_na_1}$$

Given $a_1, a_2, \ldots a_n$ is an arithmetic progression with common difference d.

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$$\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) = \tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right) = \tan^{-1}a_2 - \tan^{-1}a_1$$

$$\tan^{-1}\left(\frac{d}{1+a_2a_3}\right) = \tan^{-1}\left(\frac{a_3-a_2}{1+a_2a_3}\right) = \tan^{-1}a_3 - \tan^{-1}a_2$$

continuing this, we get

$$\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right) = \tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right)$$

$$= \tan^{-1}a_n - \tan^{-1}a_{n-1}$$

Adding,

$$\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + \cdots + \tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)$$

$$= \tan^{-1}a_2 - \tan^{-1}a_1 + \tan^{-1}a_3 - \tan^{-1}a_2 + \cdots + \tan^{-1}a_n - \tan^{-1}a_{n-1}$$

$$= \tan^{-1}a_n - \tan^{-1}a_1$$

$$= \tan^{-1}\left[\frac{a_n-a_1}{1+a_1a_n}\right]$$

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$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + \cdots \tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{a_n-a_1}{1+a_na_1}\right)\right]$$

$$= \frac{a_n-a_1}{1+a_na_1} \;//$$

Ex 4.24 Solve $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$

W.K.T $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

put $x=1$, and $y=x$.

$\therefore \tan^{-1}x - \tan^{-1}y = \tan^{-1}(1) - \tan^{-1}x$

$$= \tan^{-1}\left[\frac{1-x}{1+x}\right]$$

Now $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$

i.e. $\tan^{-1}(1) - \tan^{-1}x = \frac{1}{2}\tan^{-1}x$

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$\tan^{-1}(1) = \frac{1}{2}\tan^{-1}x + \tan^{-1}x$

$\pi/4 = \frac{3}{2}\tan^{-1}x$

$\tan^{-1}x = \frac{\pi}{4} \times \frac{2}{3} = \frac{\pi}{6}$

$\therefore x = \tan(\pi/6)$

$\boxed{x = \frac{1}{\sqrt{3}}}$

Ex 4.25 Solve $\sin^{-1}x > \cos^{-1}x$

Given $\sin^{-1}x > \cos^{-1}x$

Adding, $\sin^{-1}x$ on both sides

$\sin^{-1}x + \sin^{-1}x > \sin^{-1}x + \cos^{-1}x$

$2\sin^{-1}x > \pi/2$

$\sin^{-1}x > \pi/4$

$x > \sin \pi/4 \Rightarrow \boxed{x > \frac{1}{\sqrt{2}}}$

4.26 Show that

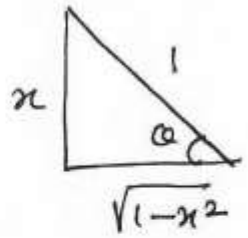
$\cot(\sin^{-1}x) = \frac{\sqrt{1-x^2}}{x}$, $-1 \le x \le 1$ and $x \neq 0$

Let $\sin^{-1}x = \alpha \Rightarrow x = \sin\alpha$.

Now $\cos\alpha = \sqrt{1-\sin^2\alpha}$

$= \sqrt{1-x^2}$

$= \frac{\sqrt{1-x^2}}{1}$



Now,

$\cot(\sin^{-1}x) = \cot\alpha$

$= \frac{\sqrt{1-x^2}}{x}$ (by diagram)

Hence $\cot(\sin^{-1}x) = \frac{\sqrt{1-x^2}}{x}$

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Ex 4.27 Solve

$\tan^{-1} 2x + \tan^{-1} 3x = \pi/4$, if $6x^2 < 1$

$\tan^{-1} 2x + \tan^{-1} 3x = \pi/4$

$\tan^{-1}\left[\dfrac{2x+3x}{1-(2x)(3x)}\right] = \pi/4$

$\tan^{-1}\left[\dfrac{5x}{1-6x^2}\right] = \pi/4$

$\dfrac{5x}{1-6x^2} = \tan \pi/4$

$\dfrac{5x}{1-6x^2} = 1 \Rightarrow 5x = 1-6x^2$

$6x^2+5x-1=0$

-6

$6 \quad -1$

$\frac{6}{6} \quad \frac{-1}{6}$

$1, \ -\frac{1}{6}$

i.e. $(x+1)(6x-1) = 0$

$x = -1, \quad 6x = 1$

$x = \dfrac{1}{6}$

$x = -1$ does not satisfies $6x^2 < 1$

$\therefore$ $x = -1$ is not possible.

$\therefore$ $\boxed{x = \frac{1}{6}}$ is the soln of the eqn.

Ex 4.28

Solve $\tan^{-1}\left(\dfrac{x-1}{x-2}\right) + \tan^{-1}\left(\dfrac{x+1}{x+2}\right) = \pi/4$

$\tan^{-1}\left[\dfrac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right] = \pi/4$

$\tan^{-1}\left[\dfrac{\frac{(x-1)(x+2)+(x+1)(x-2)}{(x+2)(x-2)}}{\frac{(x+2)(x-2)-(x-1)(x+1)}{(x+2)(x-2)}}\right] = \pi/4$

$\dfrac{(x^2+x-2)+(x^2-x-2)}{(x^2-4)-(x^2-1)} = \tan \pi/4$

$\dfrac{2x^2-4}{x^2-4-x^2+1} = 1$

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$$\frac{2x^2-4}{-3} = 1 \Rightarrow 2x^2-4 = -3$$

$$2x^2-4+3 = 0$$

$$2x^2-1 = 0$$

$$2x^2 = 1 \Rightarrow x^2 = \frac{1}{2}$$

$$x = \pm\frac{1}{\sqrt{2}}$$

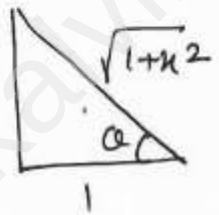
Ex 4.29 ※

Solve $\cos\left[\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right] = \sin\left[\cot^{-1}\left(\frac{3}{4}\right)\right]$

Consider the LHS

Let $\theta = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$ ——— ①

$$\sin\theta = \frac{x}{\sqrt{1+x^2}}$$



From the diagram

$$\cos\theta = \frac{1}{\sqrt{1+x^2}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) \quad ——— ②$$

From ① and ②

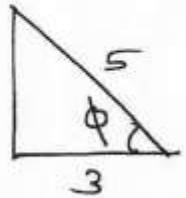
$$\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

$\therefore$ LHS $= \cos\left[\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right]$

$$= \frac{1}{\sqrt{1+x^2}} \quad ——— ③$$

Let $\phi = \cot^{-1}\frac{3}{4}$ ——— ⓐ $\Rightarrow \cot\phi = \frac{3}{4}$

From the diagram,



$$\sin\phi = \frac{4}{5}$$

$$\therefore \phi = \sin^{-1}\left(\frac{4}{5}\right) \quad ——— ⓑ$$

From ⓐ and ⓑ

$$\cot^{-1}\left(\frac{3}{4}\right) = \sin^{-1}\left(\frac{4}{5}\right)$$

$\therefore$ RHS $= \sin\left[\cot^{-1}\left(\frac{3}{4}\right)\right]$

$$= \sin\left[\sin^{-1}\left(\frac{4}{5}\right)\right]$$

$$= \boxed{4/5} \quad ——— ④$$

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From (3) and (4)

$$\frac{1}{\sqrt{1+x^2}} = \frac{4}{5} \Rightarrow \sqrt{1+x^2} = \frac{5}{4}$$

Squaring on both sides

$$1+x^2 = \frac{25}{16} \Rightarrow 16+16x^2 = 25$$

$$16x^2 = 25-16$$

$$16x^2 = 9$$

$$x^2 = \frac{9}{16}$$

$$\boxed{x = \pm \frac{3}{4}}$$

EXERCISE 4.5

1) Find the value, if it exists, if not, give the reason for non existence.

a) $\sin^{-1}(\cos \pi)$

$$\sin^{-1}(\cos \pi) = \sin^{-1}(-1)$$

$$= -\sin^{-1}(1)$$

$$= -\pi/2 \in [-\pi/2, \pi/2]$$

b) $\tan^{-1}\left[\sin\left(-5\pi/2\right)\right]$

$$\tan^{-1}\left[-\sin\frac{5\pi}{2}\right] = \tan^{-1}\left[-\sin\left(2\pi+\pi/2\right)\right]$$

$$= \tan^{-1}\left[-\sin \pi/2\right]$$

$$= \tan^{-1}[-1] = -\tan^{-1}(1)$$

$$= -\pi/4 \in (-\pi/2, \pi/2)$$

(iii) $\sin^{-1}(\sin 5)$

$\sin^{-1}(\sin 5) = 5 \notin [-\pi/2, \pi/2]$

$\therefore \sin^{-1}(\sin 5) = \sin^{-1}(\sin(\pi-5))$

$= \pi - 5$

2) Find the value of expression in terms of x, with the help of reference triangle.

(i) $\sin\left(\cos^{-1}(1-x)\right)$

Let $\cos^{-1}(1-x) = \theta$ ——— (1).

$\therefore \cos\theta = 1-x$

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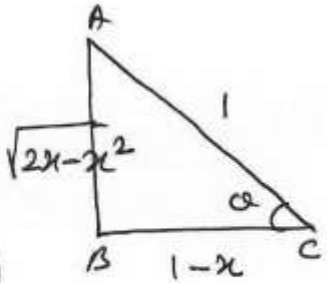
In ΔABC,



$$AB^2 = AC^2 - BC^2$$
$$= 1^2 - (1-x)^2$$
$$= 1 - (1+x^2-2x)$$
$$= 1 - 1 - x^2 + 2x$$
$$= 2x - x^2$$

$\therefore AB = \sqrt{2x-x^2}$.

Now $\sin\alpha = \sqrt{2x-x^2}$

$\therefore \alpha = \sin^{-1}\sqrt{2x-x^2}$ ——— (2).

From (1) and (2)

$$\cos^{-1}(1-x) = \sin^{-1}\sqrt{2x-x^2}$$

Now $\sin[\cos^{-1}(1-x)]$

$= \sin[\sin^{-1}(\sqrt{2x-x^2}]$

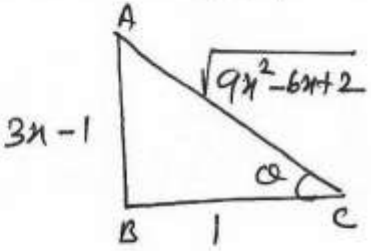
$= \sqrt{2x-x^2}$ //

(ii) $\cos[\tan^{-1}(3x-1)]$

Let $\alpha = \tan^{-1}(3x-1)$ ——— (1).

$\therefore \tan\alpha = 3x-1$ ——— (2).

In ΔABC,



$$AC = \sqrt{AB^2+BC^2}$$
$$= \sqrt{(3x-1)^2+1^2}$$
$$= \sqrt{9x^2+1-6x+1}$$
$$= \sqrt{9x^2-6x+2}.$$

Now $\cos\alpha = \frac{1}{\sqrt{9x^2-6x+2}}$

$\therefore \alpha = \cos^{-1}\left(\frac{1}{\sqrt{9x^2-6x+2}}\right)$ — (2)

From (1) and (2)

$$\tan^{-1}(3x-1) = \cos^{-1}\left(\frac{1}{\sqrt{9x^2-6x+2}}\right)$$

Now, $\cos[\tan^{-1}(3x-1)]$

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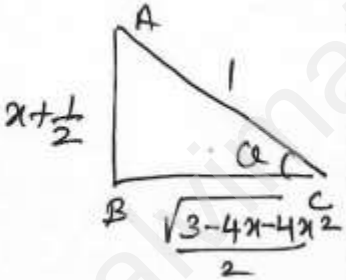
$$\cos\left[\cos^{-1}\left(\frac{1}{\sqrt{9x^2-6x+2}}\right)\right]$$

$$= \frac{1}{\sqrt{9x^2-6x+2}}$$

(iii) $\tan\left[\sin^{-1}\left(x+\frac{1}{2}\right)\right]$

Let $\theta = \sin^{-1}\left(x+\frac{1}{2}\right)$ ——— (1).

$\therefore \sin\theta = x+\frac{1}{2}$



In ΔABC,

$$BC = \sqrt{AC^2 - AB^2}$$

$$= \sqrt{1-\left(x+\frac{1}{2}\right)^2}$$

$$= \sqrt{1-\left(x^2+x+\frac{1}{4}\right)}$$

$$= \sqrt{1-x^2-x-\frac{1}{4}}$$

$$\therefore \sqrt{\frac{3}{4}-x^2-x} = \frac{\sqrt{3-4x^2-4x}}{4}$$

$$= \frac{\sqrt{3-4x-4x^2}}{2}$$

Now $\tan\theta = \dfrac{x+\frac{1}{2}}{\frac{\sqrt{3-4x-4x^2}}{2}}$

$$= \frac{\frac{2x+1}{2}}{\frac{\sqrt{3-4x-4x^2}}{2}} = \frac{2x+1}{\sqrt{3-4x-4x^2}}$$

$\therefore \theta = \tan^{-1}\left(\dfrac{2x+1}{\sqrt{3-4x-4x^2}}\right)$ ——— (2)

From (1) and (2)

$$\sin^{-1}\left(x+\frac{1}{2}\right) = \tan^{-1}\left(\frac{2x+1}{\sqrt{3-4x-4x^2}}\right)$$

$$\therefore \tan\left[\sin^{-1}\left(x+\frac{1}{2}\right)\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{2x+1}{\sqrt{3-4x-4x^2}}\right)\right]$$

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$$= \frac{2x+1}{\sqrt{3-4x-4x^2}} //$$

---

3) Find the value of

(i) $\sin^{-1}\left[\cos\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right]$

$= \sin^{-1}\left[\cos \pi/3\right]$

$= \sin^{-1}\left[\frac{1}{2}\right] = \pi/6 \in \left[-\pi/2, \pi/2\right]$

(ii) $\cot\left[\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5}\right]$

WKT, $\sin^{-1}x + \sin^{-1}y$

$= \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

$\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = \sin^{-1}\left[\frac{3}{5}\sqrt{1-\frac{16}{25}} + \frac{4}{5}\sqrt{1-\frac{9}{25}}\right]$

$= \sin^{-1}\left[\frac{3}{5}\sqrt{\frac{9}{25}} + \frac{4}{5}\sqrt{\frac{16}{25}}\right]$

$= \sin^{-1}\left[\frac{3}{5}\times\frac{3}{5} + \frac{4}{5}\times\frac{4}{5}\right]$

$= \sin^{-1}\left[\frac{9}{25} + \frac{16}{25}\right] = \sin^{-1}\left[\frac{9+16}{25}\right]$

$= \sin^{-1}\left[\frac{25}{25}\right]$

$= \sin^{-1} 1$

$= \pi/2$.

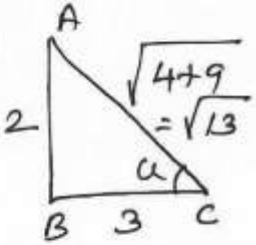
Now $\cot\left[\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5}\right]$

$= \cot\left[\pi/2\right] = 0$

(iii) $\tan\left[\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right]$

Let $\alpha = \cot^{-1}\frac{3}{2}$ — (1)

$\therefore \cot\alpha = \frac{3}{2}$



From the fig,

$\sin\alpha = \frac{2}{\sqrt{13}}$

$\therefore \alpha = \sin^{-1}\left(\frac{2}{\sqrt{13}}\right)$ — (2)

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From ① and ②

$$\cot^{-1}\left(\frac{3}{2}\right) = \sin^{-1}\frac{2}{\sqrt{13}}$$

$\therefore \sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}$

$= \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{2}{\sqrt{13}}$

WKT $\sin^{-1}x + \sin^{-1}y$

$= \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

$\therefore \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{2}{\sqrt{13}}$

$= \sin^{-1}\left[\frac{3}{5}\sqrt{1-\frac{4}{13}} + \frac{2}{\sqrt{13}}\sqrt{1-\frac{9}{25}}\right]$

$= \sin^{-1}\left[\frac{3}{5}\sqrt{\frac{9}{13}} + \frac{2}{\sqrt{13}}\sqrt{\frac{16}{25}}\right]$

$= \sin^{-1}\left[\frac{3}{5}\times\frac{3}{\sqrt{13}} + \frac{2}{\sqrt{13}}\times\frac{4}{5}\right]$

$= \sin^{-1}\left[\frac{9}{5\sqrt{13}} + \frac{8}{5\sqrt{13}}\right]$

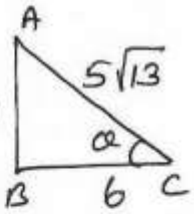
$= \sin^{-1}\left(\frac{17}{5\sqrt{13}}\right)$

Now $\tan\left[\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right]$

$= \tan\left[\sin^{-1}\frac{17}{5\sqrt{13}}\right]$

Let $\theta = \sin^{-1}\frac{17}{5\sqrt{13}}$ — ③

$\sin\theta = \frac{17}{5\sqrt{13}}$



In $\triangle ABC$, $BC = \sqrt{AC^2 - AB^2}$

$= \sqrt{25\times13 - 17^2}$

$= \sqrt{325 - 289} = \sqrt{36} = 6$

$\therefore \tan\theta = \frac{17}{6} \Rightarrow \theta = \tan^{-1}\frac{17}{6}$ — ④

From ③ and ④

$$\sin^{-1}\frac{17}{5\sqrt{13}} = \tan^{-1}\frac{17}{6}$$

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Now $\tan\left[\sin^{-1}\frac{17}{5\sqrt{13}}\right]$

$= \tan\left[\tan^{-1}\frac{17}{6}\right] = \frac{17}{6}$

$\therefore \tan\left[\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right] = \frac{17}{6}$

4(i) prove that

(i) $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\left(\frac{1}{2}\right)$

WKT $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Now, $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\left[\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\times\frac{7}{24}}\right]$

$= \tan^{-1}\left[\frac{\frac{48+77}{11\times24}}{\frac{11\times24-14}{11\times24}}\right]$

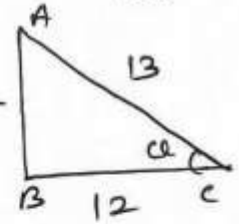
$= \tan^{-1}\left[\frac{125}{\cancel{11\times24}} \times \frac{11\times24}{264-14}\right]$

$= \tan^{-1}\left[\frac{125}{250}\right] = \tan^{-1}\left(\frac{1}{2}\right)$.

Hence proved.

(ii) $\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13} = \sin^{-1}\frac{16}{65}$

Let $\alpha = \cos^{-1}\frac{12}{13}$ — (1)



$\therefore \cos\alpha = \frac{12}{13}$

In $\triangle ABC$, $AB = \sqrt{AC^2 - BC^2}$

$= \sqrt{169-144} = \sqrt{25}$

$= 5$

$\therefore \sin\alpha = \frac{5}{13} \Rightarrow \alpha = \sin^{-1}\left(\frac{5}{13}\right)$ — (2)

From (1) and (2)

$\cos^{-1}\left(\frac{12}{13}\right) = \sin^{-1}\frac{5}{13}$

$\therefore \sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13}$

$= \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{5}{13}$

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WKT $\sin^{-1}x - \sin^{-1}y$

$= \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$

$\therefore \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{5}{13}$

$= \sin^{-1}\left[\frac{3}{5}\sqrt{1-\frac{25}{169}} - \frac{5}{13}\sqrt{1-\frac{9}{25}}\right]$

$= \sin^{-1}\left[\frac{3}{5}\sqrt{\frac{144}{169}} - \frac{5}{13}\sqrt{\frac{16}{25}}\right]$

$= \sin^{-1}\left[\frac{3}{5}\times\frac{12}{13} - \frac{5}{13}\times\frac{4}{5}\right]$

$= \sin^{-1}\left[\frac{36}{65} - \frac{20}{65}\right]$

$= \sin^{-1}\left[\frac{16}{65}\right]$ Hence proved.

---

5) prove that

$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$

$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z$

$= \tan^{-1}\left(\frac{x+y}{1-xy}\right) + \tan^{-1}z$

$= \tan^{-1}\left[\frac{\frac{x+y}{1-xy} + z}{1-\left(\frac{x+y}{1-xy}\right)(z)}\right]$

$= \tan^{-1}\left[\frac{\frac{x+y+z(1-xy)}{1-xy}}{\frac{(1-xy)-(x+y)(z)}{1-xy}}\right]$

$= \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$

$=$ RHS Hence proved.

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6) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$ Show that $x+y+z = xyz$.

From Sum No 5, WKT

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$$

Now, Sum No 6 $\Rightarrow$

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$$

i.e. $$\tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right] = \pi$$

$\therefore$ $$\frac{x+y+z-xyz}{1-xy-yz-zx} = \tan\pi = 0$$

i.e. $$x+y+z-xyz = 0$$

$\therefore$ $$x+y+z = xyz$$

Hence proved.

7) prove that

$$\tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), \quad |x| < \frac{1}{\sqrt{3}}$$

$$\tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$= \tan^{-1}\left[\frac{x+\frac{2x}{1-x^2}}{1-x\left(\frac{2x}{1-x^2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\frac{x(1-x^2)+2x}{1-x^2}}{\frac{(1-x^2)-2x^2}{1-x^2}}\right]$$

$$= \tan^{-1}\left[\frac{x-x^3+2x}{1-x^2-2x^2}\right]$$

$$= \tan^{-1}\left[\frac{3x-x^3}{1-3x^2}\right]$$ where $|3x^2| < 1$

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$|3x^2| < 1 \Rightarrow x^2 < \frac{1}{3}$

$x < \frac{1}{\sqrt{3}}$

Hence proved

8) Simplify $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$

$\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$

$= \tan^{-1}\left[\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1 + \frac{x}{y}\left(\frac{x-y}{x+y}\right)}\right]$

$= \tan^{-1}\left[\frac{\frac{x(x+y) - y(x-y)}{y(x+y)}}{\frac{y(x+y) + x(x-y)}{y(x+y)}}\right]$

$= \tan^{-1}\left[\frac{x^2+xy-yx+y^2}{yx+y^2+x^2-xy}\right]$

XII|CH4|4.5|P19

$= \tan^{-1}\left[\frac{x^2+y^2}{x^2+y^2}\right] = \tan^{-1}(1)$

$= \pi/4$

9) Solve

(1) $\sin^{-1}\frac{5}{x} + \sin^{-1}\frac{12}{x} = \pi/2$

$\sin^{-1}\frac{5}{x} = \frac{\pi}{2} - \sin^{-1}\frac{12}{x}$

$\sin^{-1}\frac{5}{x} = \cos^{-1}\frac{12}{x}$

$\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$\pi/2 - \sin^{-1}x = \cos^{-1}x.$

Let $\sin^{-1}\frac{5}{x} = \theta$ | $\cos^{-1}\frac{12}{x} = \theta$

$\sin\theta = \frac{5}{x}$ | $\cos\theta = \frac{12}{x}$

WKT

$\sin^2\theta + \cos^2\theta = 1$

i.e. $\left(\frac{5}{x}\right)^2 + \left(\frac{12}{x}\right)^2 = 1$

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$$\frac{25}{x^2} + \frac{144}{x^2} = 1$$

$$\frac{25+144}{x^2} = 1$$

$$\Rightarrow \quad x^2 = 169$$

$$\Rightarrow \quad \boxed{x = \pm 13}$$

But $x = -13$ is not possible.

$$\therefore \quad \boxed{x = 13}$$

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(ii) $2\tan^{-1}x = \cos^{-1}\frac{1-a^2}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2}$

put $a = \tan\alpha$, $b = \tan\phi$.

$2\tan^{-1}x = \cos^{-1}\left(\frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right) - \cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right)$

$= \cos^{-1}(\cos 2\alpha) - \cos^{-1}(\cos 2\phi)$

$= 2\alpha - 2\phi$

$2\tan^{-1}x = 2(\alpha - \phi)$

$\tan^{-1}x = \alpha - \phi$

$\tan^{-1}x = \tan^{-1}(a) - \tan^{-1}(b)$

$= \tan^{-1}\left(\frac{a-b}{1+ab}\right)$

$\therefore x = \frac{a-b}{1+ab}$ //

(iii) $2\tan^{-1}(\sin x) = \tan^{-1}(2\operatorname{cosec} x)$

LHS $= 2\tan^{-1}\sin x$

$= \tan^{-1}(\sin x) + \tan^{-1}(\sin x)$

$= \tan^{-1}\left(\frac{\sin x + \sin x}{1-(\sin x)(\sin x)}\right)$

$= \tan^{-1}\left(\frac{2\sin x}{1-\sin^2 x}\right)$

$= \tan^{-1}\left(\frac{2\sin x}{\cos^2 x}\right)$ ——— (1).

Given $2\tan^{-1}(\sin x) = \tan^{-1}(2\operatorname{cosec} x)$

i $\tan^{-1}\left(\frac{2\sin x}{\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x)$

$\tan^{-1}\left(\frac{2\sin x}{\cos^2 x}\right) - \tan^{-1}(2\operatorname{cosec} x) = 0$

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$$\tan^{-1}\left[\frac{\frac{2\cos x}{\sin^2 x} - 2\operatorname{cosec} x}{1 + \frac{2\cos x}{\sin^2 x}\cdot 2\operatorname{cosec} x}\right] = 0$$

$$\frac{\frac{2\cos x - 2\operatorname{cosec} x \cdot \sin^2 x}{\sin^2 x}}{\frac{\sin^2 x + 2\cos x \cdot 2\operatorname{cosec} x}{\sin^2 x}} = \tan 0$$

$$\frac{2\cos x - 2\frac{1}{\sin x}\cdot \sin^2 x}{\sin^2 x + 4\cos x \cdot \frac{1}{\sin x}} = 0$$

$$2\cos x - 2\sin x = 0$$

$$2\cos x = 2\sin x$$

$$\cos x = \sin x \Rightarrow x = \pi/4.$$

(IV) $\cot^{-1} x - \cot^{-1}(x+2) = \frac{\pi}{12}, \quad x > 0$

$$\cot^{-1} x - \cot^{-1}(x+2) = \frac{\pi}{12} = \frac{180}{12} = 15^\circ$$

$$\tan^{-1}\left(\frac{1}{x}\right) - \tan^{-1}\left(\frac{1}{x+2}\right) = 15^\circ$$

$$\tan^{-1}\left[\frac{\frac{1}{x} - \frac{1}{x+2}}{1 + \frac{1}{x}\left(\frac{1}{x+2}\right)}\right] = 15^\circ$$

$$\frac{\frac{x+2-x}{x(x+2)}}{\frac{x(x+2)+1}{x(x+2)}} = \tan 15^\circ$$

$$\frac{2}{x^2+2x+1} = \tan(45-30)$$

$$\frac{2}{(x+1)^2} = \frac{\tan 45 - \tan 30}{1 + \tan 45 \cdot \tan 30}$$

$$\frac{2}{(x+1)^2} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$$

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$$\frac{2}{(x+1)^2} = \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}}$$

$$\frac{2}{(x+1)^2} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$\frac{2}{(x+1)^2} = \frac{(\sqrt{3}-1)^2}{3-1}$$

$$\frac{2}{(x+1)^2} = \frac{(\sqrt{3}-1)^2}{2}$$

$$(x+1)^2 = \frac{4}{(\sqrt{3}-1)^2}$$

$$\therefore x+1 = \frac{2}{\sqrt{3}-1} = \frac{2}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{2(\sqrt{3}+1)}{3-1} = \frac{2(\sqrt{3}+1)}{2}$$

$$x+1 = \sqrt{3}+1$$

$$\boxed{x = \sqrt{3}}$$

XII | CH4 | 4.5 | P23

10) Find the number of solutions of the eqn

$$\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}(3x)$$

$$\Rightarrow \tan^{-1}\left[\frac{x-1+x+1}{1-(x-1)(x+1)}\right] + \tan^{-1}x = \tan^{-1}(3x)$$

$$\tan^{-1}\left[\frac{2x}{1-(x^2-1)}\right] + \tan^{-1}x = \tan^{-1}3x$$

$$\tan^{-1}\left[\frac{2x}{1-x^2+1}\right] + \tan^{-1}x = \tan^{-1}3x$$

$$\tan^{-1}\left[\frac{2x}{2-x^2}\right] + \tan^{-1}x = \tan^{-1}3x$$

$$\tan^{-1}\left[\frac{\frac{2x}{2-x^2}+x}{1-\frac{2x}{2-x^2}\times x}\right] = \tan^{-1}3x$$

$$\tan^{-1}\left[\frac{\frac{2x+x(2-x^2)}{2-x^2}}{\frac{(2-x^2)-2x^2}{2-x^2}}\right] = \tan^{-1}3x$$

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$$\frac{2x + 2x - 2x^2}{2 - x^2 - 2x^2} = 3x$$

$$\frac{4x - 2x^2}{2 - 3x^2} = 3x$$

$$4x - 2x^2 = 3x(2 - 3x^2)$$

$$4x - 2x^2 = 6x - 9x^3$$

$$9x^3 - 2x^2 + 4x - 6x = 0$$

$$9x^3 - 2x^2 - 2x = 0$$

This is cubic eqn in x

∴ The given eqn has 3 solns.

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Chapter 5 – Two Dimensional Analytical Geometry - II

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CHAPTER 5 - Two Dimensional Analytical Geometry :-

Note:

1) The eqn of circle with centre (0,0) and radius r is
$x^2+y^2=r^2$.

2) Eqn of a circle with centre C(h,k) and radius r is
$(x-h)^2+(y-k)^2=r^2$.

3) General form of eqn of a circle is $x^2+y^2+2gx+2fy+c=0$.
with centre $(-g,-f)$ and
radius $= \sqrt{g^2+f^2-c}$.

a) It is real circle if
$g^2+f^2-c>0$

b) a point circle if $g^2+f^2-c=0$

c) an imaginary circle if
$g^2+f^2-c<0$.

Ex 5.1

Find the general eqn of a circle with centre (-3,-4) and radius 3 units.

Wkt Eqn of a circle with centre (h,k) and radius r is
$(x-h)^2+(y-k)^2=r^2$

∴ The required eqn of circle with centre (-3,-4) and radius 3 is
$(x+3)^2+(y+4)^2=3^2$
i.e. $x^2+6x+9+y^2+8y+16-9=0$
$x^2+y^2+6x+8y+16=0$

Note: The circle passing through the points of interseection of the line $lx+my+n=0$ and the circle $x^2+y^2+2gx+2fy+c=0$ is the circle of the form is

$x^2+y^2+2gx+2fy+c+\lambda(lx+my+n)=0$.

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5.2 Find the eqn of the circle discribed on the chord $3x+y+5=0$ of the circle $x^2+y^2=16$ as diameter.

Given eqn of the circle is $x^2+y^2=16$.

Eqn of chord is $3x+y+5=0$.

The required eqn of the circle passing through the pt of interseection of the circle and chord is

$x^2+y^2-16+\lambda(3x+y+5)=0$.

$x^2+y^2-16+3\lambda x+\lambda y+5\lambda=0$.

$x^2+y^2+3\lambda x+\lambda y-16+5\lambda=0$ —— (1)

compare this eqn with

$x^2+y^2+2gx+2fy+c=0$

$2g=3\lambda$ | $2f=\lambda$

$g=\frac{3\lambda}{2}$ | $f=\frac{\lambda}{2}$

$\therefore$ centre $(-g,-f)=\left(-\frac{3\lambda}{2},-\frac{\lambda}{2}\right)$

Given that the chord $3x+y+5=0$ is a diameter.

$\therefore$ $-\frac{3\lambda}{2}, -\frac{\lambda}{2}$ passes through $3x+y+5=0$.

$\therefore 3\left(-\frac{3\lambda}{2}\right)-\frac{\lambda}{2}+5=0$

$-\frac{9\lambda}{2}-\frac{\lambda}{2}+\frac{10}{2}=0$

$-10\lambda+10=0$

$-10\lambda=-10$

$\boxed{\lambda=1}$

Sub $\lambda=1$ in (1)

$x^2+y^2+3x+y-11=0$ which is required eqn of circle.

Ex 5.3 Determine whether $x+y-1=0$ is the eqn of a diameter of the circle $x^2+y^2-6x+4y+c=0$ for all possible value of c.

Eqn of the line is $x+y-1=0$ —— (1)

Eqn of circle $x^2+y^2-6x+4y+c=0$ —— (2)

Centre of the circle (2) is

$2f=-6$ | $2g=4$

$f=-3$ | $g=2$

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$(-1,-9) = (3,-2)$

Sub $(3,-2)$ in ①

$3-2-1=0 \Rightarrow 3-3=0$

∴ $(3,-2)$ satisfies the eqn $x+y-1=0$

∴ The line $x+y-1$ is a diameter of the circle ① for all values of c.

Note:- The eqn of a circle with (x_1,y_1) and (x_2,y_2) as extremities of one of the diameters of the circle is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

5.4 Find the general eqn of the circle whose diameter is the line segment joining the points $(-4,-2)$ and $(1,1)$.

We have

The eqn of the circle with the line joining the pts $(-4,-2)$ and $(1,1)$ is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

Here $(x_1,y_1)=(-4,-2)$

$(x_2,y_2)=(1,1)$

∴ The required eqn of circle is

$(x+4)(x-1)+(y+2)(y-1)=0$

$x^2+3x-4+y^2+y-2=0$

$x^2+y^2+3x+y-6=0.$

Note: The position of a point $P(x_1,y_1)$ w.r to a given circle $x^2+y^2+2gx+2fy+c=0$ in the plane containing the circle is outside or on or inside the circle according as

$x_1^2+y_1^2+2gx_1+2fy_1+c$ is $\begin{cases} >0 & \text{or} \\ =0 & \text{or} \\ <0. \end{cases}$

5.5 Examine the position of the point $(2,3)$ with respect to the circle $x^2+y^2-6x-8y+12=0$.

The eqn of the circle is

$x^2+y^2-6x-8y+12=0$ —— ①

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Taking the pt $(x_1, y_1) = (2,3)$

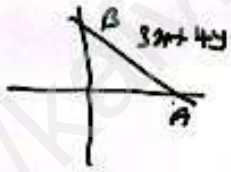
Sub $(2,3)$ in ①

$4+9-12-24+12 = 13-24 = -11 < 0$

∴ The given pt lies inside the circle.

Ex 5.6 The line $3x+4y-12=0$ meets the coordinate axes at A and B find the eqn of the circle drawn on AB as diameter.

The given line $3x+4y-12=0$ — ① meets the coordinate axis at A and B



To get the point A put $y=0$ in ①

$3x-12=0$

$3x=12$

$x=4$

∴ The point A is $(4,0)$

To get the pt B put $x=0$

$4y-12=0$

$4y=12 \Rightarrow \boxed{y=3}$

∴ The pt B is $B(0,3)$

∴ The eqn of circle with the line joining the pts A and B is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

$\Rightarrow (x-4)(x-0)+(y-0)(y-3)=0$

$x^2-4x+y^2-3y=0$

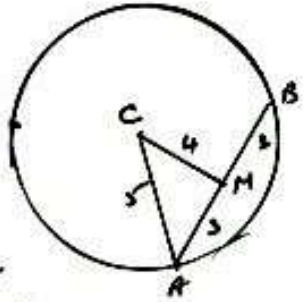
$\Rightarrow x^2+y^2-4x-3y=0$ which is the required eqn of the circle.

Ex 5.7 A line $3x+4y+10=0$ cuts a chord of length 6 units on a circle with centre of the circle $(2,1)$ find the eqn of the circle in general form.

Given centre $C = C(2,1)$

The line $3x+4y+10=0$ cuts a chord AB of length 6 units.

Draw CM ⊥r to AB



Length of the perpendicular CM from $C(2,1)$ to the line $3x+4y+10=0$ is

$CM = \frac{|3(2)+4(1)+10|}{\sqrt{3^2+4^2}} = \frac{6+4+10}{\sqrt{25}}$

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$= \frac{20}{\sqrt{25}} = \frac{20}{5} = 4$ units.

In right angled triangle BMC,
BM = 3, CM = 4.

$\therefore BC^2 = BM^2 + CM^2$

$= 3^2 + 4^2 = 9 + 16 = 25$

$\therefore BC = \sqrt{25} = 5$ units.

∴ radius of the circle = 5 units.

∴ The eqn of circle with centre C(2,1) and radius r = 5 units is $(x - x_1)^2 + (y - y_1)^2 = 5^2$

$(x-2)^2 + (y-1)^2 = 25$

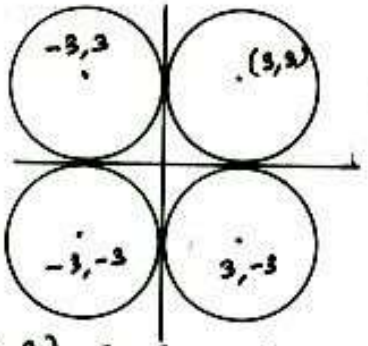
$x^2 - 4x + 4 + y^2 + 1 - 2y = 25$

$x^2 + y^2 - 4x - 2y + 5 - 25 = 0$

$x^2 + y^2 - 4x - 2y - 20 = 0$.

Ex 5.8 A circle of radius 3 units touches both the axes. Find the equations of all possible circles formed in the general form.

Given that
The radius of the circle 3 unit touches both the axes.



∴ There are four possible centres $C_1(3,3)$, $C_2(-3,3)$, $C_3(-3,-3)$, $C_4(3,-3)$.

∴ The eqn of the circle with centre $(\pm 3, \pm 3)$ and radius 3 is

$(x - \pm 3)^2 + (y - \pm 3)^2 = 3^2$

$x^2 \pm 6x + 9 + y^2 \pm 6y + 9 - 9 = 0$

$x^2 + y^2 \pm 6x \pm 6y + 9 = 0$.

Ex 5.9 Find the centre and radius of the circle $3x^2 + (a+1)y^2 + 6x - 9y + a + 4 = 0$

Given eqn $3x^2 + (a+1)y^2 + 6x - 9y + a + 4 = 0$ ——— (1)

represents a circle

⇒ coeff of x^2 = coeff of y^2

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$3 = a+1 \Rightarrow \boxed{a=2}$

Sub $a=2$ in ①

$3x^2+3y^2+6x-9y+6=0$

$(\div 3)$, $x^2+y^2+2x-3y+2=0$.

compare this eqn with

$x^2+y^2+2gx+2fy+c=0$

$2g=2 \mid 2f=-3 \mid c=2$

$\therefore g=1 \mid f=-3/2 \mid$

$\therefore$ centre $C=(-g,-f)=(-1,\frac{3}{2})$

Radius $r=\sqrt{g^2+f^2-c}=\sqrt{(-1)^2+(\frac{3}{2})^2-2}$

$=\sqrt{1+\frac{9}{4}-2}=\sqrt{\frac{9}{4}-1}$

$\sqrt{\frac{5}{2}}$

$\therefore$ centre $C=(-1,3/2)$

radius $r=\sqrt{\frac{5}{2}}$

Ex 5.10 Find the eqn of the circle passing through the points (1,1) (2,-1), (3,2)

XII | CH 5 | 5.1 | P6

The general eqn of the circle be

$x^2+y^2+2gx+2fy+c=0$ ——— ①.

Eqn ① passes through (1,1)

$\therefore 1+1+2g+2f+c=0$

$2g+2f+c=-2$ ——— ②

Eqn ① passes through (2,-1)

$\therefore 4+1+4g-2f+c=0$

$4g-2f+c=-5$ ——— ③

Eqn ① passes through (3,2)

$9+4+6g+4f+c=0$

$6g+4f+c=-13$ ——— ④.

② − ③ ⇒ $-2g+4f=3$ ——— ⑤.

③ − ④ ⇒ $-2g-6f=8$ ——— ⑥

⑤ − ⑥ ⇒ $10f=-5$

$f=-\frac{5}{10}=\boxed{-\frac{1}{2}}$

Sub $f=-\frac{1}{5}$ in ⑤

$-2g+4(-\frac{1}{2})=3$

$-2g-\frac{4}{2}=3 \Rightarrow -2g=3+\frac{4}{2}$

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$-2g = 3+2 = 5$

$\boxed{g = \frac{5}{-2}}$

Sub $f = -\frac{1}{2}$, $g = -\frac{5}{2}$ in (2)

$2(-\frac{5}{2}) + 2(-\frac{1}{2}) + c = -2$

$-5 - 1 + c = -2 \Rightarrow -6 + c = -2$

$\boxed{c = 4}$

$c = 6 - 2$
$= 4$

∴ The eqn of the circle is

$x^2 + y^2 + 2(-\frac{5}{2})x + 2(-\frac{1}{2})y + 4 = 0$

i.e. $x^2 + y^2 - 5x - y + 4 = 0$.

Note: Eqn of tgt at (x_1, y_1) on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

2) Eqn of normal at (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $yx_1 - xy_1 + g(y - y_1) - f(x - x_1) = 0$.

3) The eqn of tgt at (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

4) The eqn of normal at (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xy_1 - yx_1 = 0$.

5) The condition for the line $y = mx + c$ to be a tangent to the circle $x^2 + y^2 = a^2$ is $c^2 = a^2(1 + m^2)$

6) To find the point of contact is $\left(-\frac{a^2 m}{c}, \frac{a^2}{c}\right)$.

7) The eqn of any tangent to the circle $x^2 + y^2 = a^2$ is $y = mx \pm a\sqrt{1 + m^2}$.

Ex 5.11. Find the eqns of the tgt and normal to the circle $x^2 + y^2 = 25$ at $P(-3, 4)$.

Wkt eqn of tgt to the circle $x^2 + y^2 = a^2$ at (x_1, y_1) is $xx_1 + yy_1 = a^2$

∴ Eqn of tgt to the circle $x^2 + y^2 = 25$ at $P(-3, 4)$ is

$x(-3) + y(4) = 25$

$-3x + 4y = 25$ i.e. $\boxed{3x - 4y + 25 = 0}$

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Eqn of normal is $xy_1 - yx_1 = 0$

∴ The required eqn of normal is $x(4) - y(-3) = 0$

$$\boxed{4x + 3y = 0}$$

5.12 If $y = 4x + c$ is a tgt to the circle $x^2 + y^2 = 9$, find c

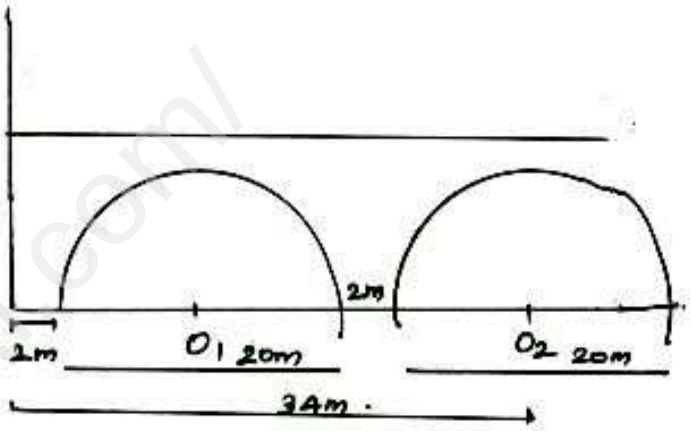
Condition for the line $y = mx + c$ to be a tgt to the circle $x^2 + y^2 = a^2$ is $c^2 = a^2(1 + m^2)$

Here $a = 3$, $m = 4$

$$\therefore c = \pm\sqrt{9(1+16)}$$

$$c = \pm 3\sqrt{17}$$

5.13 A road bridge over an irrigation canal have two semi circular vents each with a span of 20m and the supporting pillars of width 2m. Use figure to write the eqns that represents the semi vertical vents.



Let O_1, O_2 be the centre of two vents.

First vent with centre $O_1(12,0)$ and radius $r = 10$.

∴ The eqn of first semi circle is

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

$$(x - 12)^2 + (y - 0)^2 = 10^2$$

$$x^2 - 24x + 144 + y^2 - 100 = 0$$

$$x^2 + y^2 - 24x + 44 = 0.$$

Second vent with centre $O_2(34,0)$ and radius $r = 10$ cm.

∴ The eqn of second semi circle is $(x - 34)^2 + (y - 0)^2 = 10^2$

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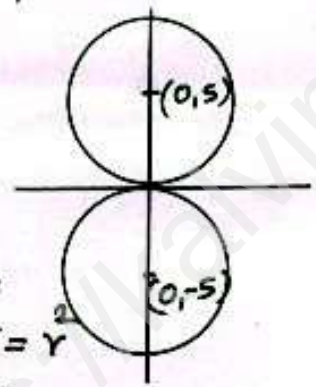
$x^2 - 68x + 1156 + y^2 = 100$

$x^2 + y^2 - 68x + 1056 = 0$.

EXERCISE - 5.1

1. obtain the eqn of the circles with radius 5 cm and touching x-axis at the origin in general form.

Given the circle touching x-axis at the origin having radius 5 unit.



∴ The possible centres are $(0, \pm 5)$.

∴ The eqn of circle with centre $(0, \pm 5)$ having radius 5 units is $(x-0)^2 + (y \pm 5)^2 = r^2$

$x^2 + y^2 \pm 10y + 25 = 5^2$

$\boxed{x^2 + y^2 \pm 10y = 0}$

2) Find the eqn of the circle with centre (2,-1) and passing through the point (3,6) in standard form.

Given centre = C(2,-1) and passing through P(3,6)

∴ radius $r = CP = \sqrt{(3-2)^2 + (6+1)^2}$

$= \sqrt{1+49} = \sqrt{50}$

$= 5\sqrt{2}$

∴ The eqn of the circle with centre C(2,-1) and radius $r = 5\sqrt{2}$ is

$(x - x_1)^2 + (y - y_1)^2 = r^2$

i.e. $(x-2)^2 + (y+1)^2 = (5\sqrt{2})^2$

$x^2 - 4x + 4 + y^2 + 2y + 1 = 50$

$x^2 + y^2 - 4x + 2y + 5 - 50 = 0$

$x^2 + y^2 - 4x + 2y - 45 = 0$.

---

3) Find the eqn of circles that touch both the axes and passes through (-4,-2) in general form

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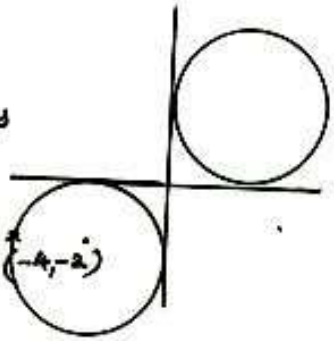

Given that the circles touches both the axes and pass through $(-4,-2)$

Let the centre be $C(-a,-a)$, radius $= a$

∴ The eqn of circle with centre $C(-a,-a)$ and radius a is

$$(x+a)^2 + (y+a)^2 = a^2$$

It passes $(-4,-2)$

$$\therefore (-4+a)^2 + (-2+a)^2 = a^2$$

$$16 + a^2 - 8a + 4 + a^2 - 4a = a^2$$

$$2a^2 - 12a + 20 - a^2 = 0$$

$$a^2 - 12a + 20 = 0$$

$$(a-2)(a-10) = 0$$

$$a = 2 \text{ or } a = 10.$$

when $a = 2$, the required eqn of the circle is

$$(x+2)^2 + (y+2)^2 = 2^2$$

$$x^2 + 4 + 4x + y^2 + 4y + 4 = 4$$

$$x^2 + y^2 + 4x + 4y + 4 = 0.$$

when $a = 10$, the eqn of the circle is

$$(x+10)^2 + (y+10)^2 = 10^2$$

$$x^2 + 20x + 100 + y^2 + 20y + 100 = 100$$

$$x^2 + y^2 + 20x + 20y + 100 = 0.$$

---

4) Find the eqn of the circle with centre $(2,3)$ and passing through the intersection of the line $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$.

Given lines are

$3x - 2y = 1$ — ① and $4x + y = 27$ — ②.

To get the point of intersection, solve the eqns ① and ②.

$$3x - 2y = 1 \quad \text{— ①}$$

② × 2 ⇒ $8x + 2y = 54$ — ③

$$11x = 55$$

$$\boxed{x = 5}$$

Sub $x = 5$ in ① $\quad 3x - 2y = 1$

$$15 - 2y = 1$$

$$2y = 14$$

$$\boxed{y = 7}$$

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∴ The required eqn of circle with centre C(2,3) and passes through the point P(5,7).

∴ radius $r = CP = \sqrt{(5-2)^2+(7-3)^2}$

$= \sqrt{3^2+4^2}$

$\therefore \sqrt{9+16} = \sqrt{25} = 5$.

∴ The eqn of circle with centre C(2,3) and radius r = 5 units is

$(x-x_1)^2 + (y-y_1)^2 = r^2$

i.e. $(x-2)^2 + (y-3)^2 = 5^2$

$x^2 - 4x + 4 + y^2 - 6y + 9 - 25 = 0$

$x^2 + y^2 - 4x - 6y - 12 = 0$.

---

5) obtain the eqn of circle for which (3,4) and (2,-7) are the ends of a diameter.

The ends of diameters are (3,4) and (2,-7).

∴ The required eqn of the circle is $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$

Here $(x_1, y_1) = (3,4)$

$(x_2, y_2) = (2,-7)$

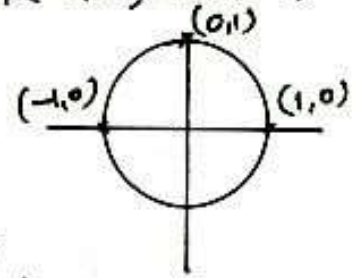
∴ The eqn is $(x-3)(x-2) + (y-4)(y+7) = 0$

$x^2 - 5x + 6 + y^2 + 3y - 28 = 0$

$x^2 + y^2 - 5x + 3y - 22 = 0$

---

6) Find the eqn of the circle through the points (1,0), (-1,0), and (0,1)



The given points are (1,0), (-1,0), (0,1).

The eqn of the circle passing through the points (1,0), (-1,0), (0,1) is nothing, the circle with centre C(0,0) and radius 1 unit.

∴ The eqn of circle is $(x-0)^2 + (y-0)^2 = 1$

i.e. $x^2 + y^2 = 1$

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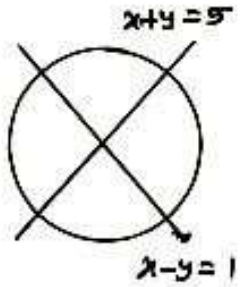
7) A circle of area 9π square units has two of its diameters along the lines $x+y=5$ and $x-y=1$ find the eqn of the circle.

Given

Area of the circle

$\pi r^2 = 9\pi$

$r^2 = 9 \Rightarrow \boxed{r=3}$



Given the eqn of the diameters are

$x+y=5$ —— ① , $x-y=1$ —— ②

The point of intersection of the diameters is centre of the circle C.

$x+y=5$ —— ①

$x-y=1$ —— ②

① + ② $\Rightarrow$ $2x=6 \Rightarrow \boxed{x=3}$

Sub $x=3$ in ① $3+y=5$

$\boxed{y=2}$

∴ The centre is C(3,2).

∴ The required circle is with centre C(3,2) and radius $r=3$.

∴ The eqn of circle is

$(x-x_1)^2 + (y-y_1)^2 = r^2$

i.e. $(x-3)^2 + (y-2)^2 = 3^2$

$x^2+9-6x+y^2+4-4y = 9$

$x^2+y^2-6x-4y+4=0$.

---

8) If $y = 2\sqrt{2}x + c$ is a tgt to the circle $x^2+y^2=16$, find the value of c.

The condition for the line $y=mx+c$ to be a tgt to the circle $x^2+y^2=a^2$ is $c^2 = a^2(1+m^2)$.

Here $x^2+y^2=16 \Rightarrow a^2=16$.

$y = 2\sqrt{2}x + c \Rightarrow m = 2\sqrt{2}$

∴ $c^2 = a^2(1+m^2)$

$= 16(1+(2\sqrt{2})^2)$

$= 16(1+8)$

$= 16 \times 9$

∴ $c = \sqrt{16\times9} = 4\times3 = 12$ units.

$\boxed{c = \pm 12}$

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9) Find the eqn of tgt and normal to the circle $x^2+y^2-6x+6y-8=0$ at $(2,2)$.

Eqn of the circle is
$x^2+y^2-6x+6y-8=0$.
compare this with $x^2+y^2+2gx+2fy+c=0$

$2g=-6 \Rightarrow g=-3$ | $2f=6 \Rightarrow f=3$ | $c=-8$.

The eqn of tgt to the circle $x^2+y^2+2gx+2fy+c=0$ at (x_1,y_1) is
$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

∴ The eqn of tgt to the circle $x^2+y^2-6x+6y-8=0$ at $(2,2)$ is
$x(2)+y(2)-3(x+2)+3(y+2)-8=0$.

i.e. $2x+2y-3x-6+3y+6-8=0$

$-x+5y-8=0$. or

$\boxed{x-5y+8=0}$

The eqn of normal is ⊥r to the eqn of tgt.

∴ The eqn of normal is

$5x+y+k=0$

It pass through $(2,2)$

∴ $5(2)+2+k=0$

$k+12=0 \Rightarrow k=-12$

∴ The eqn of normal is $5x+y-12=0$

---

10) Determine the points $(-2,1)$, $(0,0)$ and $(-4,-3)$ lie outside, on or inside the circle $x^2+y^2-5x+2y-5=0$.

Given circle is $x^2+y^2-5x+2y-5=0$ ——①

Sub $(-2,1)$ in ①

$4+1-5(-2)+2(1)-5$

$5+10+2-5=12>0$

∴ The pt $(-2,1)$ lies outside the circle.

Sub $(0,0)$ in ①

$0+0+0-5=-5<0$

∴ The point $(0,0)$ lies inside the circle.

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Sub $(-4,-3)$ in (1)

$16+9-5(-4)+2(-3)-5$

$= 25+20-6-5 = 34 > 0$

∴ The pt $(-4,-3)$ lies outside the circle.

---

11) Find the centre and radius of the following circles.

(i) $x^2+(y+2)^2=0$.

$x^2+y^2+4y+4=0$.

Compare this eqn with

$x^2+y^2+2gx+2fy+c=0$

$2g=0 \mid 2f=4 \mid c=4$

$g=0 \mid f=2$

∴ centre is $(-g,-f) = (0,-2)$

radius $= \sqrt{g^2+f^2-c} = \sqrt{0+4-4}$

$= 0$.

∴ centre is $(0,-2)$

radius $r = 0$.

(ii) $x^2+y^2+6x-4y+4=0$

Compare this eqn with

$x^2+y^2+2gx+2fy+c=0$

$2g=6 \mid 2f=-4 \mid c=4$

$g=3 \mid f=-2$

∴ centre is $(-g,-f) = (-3,2)$

radius $r = \sqrt{g^2+f^2-c} = \sqrt{9+4-4}$

$= \sqrt{9} = 3$.

(iii) $x^2+y^2-x+2y-3=0$.

Compare this eqn with $x^2+y^2+2gx+2fy+c=0$.

$2g=-1 \mid 2f=2 \mid c=-3$

$g=-\frac{1}{2} \mid f=1$

∴ The centre is $(-g,-f) = \left(\frac{1}{2},-1\right)$

radius $r = \sqrt{g^2+f^2-c} = \sqrt{\frac{1}{4}+1+3}$

$= \sqrt{\frac{1}{4}+4} = \sqrt{\frac{17}{4}}$

$= \boxed{\frac{\sqrt{17}}{2}}$

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(iv) $2x^2+2y^2-6x+4y+2=0$. —①

compare this eqn with

$x^2+y^2+2gx+2fy+c=0$.

$\frac{①}{2} \Rightarrow x^2+y^2-3x+2y+1=0$

$2g=-3 \Rightarrow g=-\frac{3}{2}$ | $2f=2 \Rightarrow f=1$ | $c=1$.

∴ The centre is $(-g,-f)=\left(\frac{3}{2},-1\right)$

radius $r=\sqrt{g^2+f^2-c}$

$=\sqrt{\frac{9}{4}+1-1}=\sqrt{\frac{9}{4}}=\boxed{\frac{3}{2}}$

---

12) If the eqn $3x^2+(3-p)xy+qy^2-2px=8pq$ represents a circle, find p and q. Also determine the centre and radius of the circle.

Given eqn $3x^2+(3-p)xy+qy^2-2px=8pq$ —①

represents a circle.

⇒ coeff. of x^2 = coeff of y^2 and coeff $xy=0$.

i.e $\boxed{q=3}$ and $3-p=0$

$\Rightarrow \boxed{p=3}$

Sub $p=3$, $q=3$ in ①

$3x^2+0xy+3y^2-6x=8(3)(3)$

$3x^2+3y^2-6x-72=0$

$(\div 3)$ $x^2+y^2-2x-24=0$.

compare this eqn with

$x^2+y^2+2gx+2fy+c=0$.

$2g=-2 \Rightarrow g=-1$ | $2f=0 \Rightarrow f=0$ | $c=-24$.

∴ centre is $C(-g,-f)=C(1,0)$

radius $r=\sqrt{g^2+f^2-c}=\sqrt{1+0+24}$

$=\sqrt{25}$

$=5$.

∴ centre is $(1,0)$

radius $r=5$ units.

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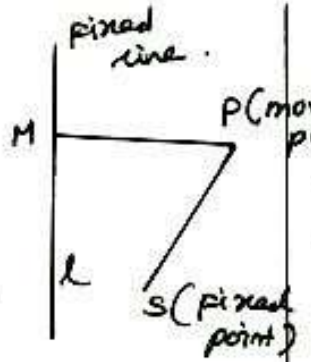
Conic :-



The locus of a point P which move on a plane such that its distance from the fixed point S bears a constant ratio to its distance from the fixed line l.

i.e. $\frac{SP}{PM} = e$.

Here the fixed point S is called focus, the fixed line is called directrix, and the constant ratio is called eccentricity.

a) If $e = 1$, the conic is parabola
b) If $e < 1$, the conic is an ellipse
c) If $e > 1$, the conic is hyperbola.

General eqn of a conic

The general eqn of a conic is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

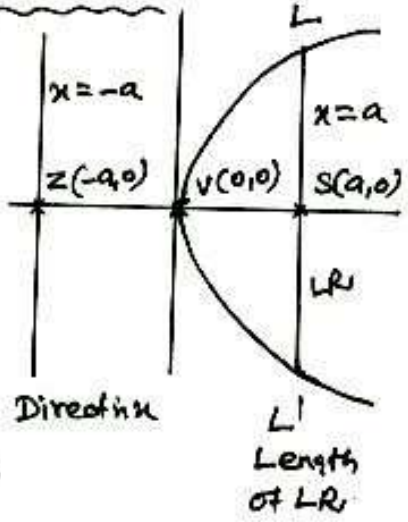
1) If $B^2 - 4AC = 0$, then the conic is called a parabola.
2) If $B^2 - 4AC < 0$, then the conic is called an ellipse.
3) If $B^2 - 4AC > 0$, then the conic is called hyperbola.
4) If the coeff. of x^2 = Coeff. of y^2 and coeff. of $xy = 0$, then the above eqn represents a circle.

Standard Eqn of parabola :-

Equation of parabola

Type I $y^2 = 4ax$.

(Open Right ward)



1) Vertex is V(0,0)
2) Focus is S(a,0)
3) Eqn of latus Rectum is $x = a$
4) Eqn of directrix is $x = -a$
5) Length of latus Rectum $= 4a$.

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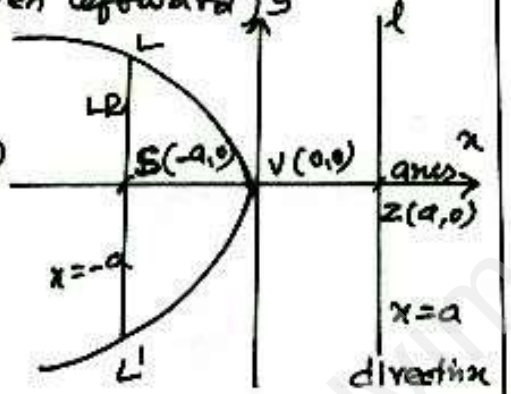
6) x-axis is axis of the parabola.

7) $a = VS = VZ$.

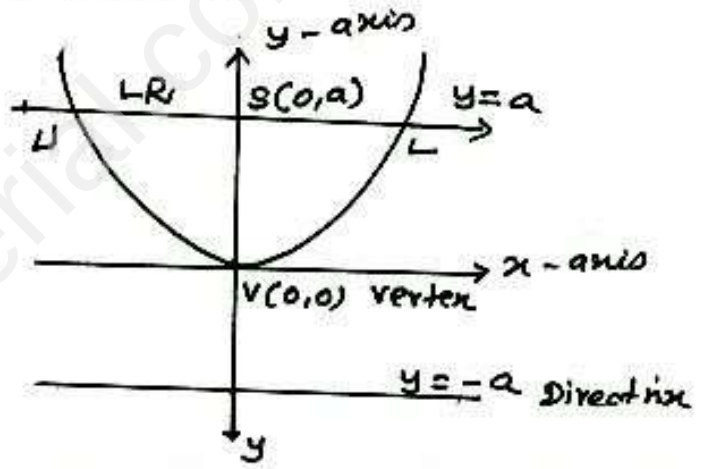
Type II Equation of parabola is $y^2 = -4ax$ (open leftward)



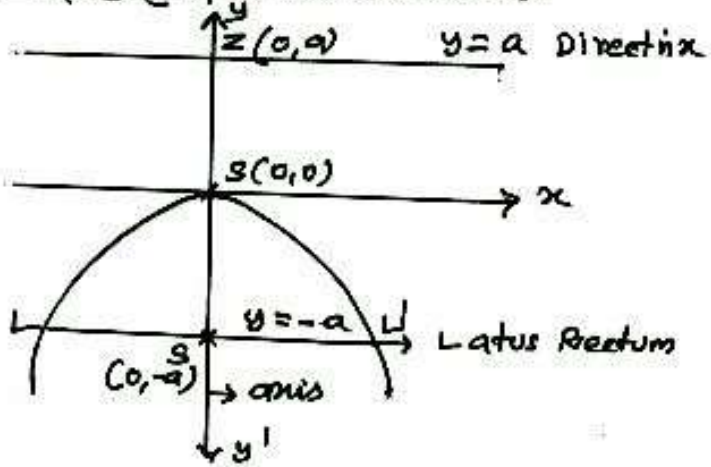
1) Vertex is V(0,0)
2) Focus is S(-a,0)
3) Equation of Latus Rectum is $x = -a$
4) Eqn of directrix is $x = a$
5) Length of LR is $LL' = 4a$
6) $a = VS = VZ$

Type III Equation of parabola is $x^2 = 4ay$ (open upward)

1) Vertex is V(0,0)
2) Focus is S(0,a)
3) Equation of latus Rectum is $y = a$
4) Eqn of directrix is $y = -a$
5) Length of latus rectum $LL' = 4a$
6) y-axis is axis of the parabola
7) $a = VS = VZ$



Type IV Equation of parabola is $x^2 = -4ay$ (open downward)



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1) Vertex is $V(0,0)$
2) Focus is $S(0,-a)$
3) Eqn of LR is $y = -a$
4) Eqn of Directrix is $y = a$
5) Length of Latus rectum $LL' = 4a$
6) y-axis is axis of parabola
7) $a = VS = VZ$.

Note: If the vertex of the parabola is $V(h,k)$ then, the standard eqn of parabola is

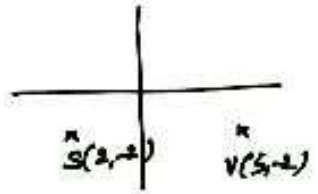
1) $(y-k)^2 = 4a(x-h)$ [open Rightward]
2) $(y-k)^2 = -4a(x-h)$ [open leftward]
3) $(x-h)^2 = 4a(y-k)$ [open upward]
4) $(x-h)^2 = -4a(y-k)$ [open downward]

Ex 5.14 Find the equation of the parabola with focus $(-\sqrt{2},0)$ and directrix $x = \sqrt{2}$.

Focus is $(-\sqrt{2},0) = F(-a,0)$
Directrix is $x = \sqrt{2} = (x = a)$
From the data, the parabola is open left and vertex is $V(0,0)$
$\therefore$ The eqn of parabola is
$(y-0)^2 = -4\sqrt{2}(x-0)$
i.e. $y^2 = -4\sqrt{2}x$

5.15 Find the eqn of parabola whose vertex is $(5,-2)$ and focus is $(2,-2)$

Given vertex $= V(5,-2)$
focus is $S(2,-2)$
Focal distance
$VS = \sqrt{(5-2)^2 + (-2+2)^2}$
$= \sqrt{3^2} = 3$
i.e. $a = 3$



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From the given data,
the parabola is open left about the line ⊥ to x-axis.

∴ The eqn of parabola with vertex V(h,k) and open left is

$(y-k)^2 = -4a(x-h)$

Here $(h,k) = V(5,-2)$ and $a=3$

∴ The required eqn is

$(y+2)^2 = -4(3)(x-5)$

$(y+2)^2 = -12(x-5)$

$y^2+4y+4 = -12x+60$

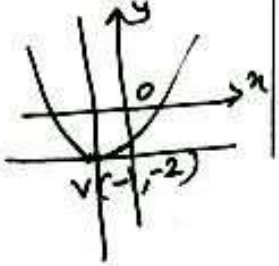
$y^2+12x+4y+4-60=0$

$y^2+12x+4y-56=0$.

Ex 5.16 Find the eqn of the parabola with vertex (-1,-2), axis parallel to y-axis and passing through (3,6)

vertex is (-1,-2)

From the data, the parabola is open upward and ⊥ to y-axis.



The eqn of parabola with vertex V(h,k) and ⊥ to y-axis is

$(x-h)^2 = 4a(y-k)$

Here $(h,k) = (-1,-2)$

∴ The eqn is $(x+1)^2 = 4a(y+2)$

But this eqn passes through (3,6)

∴ $(3+1)^2 = 4a(6+2)$

$16 = 4a(8) \Rightarrow a = \frac{16}{2 \times 8} = \frac{1}{2}$

$\boxed{a = \frac{1}{2}}$

∴ The eqn of parabola is

$(x+1)^2 = 4\left(\frac{1}{2}\right)(y+2)$

$(x+1)^2 = 2(y+2)$

$x^2+2x+1 = 2y+4$

$x^2+2x+2y+1-4=0$

$x^2+2x+2y-3=0$

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* **5.17** find the vertex, focus, directrix, and length of latus rectum of the parabola $x^2 - 4x - 5y - 1 = 0$.

Given eqn is $x^2 - 4x - 5y - 1 = 0$

i.e. $x^2 - 4x = 5y + 1$

$x^2 - 4x + 4 = 5y + 1 + 4$

$(x-2)^2 = 5y + 5$

$(x-2)^2 = 5(y+1)$

$X^2 = 5Y$ where $X = x - 2$, $Y = y + 1$

This is of the form $X^2 = 4aY$ (0.w.w)

$\therefore 4a = 5 \Rightarrow \boxed{a = 5/4}$

	Referred to x, y	Referred to x, y
vertex	V(0,0)	$X = 0$ $Y = 0$ $x - 2 = 0$ $y + 1 = 0$ $x = 2$ $y = -1$ $\therefore \boxed{V(2,-1)}$
Directrix	$Y = -a$ i.e. $y = -\frac{5}{4}$	$Y = -5/4$ i.e. $y + 1 = -5/4$ $y = -1 - \frac{5}{4}$ $y = -\frac{9}{4}$ or $\boxed{4y + 9 = 0}$
focus	S(0,a) i.e. S(0, 5/4)	$X = 0$, $Y = \frac{5}{4}$ i.e. $x - 2 = 0$ $y + 1 = \frac{5}{4}$ $x = 2$ $y = \frac{5}{4} - 1$ $y = \frac{1}{4}$ $\therefore \boxed{S(2, \frac{1}{4})}$
Latus Rectum	$y = a$ i.e. $y = \frac{5}{4}$	$y + 1 = \frac{5}{4}$ $y = \frac{5}{4} - 1$ $y = \frac{1}{4}$ or $\boxed{4y - 1 = 0}$
length of LR	$4a = 4(5/4) = 5$ $\boxed{4a = 5}$	$\boxed{4a = 5}$

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EXERCISE 5.2

1. Find the equation of the parabola

(i) focus (4,0) and directrix $x=-4$

Given that

focus is $S(4,0)$

Eqn of directrix is $x=-4$

$P(x,y)$ is the moving point

By defn, $\frac{SP}{PM}=1 \Rightarrow SP^2=PM^2$

$$\Rightarrow (x-4)^2+(y-0)^2=\left(\left|\frac{x+4}{\sqrt{1^2+0^2}}\right|\right)^2$$

$$(x-4)^2+y^2=(x+4)^2$$

$$x^2-8x+16+y^2=x^2+8x+16$$

$$y^2=16x \quad \text{or} \quad y^2-16x=0$$

which is required eqn of parabola.

(ii) passes through (2,-3) and symmetric about y-axis

Given that the parabola symmetric about y-axis.

$\therefore$ Eqn of parabola is $x^2=4ay$ —①

It passes through (2,-3).

$\therefore \quad 4=4a(-3)$

$$4=-12a \Rightarrow a=\frac{4}{-12}=\frac{-1}{3}$$

$$\boxed{a=-\frac{1}{3}}$$

$\therefore$ The required eqn is $x^2=4\left(-\frac{1}{3}\right)y$

$$x^2+\frac{4}{3}y=0$$

i.e. $\boxed{3x^2+4y=0}$

(iii) vertex (1,-2), and focus (4,-2).

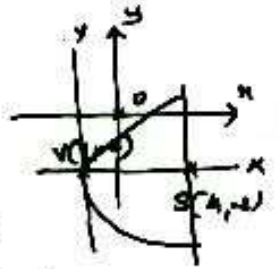
Given vertex $V=V(1,-2)$

focus $S=S(4,-2)$

$$a=VS=\sqrt{(4-1)^2+(-2+2)^2}$$

$$=\sqrt{3^2+0}=3$$

From the data, the parabola is open right.



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The eqn of parabola with vertex V(h,k) and open right is

$(y-k)^2 = 4a(x-h)$

Here $(h,k) = (1,-2)$

$\therefore (y+2)^2 = 4(3)(x-1)$

$(y+2)^2 = 12(x-1)$ (OR)

$y^2+4+4y-12x+12=0$

i.e. $y^2-12x+4y+16=0$.

(IV) end points of latus rectum $(4,-8)$ and $(4,8)$

Let end points of Latus rectum be $L(4,-8)$ and $L'(4,8)$

Focus is mid point of LL'

i.e. $S = \left(\frac{4+4}{2}, \frac{-8+8}{2}\right)$

$= (4,0) = S(a,0)$

$\therefore \boxed{a=4}$

$\therefore$ vertex is $V(0,0)$

From the data, the parabola is open right.

$\therefore$ The eqn of parabola is

$y^2 = 4ax$

i.e. $y^2 = 4(4)x$

$y^2 = 16x$.

---

4) Find the vertex, focus, eqn of directrix and length of latus rectum

(i) $y^2 = 16x$.

Given eqn is $y^2 = 16x$.

This is of the form $y^2 = 4ax$ (O.R.W)

$\therefore 4a = 16 \Rightarrow \boxed{a=4}$

Vertex $= V(h,k) = \boxed{V(0,0)}$

Focus $= S(a,0) = \boxed{S(4,0)}$

Eqn of directrix $= x = -a$

i.e. $x = -4$ or $\boxed{x+4=0}$

Length of LR $= 4a = 4(4) = \boxed{16}$

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(ii) $x^2 = 24y$

This is of the form $x^2 = 4ay$ (o.u.w)

$\therefore 4a = 24 \Rightarrow \boxed{a = 6}$

Vertex = $\boxed{V(0,0)}$

Focus = $S(0,a) = \boxed{S(0,6)}$

Eqn of directrix is $y = -a$

i.e. $y = -6$

$\boxed{y + 6 = 0}$

Length of LR = $4a$

$= 4(6) = \boxed{24}$

(iii) $y^2 = -8x$

This is of the form $y^2 = -4ax$. (o.l.w)

$\therefore 4a = 8 \Rightarrow \boxed{a = 2}$

Vertex is $\boxed{V(0,0)}$

Focus is $S(-a,0) = \boxed{S(-2,0)}$

Eqn of directrix is $x = a$

i.e. $x = 2$ or

$\boxed{x - 2 = 0}$

Length of LR = $4a$

$= 4(2) = \boxed{8}$

(iv) $x^2 - 2x + 8y + 17 = 0$

$x^2 - 2x = -8y - 17$

$x^2 - 2x + 1 = -8y - 17 + 1$

$(x-1)^2 = -8y - 16$

$(x-1)^2 = -8(y+2)$

$X^2 = -8Y$ where

$X = x - 1$

$Y = y + 2$.

This is of the form $x^2 = -4ay$ (open down word)

$4a = 8 \Rightarrow \boxed{a = 2}$

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	Referred to X, Y	Referred to x, y
Vertex	$V(0,0)$ i.e. $X=0,\ Y=0$	$x=0 \mid y=0$ $x-1=0 \mid y+2=0$ $x=1 \mid y=-2$ $\therefore V(1,-2)$
Focus	$S(0,-a)$ i.e. $X=0,\ Y=-a$	$x=0 \mid y=-a$ $x-1=0 \mid y+2=-2$ $x=1 \mid y=-4$ Focus $S(1,-4)$
Directrix	$Y=a$ i.e. $y=2$	i.e. $Y=2$ $y+2=2$ $y=0$
Length of LR	$4a=4(2)$ $=8$	$4a=8$

(V) $y^2-4y-8x+12=0$

$y^2-4y = 8x-12$

$y^2-4y+4 = 8x-12+4$

$(y-2)^2 = 8x-8$

$(y-2)^2 = 8(x-1)$

$y^2=8x$. where $X=x-1$, $Y=y-2$

This is of the form $y^2=4ax$ (O.R.W)

$\therefore 4a=8 \Rightarrow \boxed{a=2}$

	Referred to X, Y	Referred to x, y
Vertex	$V(0,0)$ i.e. $X=0,\ Y=0$	$x=0 \mid y=0$ $x-1=0 \mid y-2=0$ $x=1 \mid y=2$ $\therefore$ Vertex $V(1,2)$
Focus	$S(a,0)$ i.e. $X=a,\ Y=0$	$x=a \mid y=0$ $x-1=2 \mid y-2=0$ $x=3 \mid y=2$ $\therefore$ Focus $S(3,2)$
Directrix	$X=-a$ i.e. $X=-2$	$X=-a$ $x-1=-2$ $x=-1$ or $x+1=0$
Length of LR	$4a=4(2)$ $=8$	$4a=8$

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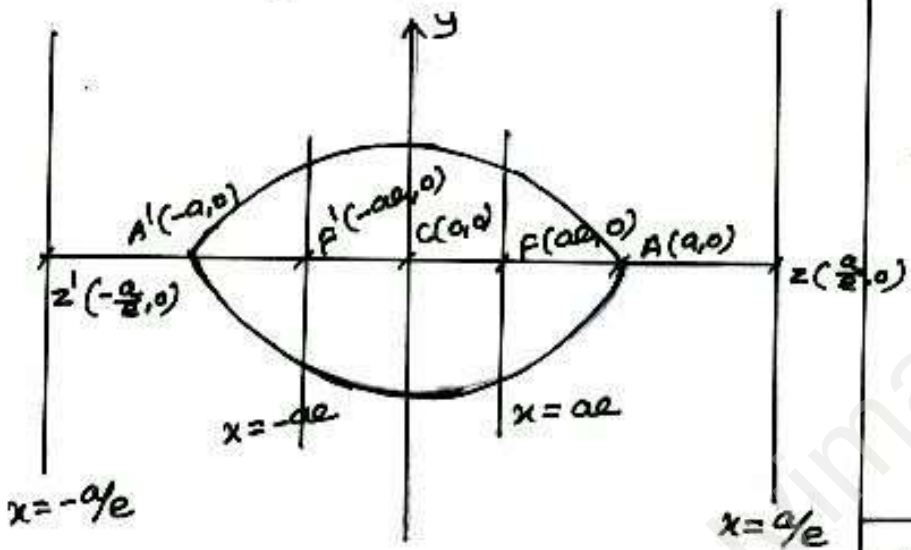
ELLIPSE ($e<1$)

Type I $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a>b)$



1) The centre is $C(0,0)$
2) Vertices are $A(a,0)$ and $A'(-a,0)$
3) The foci are $F(ae,0)$, $F'(-ae,0)$
4) Eqn of directrix are $x = a/e$ and $x = -a/e$
5) Eqn of LR are $x = ae$ and $x = -ae$.
6) Length of LR is $\frac{2b^2}{a}$
7) Eccentricity $e = \sqrt{1-\frac{b^2}{a^2}}$
8) $b^2 = a^2(1-e^2)$
9) Major axis is x-axis and Length of major axis is $2a$.
10) Minor axis is y-axis and length of minor axis is $2b$.
11) $CA = CA' = a$ and $FF' = 2ae$.

Note: The sum of the focal distance of any point on the ellipse is equal to length of the major axis

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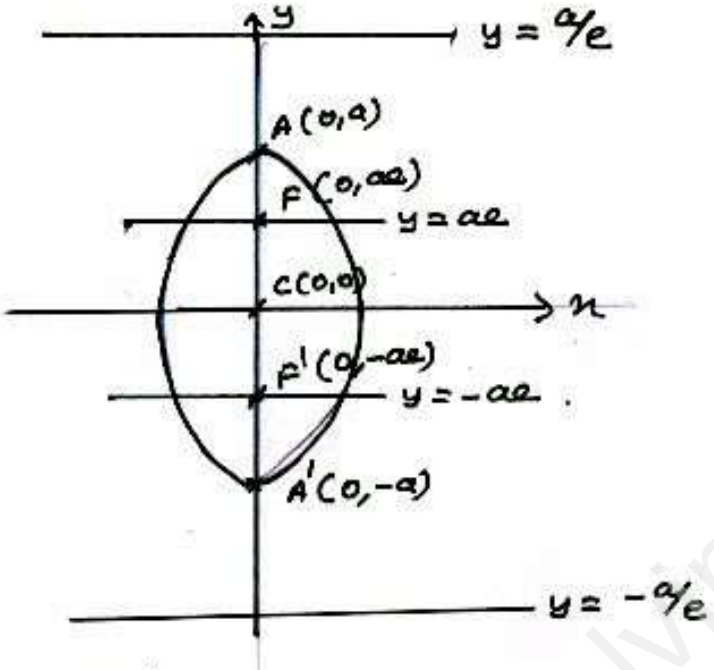
Type II $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (a>b)$



1) The centre is $C(0,0)$
2) The vertices are $A(0,a)$ and $A'(0,-a)$
3) The foci are $F(0,ae)$ and $F'(0,-ae)$
4) Eqn of directrix are $y = a/e$ and $y = -a/e$.
5) Eqn of latus rectum are $y = ae$ and $y = -ae$. XII|CH 5 / 5.2|P11
6) Length of $LR = \frac{2b^2}{a}$
7) eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}}$
8) $b^2 = a^2(1-e^2)$
9) Major axis is y-axis and length of Major axis is $2a$.
10) Minor axis is x-axis and length of minor axis is $2b$.
11) $CA = CA' = a$ and $FF' = 2ae$.

Note: standard eqn of the ellipse with centre at (h,k) and Major axis parallel to x-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \quad a>b.$$

2) standard eqn of the ellipse with centre at (h,k) and Major axis parallel to y-axis is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1, \quad a>b.$$

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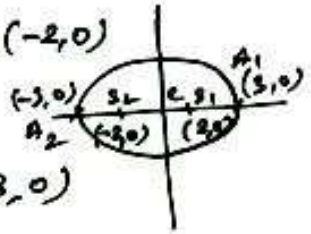
5.18 Find the eqn of the ellipse with foci $(\pm 2, 0)$, vertices $(\pm 3, 0)$

The foci are $S_1(2,0)$ $S_2(-2,0)$



vertices are

$A_1(3,0)$, and $A_2(-3,0)$

The centre is mid pt of A_1A_2

$= \left(\frac{3-3}{2}, \frac{0+0}{2}\right) = (0,0)$.

From the data, the major axis is along x-axis.

∴ The eqn of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{——} (1)$$

Here $CA = 3$ ⇒ $a = 3$

$CF = ae = 2$ ⇒ $3e = 2$, $e = \frac{2}{3}$

Now $b^2 = a^2(1-e^2)$

$= 9\left(1 - \frac{4}{9}\right) = 9\left(\frac{5}{9}\right) = 5$

∴ The eqn of ellipse is

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

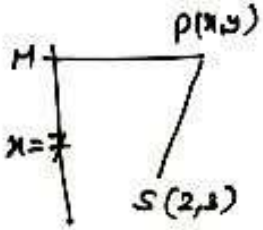
E 5.19 Find the eqn of ellipse whose eccentricity is $\frac{1}{2}$. one of the foci is $(2,3)$ and a directrix is $x = 7$. Also find the major and minor axes of the ellipse.

Given

Foci is $S(2,3)$

eccentricity $e = \frac{1}{2}$

Directrix $x - 7 = 0$



$P(x,y)$ be any point such that

$$\frac{SP}{PM} = e \Rightarrow SP^2 = e^2 PM^2$$

$$\text{i.e. } (x-2)^2 + (y-3)^2 = \frac{1}{4}\left[\frac{x-7}{\sqrt{1+0}}\right]^2$$

$$x^2 + 4 - 4x + y^2 + 9 - 6y = \frac{1}{4}(x-7)^2$$

$$x^2 + y^2 - 4x - 6y + 13 = \frac{1}{4}(x^2 + 49 - 14x)$$

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$4x^2+4y^2-16x-24y+52-x^2-49+14x=0$

$3x^2+4y^2-2x-24y+3=0$

$3x^2-2x+4y^2-24y=-3$

$3x^2-2x+4(y^2-6y)=-3$

$3\left[x^2-\frac{2}{3}x\right]+4\left[y^2-6y+9-9\right]=-3$

$3\left[x^2-\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right]+4\left[(y-3)^2-9\right]=-3$

$3\left[\left(x-\frac{1}{3}\right)^2-\frac{1}{9}\right]+4\left[(y-3)^2-9\right]=-3$

$3\left(x-\frac{1}{3}\right)^2-\frac{1}{3}+4(y-3)^2-36=-3$

$3\left(x-\frac{1}{3}\right)^2+4(y-3)^2-\frac{1}{3}-36=-3$

$3\left(x-\frac{1}{3}\right)^2+4(y-3)^2=-3+36+\frac{1}{3}$

$3\left(x-\frac{1}{3}\right)^2+4(y-3)^2=\frac{-9+108+1}{3}$

$=\frac{100}{3}$

$3\left(x-\frac{1}{3}\right)^2+4(y-3)^2=\frac{100}{3}$

$\frac{3\left(x-\frac{1}{3}\right)^2}{\frac{100}{3}}+\frac{4(y-3)^2}{\frac{100}{3}}=1$

$\frac{\left(x-\frac{1}{3}\right)^2}{\frac{100}{9}}+\frac{(y-3)^2}{\frac{100}{12}}=1$

$\frac{\left(x-\frac{1}{3}\right)^2}{\left(\frac{10}{3}\right)^2}+\frac{(y-3)^2}{\frac{25}{3}}=1$

$\frac{\left(x-\frac{1}{3}\right)^2}{\left(\frac{10}{3}\right)^2}+\frac{(y-3)^2}{\left(\frac{5}{\sqrt{3}}\right)^2}=1$

$\therefore$ Length of major axis $2a=2\times\frac{10}{3}$

$=\boxed{\frac{20}{3}}$

Length of minor axis $2b=2\left(\frac{5}{\sqrt{3}}\right)$

$=\boxed{\frac{10}{\sqrt{3}}}$

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Ex 5.20 Find the foci, vertices and Length of major and minor axis of the conic.

$4x^2 + 36y^2 + 40x - 288y + 532 = 0$.

The eqn of conic is

$4x^2 + 36y^2 + 40x - 288y + 532 = 0$

$4x^2 + 40x + 36y^2 - 288y = -532$

$4[x^2 + 10x] + 36[y^2 - 8y] = -532$

$4[x^2 + 10x + 25 - 25] + 36[y^2 - 8y + 16 - 16] = -532$

$4[(x+5)^2 - 25] + 36[(y-4)^2 - 16] = -532$

$4(x+5)^2 - 100 + 36(y-4)^2 - 576 = -532$

$4(x+5)^2 + 36(y-4)^2 = -532 + 676$

$4(x+5)^2 + 36(y-4)^2 = 144$

$\frac{4(x+5)^2}{144} + \frac{36(y-4)^2}{144} = 1$

$\frac{(x+5)^2}{36} + \frac{(y-4)^2}{4} = 1$

$\Rightarrow \frac{X^2}{36} + \frac{Y^2}{4} = 1$ where $X = x+5$, $Y = y-4$

This is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Here $a^2 = 36 \Rightarrow a = 6$

$b^2 = 4 \Rightarrow b = 2$

Eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{4}{36}}$

$= \sqrt{\frac{32}{36}} = \sqrt{\frac{16 \times 2}{36}} = \frac{4\sqrt{2}}{6}$

$\boxed{e = \frac{2\sqrt{2}}{3}}$

The major axis is ⊥ to x-axis.

Foci $S(\pm ae, 0) = S(\pm 6 \times \frac{2\sqrt{2}}{3}, 0)$

i.e. $S(\pm 4\sqrt{2}, 0)$

i.e. $X = \pm 4\sqrt{2}$, $Y = 0$

$x + 5 = \pm 4\sqrt{2}$, $y - 4 = 0$

$x = -5 \pm 4\sqrt{2}$, $y = 4$

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$\therefore$ foci are $\boxed{(-5+4\sqrt{2}, 4)}$ and $\boxed{(-5-4\sqrt{2}, 4)}$

Vertices $A(\pm a, 0) = (\pm 6, 0)$

i) $x = \pm 6, \quad y = 0$

$x+5 = \pm 6, \quad y-4 = 0$

$x = -5 \pm 6, \quad y = 4$

$\therefore$ vertices are $\boxed{(1,4), (-11,4)}$

Length of major axis $2a$

$= 2(6) = \boxed{12}$

Length of minor axis $2b$

$= 2(2) = \boxed{4}$

5.21 For the ellipse $4x^2+y^2+24x-2y+21=0$, find the centre, vertices and foci. Also prove that the length of LR is 2.

Given eqn of ellipse is

$4x^2+y^2+24x-2y+21=0$

$4x^2+24x+y^2-2y = -21$

$4[x^2+6x] + [y^2-2y] = -21$

$4[x^2+6x+9-9] + [y^2-2y+1-1] = -21$

$4[(x+3)^2-9] + [(y-1)^2-1] = -21$

$4(x+3)^2 - 36 + (y-1)^2 - 1 = -21$

$4(x+3)^2 + (y-1)^2 = 37-21$

$4(x+3)^2 + (y-1)^2 = 16$

$\frac{4(x+3)^2}{16} + \frac{(y-1)^2}{16} = 1$

$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{16} = 1$

$\frac{X^2}{4} + \frac{Y^2}{16} = 1$ where $X = x+3$, $Y = y-1$

This is of the form

$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

$\therefore$ The major axis is $\perp^l$ to y-axis.

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Here $a^2 = 16 \Rightarrow a = 4$

$b^2 = 4 \Rightarrow b = 2$

$\therefore e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{4}{16}} = \sqrt{\frac{12}{16}} = \sqrt{\frac{3}{4}}$

$= \boxed{\frac{\sqrt{3}}{2}}$

	Referred to x, y	Referred to x, y
Centre	$C(0,0)$ $x=0,\ y=0$	$x=0 \mid y=0$ $x+3=0 \mid y-1=0$ $x=-3 \mid y=1$ $\therefore$ centre is $\boxed{C(-3,1)}$
Vertices	$A(0,a)$ $A'(0,-a)$ i.e. $A(0,4)$ $A'(0,-4)$	$x=0 \mid y=\pm 4$ $x+3=0 \mid y-1=\pm 4$ $x=-3 \mid y=1\pm 4$ $\therefore$ vertices $\boxed{A(-3,5),\ A'(-3,-3)}$
Foci	$S(0,ae),\ S'(0,-ae)$ i.e. $S(0,2\sqrt{3}),\ S'(0,-2\sqrt{3})$	$x=0 \mid y=\pm 2\sqrt{3}$ $x+3=0 \mid y-1=\pm 2\sqrt{3}$ $x=-3 \mid y=1\pm 2\sqrt{3}$

$\therefore$ foci are $S(-3, 1+2\sqrt{3})$, $S'(-3, 1-2\sqrt{3})$

Length of Latus Rectum $\frac{2b^2}{a}$

i.e. $\frac{2(4)}{4} = \boxed{2}$

EXERCISE 5.2

2) Find the eqn of the ellipse

(i) foci $(\pm 3, 0)$, $e = \frac{1}{2}$.

Given foci $= S(3,0),\ S'(-3,0)$

Centre is mid point of SS'

i.e. $C = \left(\frac{3-3}{2}, \frac{0+0}{2}\right) = (0,0)$

From the data, the major axis is along x-axis.

$\therefore$ The eqn of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

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From the data,

$SS' = 2ae = 6, \quad e = \frac{1}{2}$

$2(a)(\frac{1}{2}) = 6$

$\boxed{a = 6}$

$\therefore b^2 = a^2(1-e^2)$

$= 36(1-\frac{1}{4})$

$= 36(\frac{3}{4}) = \boxed{27}$

$\therefore$ The eqn of ellipse is

$\frac{x^2}{36} + \frac{y^2}{27} = 1.$

(ii) foci $(0, \pm 4)$ and end points of major axes are $(0, \pm 5)$.

Given foci are $S(0,4)$, $S'(0,-4)$

End pts of major axis $A(0,5)$, $A'(0,-5)$

centre = mid pt of SS'

$= (\frac{0+0}{2}, \frac{4-4}{2}) = (0,0).$

$AA' = 2a \Rightarrow 2a = \sqrt{(0-0)^2+(5+5)^2}$

$2a = \sqrt{10^2} = \sqrt{100} = 10.$

$\therefore \boxed{a = 5}$

$SS' = 2ae \Rightarrow 2ae = \sqrt{(0-0)^2+(4+4)^2}$

$2ae = \sqrt{8^2} = 8$

$2(5)e = 8 \Rightarrow e = 8/10$

$\boxed{e = \frac{4}{5}}$

$b^2 = a^2(1-e^2)$

$= 25(1-\frac{16}{25}) = 25(\frac{9}{25})$

$= \boxed{9}$

From the data, the major axis is along y-axis.

$\therefore$ The eqn of ellipse is

$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

i.e. $\frac{x^2}{9} + \frac{y^2}{25} = 1$

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(III) Length of latus rectum 8, eccentricity $e = \frac{3}{5}$ and major axis is on x-axis.

Given length of latus rectum = 8

eccentricity $e = \frac{3}{5}$

i.e. $\frac{2b^2}{a} = 8 \Rightarrow b^2 = \frac{8a}{2} = 4a$ — (1)

eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}}$

i.e. $\frac{3}{5} = \sqrt{1 - \frac{4a}{a^2}}$

$\frac{3}{5} = \sqrt{1 - \frac{4}{a}} = \sqrt{\frac{a-4}{a}}$

$\frac{9}{25} = \frac{a-4}{a}$

$9a = 25a - 100$

$25a - 9a = 100$

$16a = 100$

$a = \frac{100}{16} = \boxed{\frac{25}{4}}$

Sub $a = \frac{25}{4}$ in (1)

$b^2 = 4\left(\frac{25}{4}\right) = 25$

Given that Major axis is on x-axis

∴ The eqn of ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

i.e. $\frac{x^2}{\left(\frac{25}{4}\right)^2} + \frac{y^2}{25} = 1$

$\frac{16x^2}{625} + \frac{y^2}{25} = 1$

(IV) Length of latus rectum 4, distance betn foci $4\sqrt{2}$ and major axis as y-axis.

Distance betn foci $= 4\sqrt{2}$

i.e. $2ae = 4\sqrt{2}$

$ae = 2\sqrt{2}$ — (1)

Length of latus rectum $\frac{2b^2}{a} = 4$

$\frac{b^2}{a} = 2 \Rightarrow b^2 = 2a$ — (2)

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But $b^2 = a^2(1-e^2)$

$= a^2 - a^2e^2$

$= a^2 - 8$ — from ① $ae = 2\sqrt{2}$, $a^2e^2 = 8$

$\therefore a^2 - b^2 = 8$ — ③

Sub ② in ③

$a^2 - 2a - 8 = 0$

$(a-4)(a+2) = 0$

$a = 4$ or $a = -2$

$a = -2$ is not possible

$\therefore a = 4$, $\therefore b^2 = 2a$

$= 2(4) = 8$

Given that the major axis is along y-axis

∴ The eqn of ellipse is

$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

i.e. $\frac{x^2}{8} + \frac{y^2}{16} = 1$

XII|CH5|5.2|P19

5) Identify the type of conic and find centre, foci, vertices and directrices of each of the following

(i) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

Given eqn is $\frac{x^2}{25} + \frac{y^2}{9} = 1$

This is in the form of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

∴ The given conic is an ellipse.

Here $a^2 = 25$ | $b^2 = 9$

$a = 5$ | $b = 3$

centre C = C(0,0)

foci = S(±ae, 0)

Now $b^2 = a^2(1-e^2)$

$9 = 25(1-e^2)$

$\frac{9}{25} = 1 - e^2 \Rightarrow e^2 = 1 - \frac{9}{25}$

$= \frac{16}{25}$

∴ $e = \frac{4}{5}$

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Vertices = $A(\pm 9, 0)$

= $\boxed{A(\pm 5, 0)}$

foci are $S(\pm ae, 0)$

$S\left(\pm 5 \times \frac{4}{5}, 0\right)$

$\boxed{S(\pm 4, 0)}$

Eqn of directrices $x = \pm \frac{a}{e}$

$x = \pm \frac{5}{4/5} = \pm \frac{25}{4}$

$\therefore \boxed{x = \pm \frac{25}{4}}$

---

(11) $\frac{x^2}{3} + \frac{y^2}{10} = 1$

The given conic is in the form of $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

$\therefore$ The conic is an ellipse.

$a^2 = 10, \quad b^2 = 3$

$\therefore a = \sqrt{10}, \quad b = \sqrt{3}$

Centre = $C(0,0)$

WKT, $b^2 = a^2(1 - e^2)$

$3 = 10(1 - e^2)$

$\frac{3}{10} = 1 - e^2 \Rightarrow e^2 = 1 - \frac{3}{10} = \frac{7}{10}$

$\therefore \boxed{e = \sqrt{\frac{7}{10}}}$

Vertices = $A(0, \pm a)$

= $\boxed{A(0, \pm \sqrt{10})}$

foci are $S(0, \pm ae)$

$S\left(0, \pm \sqrt{10} \times \frac{\sqrt{7}}{\sqrt{10}}\right)$

$\boxed{S(0, \pm \sqrt{7})}$

Eqn of directrices $y = \pm a/e$

$y = \pm \frac{\sqrt{10}}{\frac{\sqrt{7}}{\sqrt{10}}} = \pm \frac{10}{\sqrt{7}}$

$\boxed{y = \pm \frac{10}{\sqrt{7}}}$

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8) Identify the type of conic and find centre, foci, vertices and directrices of

(i) $\frac{(x-3)^2}{225} + \frac{(y-4)^2}{289} = 1$

Given conic is in the form of

$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

∴ The conic is an ellipse

where $X = x-3$, $Y = y-4$

Here $a^2 = 289$ | $b^2 = 225$

$\therefore a = \sqrt{289}$ | $\therefore b = \sqrt{225}$

$a = 17$ | $b = 15$

$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{225}{289}} = \sqrt{\frac{64}{289}}$

$= \boxed{\frac{8}{17}}$

$ae = 17\left(\frac{8}{17}\right) = \boxed{8}$

$\therefore \frac{a}{e} = \frac{17}{8/17} = \frac{17 \times 17}{8} = \boxed{\frac{289}{8}}$

	Referred to X, Y	Referred to x, y
Centre	$C(0,0)$ $X = 0$ $Y = 0$	$x = 0$ | $y = 0$ $x-3 = 0$ | $y-4 = 0$ $x = 3$ | $y = 4$ ∴ Centre $\boxed{C(3,4)}$
Foci	$S(0, \pm ae)$ $S(0, \pm 8)$	$x = 0$ | $Y = \pm 8$ $x-3 = 0$ | $y-4 = \pm 8$ $x = 3$ | $y = 4 \pm 8$ ∴ Foci are $\boxed{S(3,12),\ S'(3,-4)}$
Vertices	$A(0, \pm a)$ i.e. $A(0, \pm 17)$	$x = 0$ | $Y = \pm 17$ $x-3 = 0$ | $y-4 = \pm 17$ $x = 3$ | $y = 4 \pm 17$ ∴ $\boxed{A(3,21)\ A'(3,-13)}$
Directrix	$Y = \pm \frac{a}{e}$	i.e. $y-4 = \pm \frac{289}{8}$ $y = 4 \pm \frac{289}{8}$ $\boxed{y = \frac{32 \pm 289}{8}}$

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(ii) $\frac{(x+1)^2}{100} + \frac{(y-2)^2}{64} = 1$

The given conic is in the form of

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

∴ The conic is an ellipse and the axis ⊥ to x-axis.

Here $X = x+1$, $Y = y-2$

$a^2 = 100$ | $b^2 = 64$

$a = 10$ | $b = 8$

$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{64}{100}} = \sqrt{\frac{36}{100}}$

$\therefore e = \frac{6}{10} = \boxed{\frac{3}{5}}$

$\therefore ae = 10 \times \frac{3}{5} = \boxed{6}$

$\frac{a}{e} = \frac{10}{3/5} = \boxed{\frac{50}{3}}$

	Referred to X, Y	Referred to x, y
Centre	C(0,0) X = 0, Y = 0	x = 0 | y = 0 x+1 = 0 | y-2 = 0 x = -1 | y = 2 ∴ Centre C(-1,2)
Foci	S(±ae, 0) S(±6, 0)	x = ±6 | y = 0 x+1 = ±6 | y-2 = 0 x = -1±6 | y = 2 ∴ Foci are S(5,2), S'(-7,2)
Vertices	A(±a, 0) A(±10, 0)	x = ±10 | y = 0 x+1 = ±10 | y-2 = 0 x = -1±10 | y = 2 ∴ vertices are A(9,2), A'(-11,2)
Direction	X = ± a/e X = ± 50/3	x+1 = ± 50/3 x = -1 ± 50/3 x = 49/3 or x = -51/3

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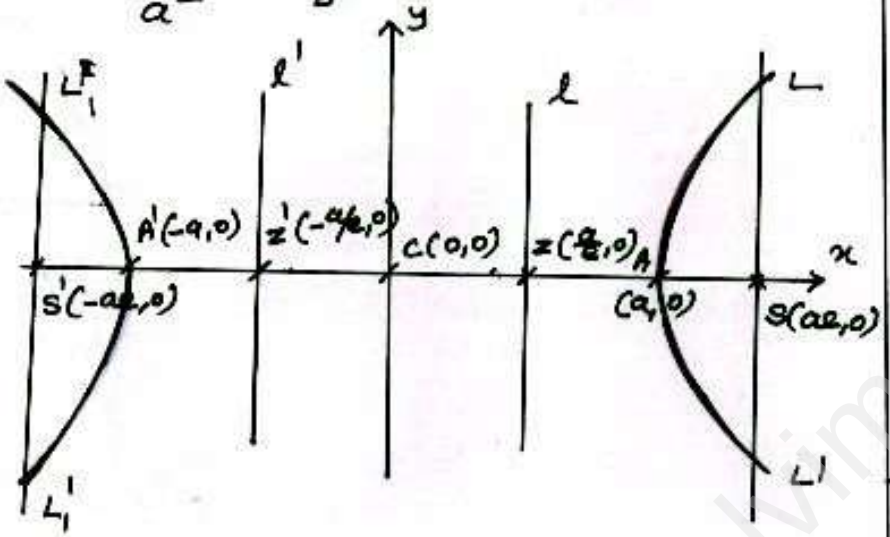
Hyperbola

Type I Equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.



1) Centre $C(0,0)$
2) Vertices are $A(a,0)$, $A'(-a,0)$
3) Foci are $S(ae,0)$ and $S'(-ae,0)$
4) Equation of Latus rectum
   $x = ae$, $x = -ae$
5) Eqn of directrices are $x = a/e$,
   $x = -a/e$.
6) Length of Latus rectum $= \frac{2b^2}{a}$
7) Eccentricity $e = \sqrt{1+\frac{b^2}{a^2}}$
8) $b^2 = a^2(e^2-1)$
9) x - axis is called Transverse axis and length of transverse axis is $AA' = 2a$.
10) y - axis is called conjugate axis and the length of conjugate axis is $BB' = 2b$.

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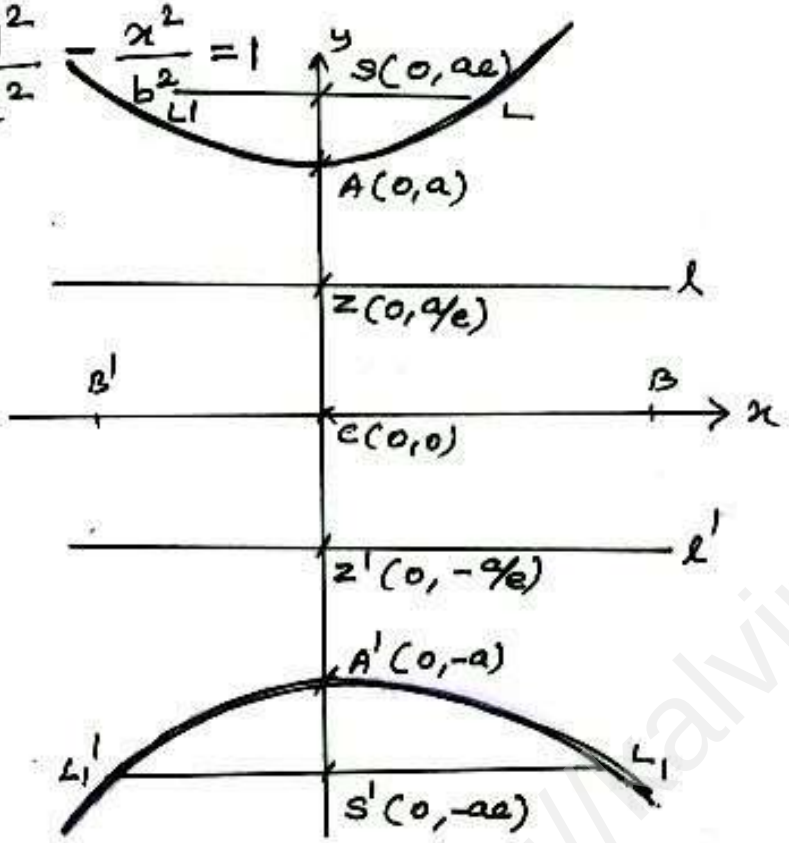
Type II Equation of hyperbola

$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$



1) Centre $C(0,0)$
2) Vertices are $A(0,a)$, $A'(0,-a)$
3) Foci are $S(0,ae)$, $S'(0,-ae)$
4) Eqn of latus rectum are $y = ae$, $y = -ae$.
5) Eqn of directrices are $y = a/e$, $y = -a/e$.
6) Length of latus rectum $= \frac{2b^2}{a}$
7) Eccentricity $e = \sqrt{1+\frac{b^2}{a^2}}$
8) $b^2 = a^2(e^2-1)$
9) y-axis is called Transverse axis and length of the transverse axis is $AA' = 2a$.
10) x-axis is called conjugate axis and length of conjugate axis is $BB' = 2b$.

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5.22 Find the eqn of hyperbola with vertices $(0, \pm 4)$ and foci $(0, \pm 6)$.

Given

vertices $(0, \pm 4) \Rightarrow A(0,4), A'(0,-4)$

foci $(0, \pm 6) \Rightarrow S(0,6)$ and $S'(0,-6)$

centre is mid pt of AA'

$$\text{i.e. } C = \left(\frac{0+0}{2}, \frac{4-4}{2}\right) = C(0,0)$$

From the data, the transverse axis is along y-axis.

Now $AA' = 2a$

$$\text{i.e. } 2a = \sqrt{(0-0)^2+(4+4)^2}$$

$$= \sqrt{8^2} = 8$$

$$\therefore \boxed{a = 4}$$

$SS' = 2ae$

$$\therefore 2ae = \sqrt{(0-0)^2+(6+6)^2}$$

$$= \sqrt{12^2} = 12$$

$$\therefore \boxed{ae = 6}$$

$$b^2 = a^2(e^2-1)$$

$$= a^2e^2 - a^2$$

$$= 36 - 16 = 20.$$

$\therefore$ The eqn of hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$\text{i.e. } \frac{y^2}{16} - \frac{x^2}{20} = 1.$$

Ex 5.23 Find the vertices, foci, for the hyperbola $9x^2 - 16y^2 = 144$

$$9x^2 - 16y^2 = 144$$

$$\frac{9x^2}{144} - \frac{16y^2}{144} = 1$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Here $a^2 = 16 \Rightarrow a = 4$ | $b^2 = 9$, $b = 3$

From the data, x-axis is transverse axis.

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eccentricity $e = \sqrt{1+\frac{b^2}{a^2}}$

$= \sqrt{1+\frac{9}{16}} = \sqrt{\frac{25}{16}}$

$= \boxed{\frac{5}{4}}$

Vertices $A(\pm a, 0)$

i $\boxed{A(4,0) \text{ and } (-4,0)}$

foci $= S(\pm ae, 0)$

$= S\left(\pm 4\left(\frac{5}{4}\right), 0\right)$

$= S(\pm 5, 0)$

i $\boxed{S(5,0) \text{ and } S'(-5,0)}$

Ex 5.24 Find the centre, foci, and eccentricity of the hyperbola

$11x^2 - 25y^2 - 44x + 50y - 256 = 0$

The eqn of hyperbola is

$11x^2 - 25y^2 - 44x + 50y - 256 = 0$

$11x^2 - 44x - 25y^2 + 50y = 256$

$11(x^2 - 4x) - 25(y^2 - 2y) = 256$

$11[x^2 - 4x + 4 - 4] - 25[y^2 - 2y + 1 - 1] = 256$

$11[(x-2)^2 - 4] - 25[(y-1)^2 - 1] = 256$

$11(x-2)^2 - 44 - 25(y-1)^2 + 25 = 256$

$11(x-2)^2 - 25(y-1)^2 = 256 + 44 - 25$

$11(x-2)^2 - 25(y-1)^2 = 275$

$\frac{11(x-2)^2}{275} - \frac{25(y-1)^2}{275} = 1$

$\frac{(x-2)^2}{\frac{275}{11}} - \frac{(y-1)^2}{11} = 1$

$\frac{X^2}{\frac{275}{11}} - \frac{Y^2}{11} = 1$ where $X = x-2$, $Y = y-1$

This is of the form

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Here $a^2 = \frac{275}{11} = 25$, $b^2 = 11$.

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Eccentricity $e = \sqrt{1+\frac{b^2}{a^2}}$

$= \sqrt{1+\frac{11}{25}} = \sqrt{\frac{36}{25}}$

$= \boxed{\frac{6}{5}}$

Now $ae = 5\left(\frac{6}{5}\right) = \boxed{6}$

centre	$C(0,0)$	$x=0 \quad y=0$ $x-2=0 \quad y-1=0$ $x=2 \quad y=1$ $\therefore$ centre $\boxed{C(2,1)}$
foci	$S(\pm ae,0)$ $S(\pm 6,0)$	$x=\pm 6 \mid y=0$ $x-2=\pm 6 \mid y-1=0$ $x=2\pm 6 \mid y=1$ $\therefore$ foci are $S(8,1),\ S'(-4,1)$
eccentricity	$e=\frac{6}{5}$	

3) find the eqn of the hyperbola

(i) foci $(\pm 2,0)$, eccentricity $= \frac{3}{2}$

Given foci $= S(\pm 2,0)$, $e = \frac{3}{2}$

i.e. $S(\pm ae,0) = S(\pm 2,0)$

$ae = 2$

$a\left(\frac{3}{2}\right) = 2 \Rightarrow a = \frac{4}{3}$

$b^2 = a^2(e^2-1)$

$= \frac{16}{9}\left(\frac{9}{4}-1\right)$

$= \frac{16}{9}\left(\frac{5}{4}\right) = \frac{20}{9}$

From the data, the transverse axis is x-axis

$\therefore$ The eqn of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

i.e. $\frac{x^2}{\frac{16}{9}} - \frac{y^2}{\frac{20}{9}} = 1 \Rightarrow \frac{9x^2}{16} - \frac{9y^2}{20} = 1$

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(11) centre (2,1), one of the foci (8,1) and corresponding directrix $x = 4$.

Given centre $C = C(2,1)$

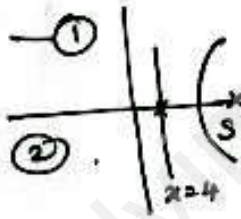
foci $= S(8,1)$

Distance betn centre and foci $= ae$

i.e. $\sqrt{(8-2)^2+(1-1)^2} = ae$

$\sqrt{6^2} = ae$

$\therefore \boxed{ae = 6}$ — ①



Now $\frac{a}{e} = 6-4 = 2$ — ②.

① × ② $\Rightarrow$ $ae \times \frac{a}{e} = 6 \times 2$

$\boxed{a^2 = 12}$

$\frac{①}{②} \Rightarrow \frac{ae}{a/e} = \frac{6}{2}$

$e^2 = 3 \Rightarrow \boxed{e = \sqrt{3}}$

Now $b^2 = a^2(e^2-1)$

$= 12(3-1) = \boxed{24}$

If the centre is (h,k) then the eqn of hyperbola is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Here $(h,k) = (2,1)$

$\therefore$ The eqn of hyperbola is

$\frac{(x-2)^2}{12} - \frac{(y-1)^2}{24} = 1.$

---

(111) passing through (5,-2), and length of transverse axis along x-axis and length 8 units.

Given that length of transverse axis is x-axis

$\therefore$ The eqn of hyperbola is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ — ①.

Also length of transverse axis

$2a = 8 \Rightarrow \boxed{a = 4}$

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$\therefore$ ① $\Rightarrow \frac{x^2}{16} - \frac{y^2}{b^2} = 1$

This eqn passes through $(5,-2)$

$$\frac{25}{16} - \frac{4}{b^2} = 1$$

$$\frac{25}{16} - 1 = \frac{4}{b^2}$$

$$\frac{9}{16} = \frac{4}{b^2} \Rightarrow b^2 = \frac{4 \times 16}{9}$$

$$= \boxed{\frac{64}{9}}$$

$\therefore$ The eqn of hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{\frac{64}{9}} = 1$$

(i) $\frac{x^2}{16} - \frac{9y^2}{64} = 1$

---

5 (iii) Identify the type of conic and find centre, foci, vertices and directrices of

$$\frac{x^2}{25} - \frac{y^2}{144} = 1$$

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Given eqn is $\frac{x^2}{25} - \frac{y^2}{144} = 1$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

with centre $C(0,0)$.

Here $a^2 = 25$ | $b^2 = 144$

$a = 5$ | $b = 12$

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{144}{25}} = \sqrt{\frac{169}{25}} = \boxed{\frac{13}{5}}$$

Also $ae = 5\left(\frac{13}{5}\right) = 13$.

centre is $C(0,0)$

foci : $(\pm ae, 0)$ (i) $(\pm 13, 0)$

vertices $(\pm 9, 0)$ (i) $(\pm 5, 0)$

Directrices $x = \pm a/e$

$$x = \pm \frac{5}{13/5} = \pm \frac{25}{13}$$

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(iv) $\frac{y^2}{16} - \frac{x^2}{9} = 1$

The given eqn represents an hyperbola with transverse axis along y-axis.

Here $a^2 = 16$ | $b^2 = 9$

$\therefore a = 4$ | $b = 3$

$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{16}} = \sqrt{\frac{25}{16}}$

$= \boxed{\frac{5}{4}}$

$ae = 4 \times \frac{5}{4} = \boxed{5}$

$\therefore$ centre is $\boxed{C(0,0)}$

foci $= (0, \pm ae)$

$= \boxed{(0, \pm 5)}$

Vertices $= (0, \pm a)$ i.e. $\boxed{(0, \pm 4)}$

Directrices $y = \pm a/e$

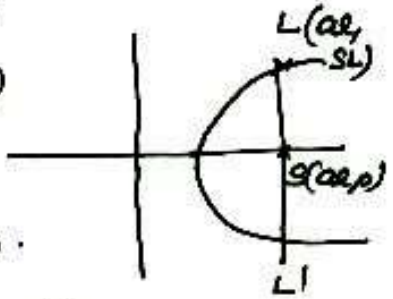
i.e. $y = \pm \frac{4}{5/4} = \boxed{\pm \frac{16}{5}}$

---

6) prove that the length of latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{2b^2}{a}$.

The eqn of hyperbola is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ——— (1)



L and L' be the end points of Latus rectum.

Then coordinates of L is L(ae, SL).

But L(ae, SL) lies on the hyperbola.

$\therefore$ (1) $\Rightarrow \frac{(ae)^2}{a^2} - \frac{(SL)^2}{b^2} = 1$

$e^2 - \frac{(SL)^2}{b^2} = 1$

$\frac{(SL)^2}{b^2} = e^2 - 1$

$SL^2 = b^2(e^2 - 1)$ ——— (2)

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WKT $b^2 = a^2(e^2-1)$

$\therefore e^2-1 = \frac{b^2}{a^2}$ — (3)

Sub (3) in (2)

$SL^2 = b^2\left(\frac{b^2}{a^2}\right)$

$= \frac{b^4}{a^2}$

$\therefore SL = \frac{b^2}{a}$

$\therefore LL' = 2(SL) = \frac{2b^2}{a}$

(i) Length of Latus rectum

$LL' = \frac{2b^2}{a}$.

(8) (iii) Identify the type of conic and find centre, foci, vertices and directrix of

$\frac{(x+3)^2}{225} - \frac{(y-4)^2}{64} = 1$

Given conic is

$\frac{(x+3)^2}{225} - \frac{(y-4)^2}{64} = 1$

$\frac{X^2}{225} - \frac{Y^2}{64} = 1$ where $X = x+3$, $Y = y-4$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Here the transverse axis is along X axis.

Here $a^2 = 225$ | $b^2 = 64$

$\therefore a = 15$ | $b = 8$

$\therefore e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{64}{225}}$

$= \sqrt{\frac{289}{225}} = \boxed{\frac{17}{15}}$

$ae = 15\left(\frac{17}{15}\right) = 17$

$a/e = \frac{15}{17/15} = \frac{225}{17}$

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(IV) $\frac{(y-2)^2}{25} - \frac{(x+1)^2}{16} = 1$

$\frac{Y^2}{25} - \frac{X^2}{16}$ where $X = x+1$, $Y = y-2$

This is of the form $\frac{Y^2}{a^2} - \frac{X^2}{b^2} = 1$

Here $a^2 = 25$ | $b^2 = 16$

$\boxed{a = 5}$ | $\boxed{b = 4}$

$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{16}{25}} = \sqrt{\frac{41}{25}} = \boxed{\frac{\sqrt{41}}{5}}$

$ae = 5\left(\frac{\sqrt{41}}{5}\right) = \boxed{\sqrt{41}}$

$a/e = \frac{5}{\frac{\sqrt{41}}{5}} = \boxed{\frac{25}{\sqrt{41}}}$

The transverse axis is along y-axis

	Referred to X, Y	Referred to x, y
Centre	C(0, 0)	$X = 0$ | $Y = 0$ $x+1 = 0$ | $y-2 = 0$ $x = -1$ | $y = 2$ ∴ centre is $\boxed{C(-1, 2)}$

	Referred to X, Y	Referred to x, y
Centre	C(0, 0)	$X = 0$ | $Y = 0$ $x+3 = 0$ | $y-4 = 0$ $x = -3$ | $y = 4$ ∴ centre is $\boxed{C(-3, 4)}$
Foci	$(\pm ae, 0)$ i.e. $(\pm 17, 0)$	$X = \pm 17$, | $Y = 0$ $x+3 = \pm 17$ | $y-4 = 0$ $x = -3 \pm 17$ | $y = 4$ ∴ Foci are $\boxed{(14, 4), (-20, 4)}$
Vertices	$(\pm a, 0)$ i.e. $(\pm 15, 0)$	$X = \pm 15$ | $Y = 0$ $x+3 = \pm 15$ | $y-4 = 0$ $x = -3 \pm 15$ | $y = 4$ ∴ Vertices are $\boxed{A(12, 4),\ A'(-18, 4)}$
Eqn of Directrix	$X = \pm \frac{a}{e}$ $X = \pm \frac{225}{17}$	$x+3 = \pm \frac{225}{17}$ $x = -3 \pm \frac{225}{17}$ $x = \frac{174}{17}, -\frac{276}{17}$

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(VI) $9x^2 - y^2 - 36x - 6y + 18 = 0$

$(9x^2 - 36x) - (y^2 + 6y) = -18$

$9[x^2 - 4x] - [y^2 + 6y] = -18$

$9[x^2 - 4x + 4 - 4] - [y^2 + 6y + 9 - 9] = -18$

$9[(x-2)^2 - 4] - [(y+3)^2 - 9] = -18$

$9(x-2)^2 - 36 - (y+3)^2 + 9 = -18$

$9(x-2)^2 - (y+3)^2 = -18 + 27$

$9(x-2)^2 - (y+3)^2 = 9$

$\frac{9(x-2)^2}{9} - \frac{(y+3)^2}{9} = 1$

$\frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1$

$\frac{X^2}{1} - \frac{Y^2}{9} = 1$ where $X = x-2$, $Y = y+3$

This is of the form $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$

The transverse axis is along X-axis.

	Referred to X, Y	Referred to x, y
Foci	$(0, \pm ae)$ i.e. $(0, \pm\sqrt{41})$	$x = 0$ | $y = \pm\sqrt{41}$; $x+1 = 0$ | $y-2 = \pm\sqrt{41}$; $x = -1$ | $y = 2 \pm \sqrt{41}$
		∴ Foci are $S(-1, 2+\sqrt{41})$, $S'(-1, 2-\sqrt{41})$
Vertices	$(0, \pm a)$ i.e. $(0, \pm 5)$	$x = 0$ | $y = \pm 5$; $x+1 = 0$ | $y-2 = \pm 5$; $x = -1$ | $y = 2 \pm 5$
		∴ Vertices are $(-1, 7)$ and $(-1, -3)$
Eqn of Directrix	$Y = \pm \frac{a}{e}$; $Y = \pm \frac{25}{\sqrt{41}}$	$y - 2 = \pm \frac{25}{\sqrt{41}}$; $y = 2 \pm \frac{25}{\sqrt{41}}$; $y = \frac{2\sqrt{41} + 25}{\sqrt{41}}$; $y = \frac{2\sqrt{41} - 25}{\sqrt{41}}$

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Here $a^2 = 1$ | $b^2 = 9$

$\therefore$ $\boxed{a = 1}$ | $\boxed{b = 3}$

$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{1}} = \boxed{\sqrt{10}}$

$ae = 1(\sqrt{10}) = \boxed{\sqrt{10}}$

$a/e = \boxed{\frac{1}{\sqrt{10}}}$

	Referred to x, y	Referred to x, y
centre	$C(0,0)$	$x = 0$ | $y = 0$ $x - 2 = 0$ | $y + 3 = 0$ $x = 2$ | $y = -3$ $\therefore$ centre is $\boxed{(2,-3)}$
foci	$(\pm ae, 0)$ i.e. $(\pm\sqrt{10}, 0)$	$x = \pm\sqrt{10}$ | $y = 0$ $x - 2 = \pm\sqrt{10}$ | $y + 3 = 0$ $x = 2 \pm \sqrt{10}$ | $y = -3$ Foci are $\boxed{S(2+\sqrt{10}, -3)}$ $\boxed{S'(2-\sqrt{10}, -3)}$
vertices	$(\pm a, 0)$ i.e. $(\pm 1, 0)$	$x = \pm 1$ | $y = 0$ $x - 2 = \pm 1$ | $y + 3 = 0$ $x = 2 \pm 1$ | $y = -3$ $\therefore$ vertices are $\boxed{A(3,-3),\ A'(1,-3)}$
Eqn of Directrix	$x = \pm a/e$ $x = \pm \frac{1}{\sqrt{10}}$	$x - 2 = \pm \frac{1}{\sqrt{10}}$ $x = 2 \pm \frac{1}{\sqrt{10}}$ $x = 2 + \frac{1}{\sqrt{10}}$, $x = 2 - \frac{1}{\sqrt{10}}$

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CONIC SECTIONS:-

Identifying the conics from the general eqn of the conic $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$.

when

1) $A=C=1$, $B=0$, $D=-2h$, $E=-2k$ $F=h^2+k^2-r^2$ the general eqn represents circle.

2) $B=0$ and either A or $C=0$ the general eqn represents parabola.

3) $A\neq C$ and A and C are same sign, the general eqn represents an ellipse.

4) $A\neq C$ and A and C are opposite sign, the general eqn represents a hyperbola.

5) $A=C$ and $B=D=E=F=0$ the general eqn represents a point.

6) $A=C=F$ and $B=D=E=0$ the general eqn yields an empty set $x^2+y^2+1=0$ and no real soln.

7) $A\neq 0$ or $C\neq 0$ and others are zeros, the general eqn represents coordinate axes.

8) $A=-C$ and rest are zero, the general eqn yields a pair of lines $x^2-y^2=0$.

Ex 5.26 Identify the type of conic

(i) $16y^2=-4x^2+64$

The given eqn is $4x^2+16y^2-64=0$.

Compare this eqn with

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$.

Here $A=4$, $B=0$, $C=16$, $D=0$, $E=0$, $F=-64$.

A and C are not equal.

A and C are same sign.

∴ The given eqn represents an ellipse.

(ii) $x^2+y^2=-4x-y+4$

The given eqn is $x^2+y^2+4x+y-4=0$

compare this with

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$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$.

$A=1,\ B=0,\ C=1,\ D=4,\ E=1,\ F=-4$

Here $A=C$

∴ The given eqn represents a circle.

(III) $x^2-2y=x+3$

$x^2-x-2y-3=0$

compare this eqn with

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$.

Here $A=1,\ B=0,\ C=0,\ D=-1,$

$E=-2,\ F=-3$.

Here $B=0$ and either A or $C=0$

the given eqn represents a parabola.

(IV) $4x^2-9y^2-16x+18y-29=0$.

Compare this with

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$.

Here $A=4,\ B=0,\ C=-9,\ D=-16$

$E=18,\ F=-29$

$A\neq C$ and A and C opposide sign.

∴ The given eqn represents a hyperbola.

EXERCISE 5.3

Identify the type of conic for

(1) $2x^2-y^2=7$.

Given eqn is $2x^2-y^2-7=0$

compare this with

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$.

Here $A=2,\ B=0,\ C=-1,\ D=0,\ E=0,\ F=-7$

Here $A\neq C$ and

A and C are opposite sign.

∴ The eqn represents a hyperbola.

---

2) $3x^2+3y^2-4x+3y+10=0$.

Compare this with

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

Here $A=3,\ B=0,\ C=3,\ D=-4,\ E=3,\ F=10$.

Here $A=C$ and same sign. $B=0$

∴ The eqn represents a circle.

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3) $3x^2+2y^2=14$.

The given eqn is $3x^2+2y^2-14=0$.

compare this eqn with

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

Here $A=3$, $B=0$, $C=2$, $D=0$, $E=0$

$F=-14$.

Here $A \neq C$ & A and C are of same sign.

∴ The given eqn represents an ellipse.

---

4) $x^2+y^2+x-y=0$.

compare this eqn with

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$.

Here $A=1$, $B=0$, $C=1$, $D=1$, $E=-1$

$F=0$.

Here $A=C=1$, $B=0$

∴ The given eqn represents a circle.

---

5) $y^2+4x+3y+4=0$.

compare this eqn with

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$.

Here $A=0$, $B=0$, $C=1$, $D=4$, $E=3$, $F=4$.

$B=0$, either A or $C=0$.

The given eqn represents a parabola.

---

6) $11x^2-25y^2-44x+50y-256=0$

compare this eqn with

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$.

Here $A=11$, $B=0$, $C=-25$, $D=-44$, $E=50$

$F=-256$.

$A \neq C$

A and C opposite sign.

∴ The given eqn represents a hyperbola.

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Conic	Parametric Equations	parameter	Range of parameter	Any point on the conic
Circle	$x = a\cos\alpha$ $y = a\sin\alpha$	α	$0 \le \alpha \le 2\pi$	α or $(a\cos\alpha, a\sin\alpha)$
parabola	$x = at^2$ $y = 2at$	t	$-\infty < t < \infty$	't' or $(at^2, 2at)$
Ellipse	$x = a\cos\alpha$ $y = b\sin\alpha$	α	$0 \le \alpha \le 2\pi$	'α' or $(a\cos\alpha, b\sin\alpha)$
Hyperbola	$x = a\sec\alpha$ $y = b\tan\alpha$	α	$-\pi \le \alpha \le \pi$ except $\alpha = \pm\frac{\pi}{2}$	'α' or $(a\sec\alpha, b\tan\alpha)$

Note:-

Eqn of tgt to the parabola $y^2 = 4ax$ at (x_1, y_1) in Cartesian form is $\boxed{yy_1 = 2a(x+x_1)}$

2) Eqn of tgt to the parabola $y^2 = 4ax$ in parametric form

$$\boxed{yt = x + at^2}$$

3) Eqn of the normal to the parabola $y^2 = 4ax$ at (x_1, y_1) in cartesian form is

$$\boxed{xy_1 + 2ay = x_1y_1 + 2ay_1}$$

4) Eqn of normal in parametric form

$$y + xt = at^3 + 2at$$

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Equations of tangent and normal to ellipse and Hyperbola.

1) Equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

a) at (x_1,y_1) $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ (cf)

b) at 'α' $\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1$ (pf)

2) Eqn of normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

a) at (x_1,y_1) is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2-b^2$ (cf)

b) at 'α' is $\frac{ax}{\cos\alpha}-\frac{by}{\sin\alpha}=a^2-b^2$ (pf)

cf - cartesian form

pf - parametric form.

3) Eqn of tgt to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

a) at (x_1,y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ (cf)

b) at 'α' is $\frac{x\sec\alpha}{a}-\frac{y\tan\alpha}{b}=1$ (pf)

4) Eqn of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

a) at (x_1,y_1) is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$ (cf)

b) at 'α' is $\frac{ax}{\sec\alpha}+\frac{by}{\tan\alpha}=a^2+b^2$ (pf)

Condition for the line $y=mx+c$ to be a tgt to the parabola $y^2=4ax$ is $c=\frac{a}{m}$

The point of contact is $\left(\frac{a}{m^2},\frac{2a}{m}\right)$ and eqn of tgt to the parabola is $y=mx+\frac{a}{m}$.

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(ii) The condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$c^2 = a^2m^2 + b^2$.

point of contact is $\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$

The eqn of tgt is $y = mx \pm \sqrt{a^2m^2 + b^2}$

(iii) The condition for $y = mx + c$ to be a tgt to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 - b^2$,

The point of contact is $\left(-\frac{a^2m}{c}, -\frac{b^2}{c}\right)$.

The eqn of tgt is $y = mx \pm \sqrt{a^2m^2 - b^2}$

Results:-

1) Two tangents can be drawn to (i) a parabola (ii) an ellipse (iii) a hyperbola. from any external point on the plane.

2) Four normals can be drawn to a) an ellipse b) a hyperbola, from an external point on the plane.

3) The locus of the point of intersection of perpendicular tgts to

a) the parabola $y^2 = 4ax$ is $x = -a$

b) the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$x^2 + y^2 = a^2 + b^2$

c) the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 - b^2$

Eqn of tgt to conic

The eqn of tgt at (x_1, y_1) is obtained from the eqn of the curve by replacing

x^2	by	xx_1	x by $\frac{1}{2}(x+x_1)$
y^2	by	yy_1	y by $\frac{1}{2}(y+y_1)$
xy	by	$\frac{1}{2}(xy_1 + yx_1)$	

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5.27 Find the eqns of tgt and normal to the parabola $x^2+6x+4y+5=0$ at $(1,-3)$

The eqn of the parabola is $x^2+6x+4y+5=0$.

The eqn of tgt at (x_1,y_1) is

$$xx_1+6\times\frac{1}{2}(x+x_1)+4\times\frac{1}{2}(y+y_1)+5=0$$

$$xx_1+3(x+x_1)+2(y+y_1)+5=0$$

∴ eqn of tgt at $(1,-3)$ is

$$x(1)+3(x+1)+2(y-3)+5=0$$

$$x+3x+3+2y-6+5=0$$

$$4x+2y+2=0.$$

$(\div 2)$ $\quad 2x+y+1=0.$

Eqn of normal is of the form

$$x-2y+k=0$$

It passes through $(1,-3)$

∴ $\quad 1-2(-3)+k=0$

$$1+6+k=0 \Rightarrow k=-7.$$

∴ Eqn of normal is

$$x-2y-7=0$$

Ex 5.28 Find the eqns of tgt and normal to the ellipse $x^2+4y^2=32$ when $\omega=\pi/4$.

Eqn of the ellipse is

$$x^2+4y^2=32$$

$$\frac{x^2}{32}+\frac{4y^2}{32}=1$$

$$\frac{x^2}{32}+\frac{y^2}{8}=1$$

compare this eqn with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

we get $a^2=32 \Rightarrow a=\sqrt{32}=4\sqrt{2}$

$b^2=8 \Rightarrow b=\sqrt{8}=2\sqrt{2}$

ω represents the point

$(a\cos\omega, b\sin\omega)$

i $(4\sqrt{2}\cos\omega, 2\sqrt{2}\sin\omega)_{\omega=\pi/4}$

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$(4\sqrt{2}\cos\frac{\pi}{4}, \ 2\sqrt{2}\sin\frac{\pi}{4})$

$(4\sqrt{2}\times\frac{1}{\sqrt{2}}, \ 2\sqrt{2}\times\frac{1}{\sqrt{2}})$

$(4, 2)$

Eqn of tgt at (x_1, y_1) is

$xx_1 + 4yy_1 = 32$

Here $(x_1, y_1) = (4, 2)$

$\therefore$ Eqn of tgt is $4x + 8y = 32$

$(\div 4)$ $x + 2y = 8$ or

$\boxed{x + 2y - 8 = 0}$

Also Eqn of normal is of the form. $2x - y + k = 0$

It passes through $(4, 2)$

$\therefore 2(4) - 2 + k = 0$

$8 - 2 + k = 0$

$\boxed{k = -6}$

$\therefore$ Eqn of normal is $\boxed{2x - y - 6 = 0}$

EXERCISE 5.4

1) Find the equations of the two tangents that can be drawn from $(5, 2)$ to the ellipse $2x^2 + 7y^2 = 14$

The eqn of the ellipse is

$2x^2 + 7y^2 = 14$

i.e. $\frac{2x^2}{14} + \frac{7y^2}{14} = 1$

$\frac{x^2}{7} + \frac{y^2}{2} = 1$ ——— (1).

$\frac{x^2}{(\sqrt{7})^2} + \frac{y^2}{(\sqrt{2})^2} = 1$ ——— (1).

Eqn of any tgt to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $y = mx \pm \sqrt{a^2m^2 + b^2}$

$\therefore$ Eqn of any tgt to the ellipse (1) is $y = mx \pm \sqrt{7m^2 + 2}$ — (2).

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The tgt passes through the point (5,2)

$\therefore \quad 2 = m(5) \pm \sqrt{7m^2+2}$

$2 = 5m \pm \sqrt{7m^2+2}$

$2-5m = \sqrt{7m^2+2}$

squaring $(2-5m)^2 = 7m^2+2$

$4+25m^2-20m-7m^2-2=0$

$18m^2-20m+2=0$

$(\div 2) \quad 9m^2-10m+1=0$

$(m-1)(9m-1)=0$

$\Rightarrow \quad m=1$ or $m=\frac{1}{9}$

9

$\frac{-9}{9} \quad \frac{-1}{9}$

$-1, -\frac{1}{9}$

when $m=1$, the eqn of tgt is

$y = x \pm \sqrt{9}$

$y = x \pm 3$

$y = x+3$ or $y = x-3$

$y = x+3$ does not pass through (5,2).

$\therefore \quad \boxed{y = x-3}$

$\therefore$ The required tgt is $y = x-3$

Sub $m = \frac{1}{9}$ in (2).

$y = \frac{1}{9}x \pm \sqrt{\frac{7}{81}+2}$

$y = \frac{x}{9} \pm \sqrt{\frac{169}{81}}$

$y = \frac{x}{9} \pm \frac{13}{9}$

$9y = x \pm 13$

$9y = x+13$ or $9y = x-13$

$9y = x-13$ does not passes through (5,2)

$\therefore \quad 9y = x+13$

$\therefore$ eqn of tgt is $\boxed{9y = x+13}$

Hence the required tgt drawn from the point (5,2) to the given ellipse are

$y = x-3$ and $9y = x+13$

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2) Find the eqns of tgts to the hyperbola $\frac{x^2}{16} - \frac{y^2}{64} = 1$ which are parallel to $10x - 3y + 9 = 0$.

Eqn of hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{64} = 1 \quad \text{——} \quad (1)$$

Here $a^2 = 16 \Rightarrow a = 4$

$b^2 = 64 \Rightarrow b = 8$

Eqn of any tgt to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$y = mx \pm \sqrt{a^2m^2 - b^2} \quad \text{——} \quad (2)$$

Here m is slope of tgt.

Given that

Slope of tgt m = Slope of the line $10x - 3y + 9 = 0$

$\therefore m = \frac{-10}{-3} = \boxed{\frac{10}{3}}$

Sub $m = \frac{10}{3}$ in (2).

$$y = \frac{10}{3}x \pm \sqrt{16 \times \frac{100}{9} - 64}$$

$$y = \frac{10}{3}x \pm \sqrt{\frac{1600 - 576}{9}}$$

$$y = \frac{10}{3}x \pm \sqrt{\frac{1024}{9}}$$

$$y = \frac{10}{3}x \pm \frac{32}{3}$$

$$3y = 10x \pm 32$$

$\therefore$ The tgts are $10x - 3y + 32 = 0$ and $10x - 3y - 32 = 0$.

---

3) Show that the line $x - y + 4 = 0$ is a tgt to the ellipse $x^2 + 3y^2 = 12$. Also find the coordinates of the point of contact.

Given line is $x - y + 4 = 0$

i.e. $y = x + 4$

compare this eqn with $y = mx + c$

$\therefore m = 1, \; c = 4$.

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The eqn of the given ellipse is $x^2+3y^2=12$

$$\frac{x^2}{12}+\frac{3y^2}{12}=1$$

$$\frac{x^2}{12}+\frac{y^2}{4}=1$$

Here $a^2=12 \Rightarrow a=\sqrt{4\times3}=2\sqrt{3}$

$b^2=4 \Rightarrow b=2$.

The condition for the line $y=mx+c$ to be the tgt to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$c^2=a^2m^2+b^2$$

Here $c^2=4^2=16$ —— ①.

$a^2m^2+b^2=12(1)+4$

$=16$ —— ②.

from ① and ② $c^2=a^2m^2+b^2$

∴ The given line $x-y+4=0$ is a tgt to the ellipse $x^2+3y^2=12$.

The point of contact is $\left(-\frac{a^2m^2}{c}, \frac{b^2}{c}\right)$

$=\left(-\frac{12(1)}{4}, \frac{4}{4}\right)=\boxed{(-3,1)}$

---

4) Find the eqn of the tgt to the parabola $y^2=16x$ perpendicular to $2x+2y+3=0$.

Eqn of the parabola is $y^2=16x$.

This is of the form $y^2=4ax$.

∴ $4a=16 \Rightarrow \boxed{a=4}$

Given that the tgt is $\perp_r$ to the line $2x+2y+3=0$.

Slope of the line $2x+2y+3=0$ is $-\frac{a}{b}=-\frac{2}{2}=\boxed{-1}$

∴ Slope of tgt $=-\frac{1}{-1}=\boxed{1}$

Eqn of any tgt to the parabola is of the form $y=mx+\frac{a}{m}$

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$y = 1 \cdot x + \frac{4}{1}$

$y = x + 4$ or

$\boxed{x - y + 4 = 0}$

---

5. Find the eqn of tgt to the parabola at $t = 2$

Eqn of the parabola $y^2 = 8x$

compare this with $y^2 = 4ax$

$\therefore 4a = 8 \Rightarrow \boxed{a = 2}$

t represents the pt $(at^2, 2at)_{t=2}$

i.e. $[2(4), 2(2)(2)]$

$(8, 8)$.

Eqn of tgt at (x_1, y_1) is

$yy_1 = 8 \times \frac{1}{2}(x + x_1)$

$y(8) = 4(x + 8)$

$8y = 4x + 32$

$4x - 8y + 32 = 0$

$(\div 4)$ $\boxed{x - 2y + 8 = 0}$

---

6) Find the eqns of the tgt and normal to hyperbola $12x^2 - 9y^2 = 108$ at $\theta = \pi/3$

Eqn of the hyperbola is $12x^2 - 9y^2 = 108$

$\frac{12x^2}{108} - \frac{9y^2}{108} = 1$

$\frac{x^2}{9} - \frac{y^2}{12} = 1$

compare this with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Here $a^2 = 9 \Rightarrow a = 3$

$b^2 = 12 \Rightarrow b = 2\sqrt{3}$

θ represents the pt $(a\sec\theta, b\tan\theta)_{\theta = \pi/3}$

i.e. $[3\sec\pi/3, 2\sqrt{3}\tan\pi/3]$

$[3(2), 2\sqrt{3}(\sqrt{3})]$

$(6, 6)$.

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∴ Eqn of tgt at (x_1, y_1) is

$12xx_1 - 9yy_1 = 108$

Here $x_1, y_1 = (6,6)$

∴ eqn of tgt is $12x(6) - 9y(6) = 108$

$72x - 44y = 108$

$(\div 6)$ $12x - 9y = 18$

$(\div 3)$ $\boxed{4x - 3y - 6 = 0}$

Eqn of normal is of the form

$3x + 4y + k = 0$

It passes through the pt (6,6)

$3(6) + 4(6) + k = 0$

$18 + 24 + k = 0$

$k = -42$

∴ eqn of normal is

$\boxed{3x + 4y - 42 = 0}$

7) prove that the point of interseation of the tgts at t_1 and t_2 on the parabola $y^2 = 4ax$ is $[at_1t_2, a(t_1+t_2)]$

Eqn of the parabola is $y^2 = 4ax$

Eqn of the tgt to the parabola at t_1 is $yt_1 = x + at_1^2$ —— ①

Eqn of tgt at t_2 is $yt_2 = x + at_2^2$ —— ②.

① − ② ⇒ $y(t_1 - t_2) = a(t_1^2 - t_2^2)$

$y = \dfrac{a(t_1+t_2)(t_1-t_2)}{t_1 - t_2}$

$y = a(t_1 + t_2)$ —— ③.

Sub ③ in ①.

$at_1(t_1+t_2) = x + at_1^2$

∴ $x = at_1(t_1+t_2) - at_1^2$

$x = at_1^2 + at_1t_2 - at_1^2$

$x = at_1t_2$ —— ④

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∴ The pt of interseection is
$(at_1t_2, a(t_1+t_2))$

---

8) If the normal at the point t_1 on the parabola $y^2 = 4ax$ meets the parabola again at the point t_2 then prove that $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

Eqn of normal to the parabola $y^2 = 4ax$ at t is $y + xt = 2at + at^3$

∴ The eqn of normal at t_1
is $y + xt_1 = 2at_1 + at_1^3$ —— (1)

The normal meets the parabola at t_2 is $(at_2^2, 2at_2)$.

$\Rightarrow 2at_2 + at_2^2 t_1 = 2at_1 + at_1^3$

$2at_2 - 2at_1 = at_1^3 - at_2^2 t_1$

$2a(t_2 - t_1) = at_1(t_1^2 - t_2^2)$

$2(t_2 - t_1) = t_1(t_1 + t_2)(t_1 - t_2)$

$-2(t_1 - t_2) = t_1(t_1 + t_2)(t_1 - t_2)$

$-2 = t_1(t_1 + t_2)$

$\therefore t_1 + t_2 = -\frac{2}{t_1}$

$t_2 = -t_1 - \frac{2}{t_1}$

$t_2 = -\left(t_1 + \frac{2}{t_1}\right)$

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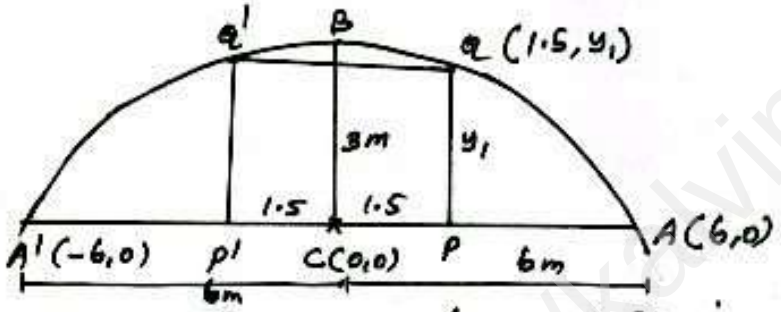
Real life Applications of conic :-

Ex 5.30:- A semi elliptical arch-way over a one-way road has a height of 3m and a width of 12m. The truck has a width of 3m and a height of 2.7m. will the truck clear the opening of the archway?



An arch is in the form of semi ellipse

∴ Its eqn is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ — ①

The horizantal width is $AA' = 12m$

∴ The vertices are $A(6,0)$ and $A'(-6,0)$

The mid point of AA' is $C(0,0)$

The height of the arch at the centre is 3m

∴ clearly $CA = 6m$, $CB = 3m$

∴ The eqn of ellipse is

$$\frac{x^2}{6^2} + \frac{y^2}{3^2} = 1 \quad \text{——} \quad (2)$$

To find the height PQ, which is 1.5m left or right of the centre.

Take $PQ = y_1$

clearly the point $P(1.5, y_1)$ is on the ellipse.

∴ Substitute $x = 1.5$, $y = y_1$ in (2)

$$\frac{(3/2)^2}{36} + \frac{y_1^2}{9} = 1$$

$$\frac{y_1^2}{9} = 1 - \frac{9}{4 \times 36}$$

$$\frac{y_1^2}{9} = 1 - \frac{1}{16}$$

$$\frac{y_1^2}{9} = \frac{15}{16}$$

$$y_1^2 = \frac{15 \times 9}{16}$$

$$y_1 = \frac{\sqrt{135}}{4} = \frac{11.62}{4}$$

$$= 2.90$$

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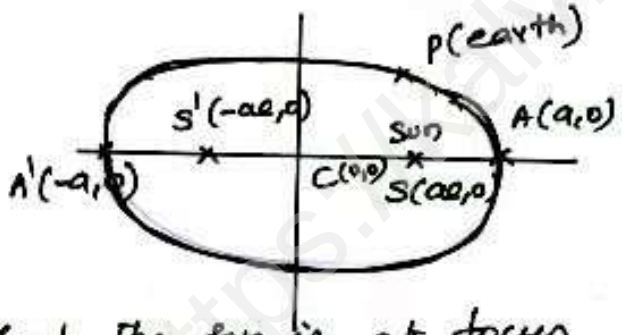
∴ The height of the arch was 1.5 m from the centre is 2.9 m.
The height of truck is 2.7 m
∴ The truck will clear the arch way.

5.31 The maximum and minimum distances of the earth from the sun respectively are 152×10^6 km and 94.5×10^6 km. The sun is at one focus of the elliptical orbit find the distance from the sun to the other focus.



Given that the sun is at focus of the elliptical orbit.

XII/CH5/5.5/P2

Let P be the position of the earth at any point.

The Maximum distance from the sun and the earth is $SA' = 152 \times 10^6$

i.e. $SC + CA' = 152 \times 10^6$

$ae + a = 152 \times 10^6$ — (1)

The minimum distance from the sun and the earth is $SA = 94.5 \times 10^6$

i.e. $CA - CS = 94.5 \times 10^6$

$a - ae = 94.5 \times 10^6$ — (2).

(1) − (2) $\Rightarrow$ $2ae = 57.5 \times 10^6$

$= 575 \times 10^5$

∴ The distance betn the sun and the other focus S' is

$SS' = 2ae = 575 \times 10^5$ km.

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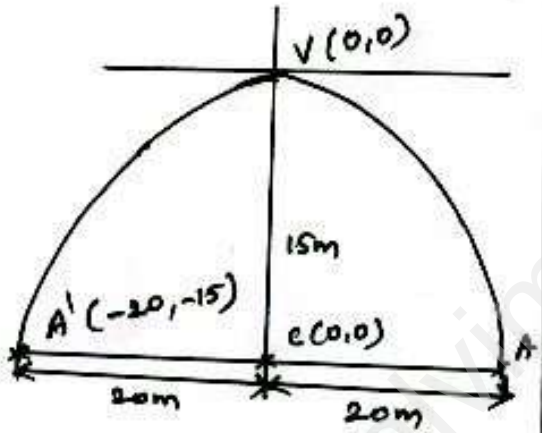
5.32 A concrete bridge is designed as a parabolic arch. The road over bridge is 40m. long and the max. height of the arch is 15m. write the eqn of the parabolic arch.



Consider the parabolic arch to be open downward.

The eqn of the parabola is

$$x^2 = -4ay \quad \text{——} \quad (1)$$

Given that the road over the bridge is 40m.

i.e $AA' = 40m$.

The vertex $V(0,0)$ is 15m above the road. A and A' are clearly the points $A(20,-15)$ and $A'(-20,-15)$

The point $A(20,-15)$ lies on the parabola (1)

$\therefore$ $20^2 = -4a(-15)$

$400 = 60a \Rightarrow a = \frac{400}{60} = \frac{20}{3}$

$a = \frac{20}{3}$

$\therefore$ The eqn of parabola is

$$x^2 = -4\left(\frac{20}{3}\right)y$$

i.e $\boxed{3x^2 = -80y}$

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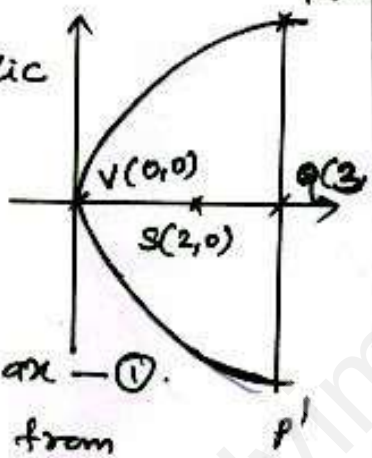
5.33 The parabolic communication antenna has a focus at 2m distance from the vertex of the antenna. Find the width of the antenna 3m from the vertex.



Consider the parabolic communication antenna tobe open Rightward.

∴ The eqn of parabola is $y^2 = 4ax$ — ①.

The focus is 2m from the vertex. ∴ $a = 2$

i.e. focus is $S(2,0)$.

Let P be any point on the parabola, whose x – coordinate is 3m from the vertex. Take $PQ = y$

∴ The point P is $P(3,y)$

The point $P(3,y)$ lies on the parabola.

∴ ① $\Rightarrow y^2 = 4a(3)$

$y^2 = 12a$ But $a = 2$

∴ $y^2 = 12(2) = 24$

$y^2 = 4 \times 2 \times 3$

$y = 2\sqrt{6}$

∴ The width of the antena $PP' = 2PQ = 2(2\sqrt{6})$

$= \boxed{4\sqrt{6}\,m}$

Ex 5.34 The eqn $y = \frac{1}{32}x^2$ model cross section of parabolic mirrors that are used for solar energy. There is a heating tube located at the focus of each parabola, how high is this tobe located above the vertex of the parabola.

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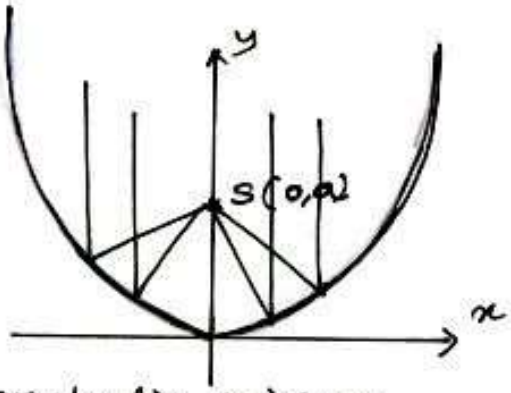


Consider the parabolic mirror tobe open upward.

The eqn of the parabola is

$x^2 = 4ay$ ——— ①.

But the given parabola is

$y = \frac{1}{32}x^2$

$\therefore x^2 = 32y$ ——— ②.

compare ① and ②

$4a = 32 \Rightarrow \boxed{a = 8}$

The heating tobe is located at the focus S(0,a) ie S(0,8)

$\therefore$ The heating tobe is placed at S(0,8).

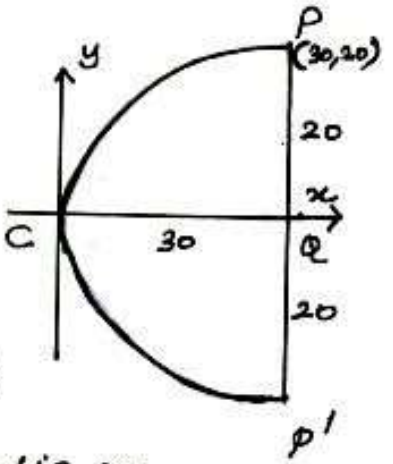
ie 8 unit above the vertex.

X11/CH5/5.5/P5

5.34 A search light has a parabolic reflector (has a cross section that forms a bowl). The parabolic bowl is 40cm wide from rim to rim and 30cm deep. The bulb is located at the focus.

1) what is the eqn of the parabola used for reflector?

2) How far from the vertex is the bulb tobe placed so that the maximum distance covered?



Consider the given parabolic reflector tobe open rightward.

$\therefore$ The eqn of parabola is $y^2 = 4ax$ ——— ①

The width of the reflector $PP' = 40cm$ and depth $CQ = 30cm$.

$\therefore$ The point P(30,20) lie on the parabola

$\therefore$ ① $\Rightarrow 20^2 = 4a(30)$

$\frac{40\cancel{0}}{12\cancel{0}} = a \Rightarrow \boxed{a = \frac{10}{3}}$

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q) ∴ The eqn of parabola is

$$y^2 = 4\left(\frac{10}{3}\right)x$$

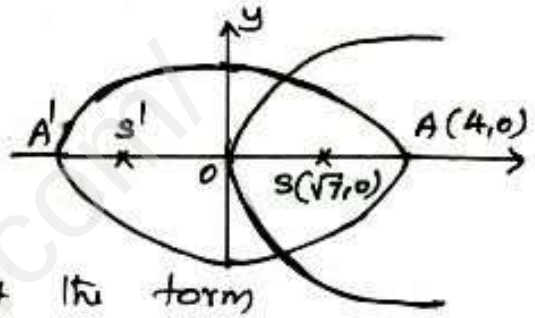
i.e. $\boxed{y^2 = \frac{40}{3}x}$

a) The focus is at S(a,0)

∴ The bulb has to be placed at a distance of $\frac{10}{3}$ cm from the vertex.

Ex 5.36 An equation of the elliptical part of an optical lens system is $\frac{x^2}{16} + \frac{y^2}{9} = 1$. The parabolic part of the system has a focus in common with the right focus of the ellipse. The vertex of the parabola is at the origin and the parabola opens to the right. Determine the eqn of the parabola.

Given that the eqn of the elliptical part of an optical lens is $\frac{x^2}{16} + \frac{y^2}{9} = 1$



This is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Here $a^2 = 16 \Rightarrow a = 4$

$b^2 = 9 \Rightarrow b = 3$

$$\therefore e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{16}} = \sqrt{\frac{7}{16}}$$

$$= \boxed{\frac{\sqrt{7}}{4}}$$

∴ The foci F(ae,0) and F'(-ae,0) are $F\left(4 \times \frac{\sqrt{7}}{4}, 0\right)$ and $F'\left(-\frac{4\sqrt{7}}{4}, 0\right)$

i.e. F(√7,0) and F'(-√7,0)

Given, The focus of the parabola is same as right focus of ellipse.

i.e. S(a,0) = S(√7,0)

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$\therefore a = \sqrt{7}$

Hence the eqn of parabola is

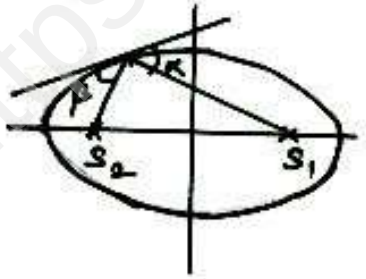
$y^2 = 4ax$

i.e. $\boxed{y^2 = 4\sqrt{7}x}$

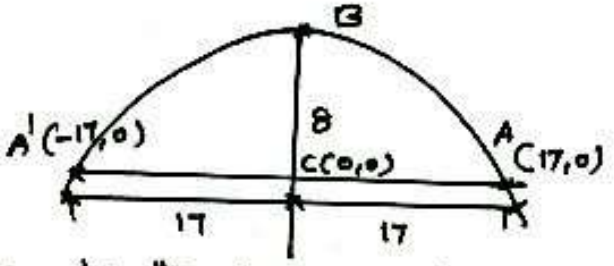
Note:- Reflective property of an ellipse

The line from the foci to a point on an ellipse make equal angles with the tangent line at the point.

The light or sound or radio waves emitted from one focus hits any point P on the ellipse is received at the other focus.



5.37 A room 34m long is Constructed tobe whispering gallery. The room has an elliptical ceiling as shown in fig 5.64. If the maximum height of the ceiling is 8m, determine where the foci are located.



The ceiling in the room is in the shape of semi ellipse.
The width of the ceiling is 34m.
The Height of the ceiling is 8m.

Here $a = 17\,m$, $b = 8\,m$.

$\therefore e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{64}{289}}$

$= \sqrt{\frac{225}{289}} = \boxed{\frac{15}{17}}$

$\therefore ae = 17 \times \frac{15}{17} = \boxed{15\,m}$

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∴ The foci are located 15 m right or left of the centre.

Ex 5.38:- If the eqn of ellipse is $\frac{(x-11)^2}{484} + \frac{y^2}{64} = 1$ (x and y are measured in cm) where to the nearest centimeter, should the patient's kidney stone be placed so that the reflected sound hits the kidney stone?

The eqn of the ellipse is

$$\frac{(x-11)^2}{484} + \frac{y^2}{64} = 1$$

This is in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Here $a^2 = 484 \Rightarrow a = 22$

$b^2 = 64 \Rightarrow b = 8$

$$\therefore e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{64}{484}}$$

$$= \sqrt{\frac{420}{484}}$$

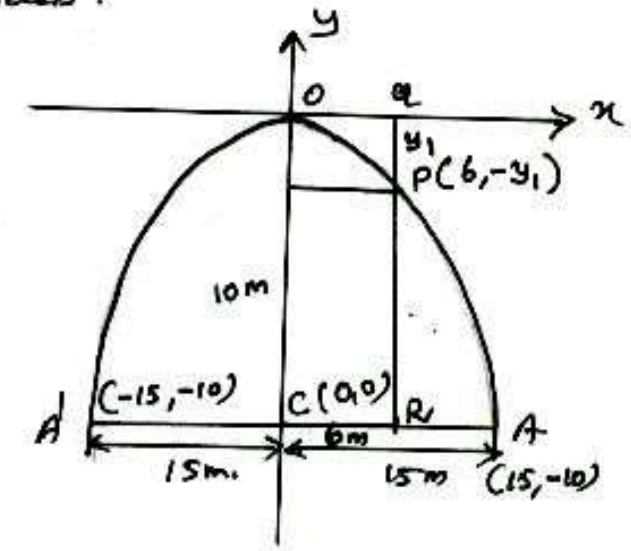
$$\therefore ae = 22 \times \frac{\sqrt{420}}{22} = \sqrt{420}$$

$$= 20.5$$

∴ The kidney stone should be placed at 20.5 cm from the centre of the ellipse.

EXERCISE 5.5

1. A bridge has a parabolic arch that is 10m high in the centre and 30m wide at the bottom. Find the height of the arch 6m the centre, on either sides.



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Given that the bridge is parabolic arch.

∴ The eqn of parabola is

$x^2 = -4ay$ —— ①

The height of the ceiling at the centre is 10m and the width is 30m.

The point A and A' are

A (15, -10) and A' (-15, -10)

clearly the pt A(15, -10) is on the parabola.

∴ sub $x = 15$, $y = -10$ in ①

$225 = -4a(-10)$

$\therefore a = \frac{225}{40}$

∴ The eqn of the parabola is

$x^2 = -4\left(\frac{225}{40}\right)y$

$= -\frac{225}{10}y$

$= -\frac{45}{2}y$ —— ②.

PR is the required height of the arch, which is 6m from the centre on either side.

Take $PQ = y_1$

∴ The point $P(6, y_1)$ lies on the parabola

sub $x = 6$, $y = y_1$ in ③

$36 = -\frac{45}{2}y_1$

$\therefore y_1 = \frac{36 \times 2}{-45} = \frac{4 \times 2}{-5} = \frac{8}{-5}$

$= 1.6$.

∴ The required height $PR = QR - PQ$

$= 10m - 1.6$

$= 8.4m$

2) A tunnel through a mountain for a four lane highway is to have a elliptical opening. The total width of the highway is to be 16m and the height at the edge of the road must be sufficient for a truk 4m high to clear if the highest

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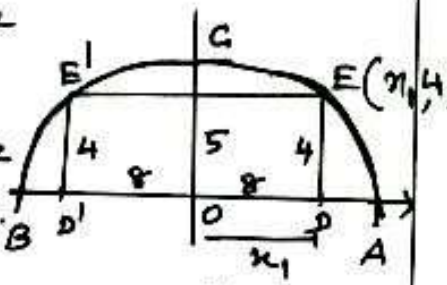
point of the opening is tobe 5m approximately. How wide must the opening be

The tunnel is in the shape of semi ellipse.



∴ The eqn of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ — ①.

Width of the highway AB = 16m.

Highest point of the opening OC = 5m.

DE is end of the opening at which the truck of 4m high has to clear.

To find width of the opening DD'

Here DE = 4m = D'E'.

Take OD = x.

clearly the point E(x1,4) lie on the ellipse.

∴ The eqn of ellipse is

$\frac{x^2}{8^2}+\frac{y^2}{5^2}=1$

$\Rightarrow \frac{x^2}{64}+\frac{y^2}{25}=1$ — ②.

Sub $x=x_1$, $y=4$ in ②

$\frac{x_1^2}{64}+\frac{16}{25}=1$

$\frac{x_1^2}{64}=1-\frac{16}{25}$

$\frac{x_1^2}{64}=\frac{9}{25} \Rightarrow x_1^2=\frac{9}{25}\times 64$

$\therefore x_1=\frac{3\times 8}{5}=\frac{24}{5}$

$x_1 = 4.8$

i.e. $OD = 4.8$ m

$\therefore DD' = 2(OD)$

$= 2(4.8) = 9.6$ m

3) At a water fountain, water attains a maximum height of 4m at horizontal distance of 0.5m from its origin. If the path of water is parabola, find the height of water at a horizontal distance

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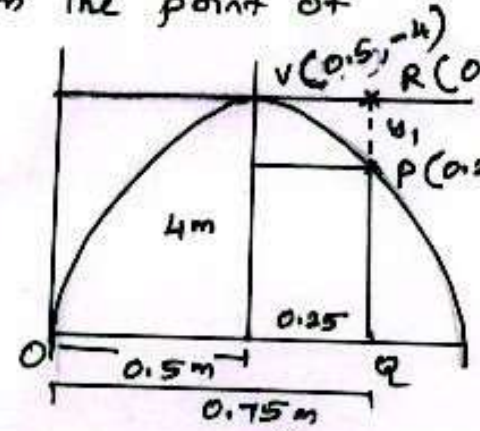
of 0.75m from the point of origin.



Consider the parabolic flow of water to be open downward.

∴ The eqn of parabola is

$$x^2 = -4ay \quad \text{——} (1)$$

From the fig, the vertex is V(0.5, -4)

∴ The parabola is

$$(0.5)^2 = -4a(-4)$$

$$0.25 = +16a$$

$$\Rightarrow a = \frac{0.25}{+16} = \frac{1}{4\times16} = \frac{1}{64}$$

∴ The eqn of parabola is

$$x^2 = -4\left(\frac{1}{64}\right)y$$

i.e. $$x^2 = -\frac{y}{16}$$

PQ is required height of water which is 2.5m from the centre.

Take PR = y_1

The point P(0.25, $-y_1$) lies on the parabola

∴ The eqn of parabola is

$$(0.25)^2 = -\frac{(-y_1)}{16}$$

$$\left(\frac{1}{4}\right)^2 = \frac{y_1}{16} \Rightarrow \frac{1}{16}\times16 = y_1$$

$$\therefore \boxed{y_1 = 1}$$

∴ PQ = QR − PR

= 4 − 1 = 3m

∴ Height of water fountain at a distance 0.75m is 3m.

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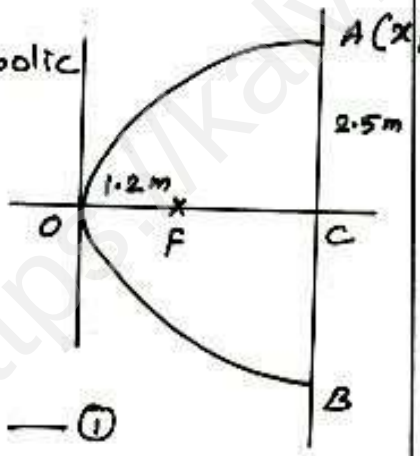
4) An engineer disigns a satellite dish with a parabolic cross section. The dish is 5m wide at the opening and the focus is placed 1.2m from the vertex

a) position a coordinates system with the origin at the vertex and the x-axis on the parabola's axis of symmetry and find an equation of the parabola.

b) find the depth of the satellite dish at the vertex.

Consider the parabolic cross section tobe open right ward.



∴ The eqn of parabola is

$y^2 = 4ax$ — ①

focus is placed 1.2m from the vertex.

∴ $a = 1.2m$.

∴ The eqn of parabola is

$y^2 = 4(x)(1.2)$

$y^2 = 4.8x$ ——— ②.

The width of the dish AC = 2.5m

Let the depth of the dish OC = x_1

clearly the point $A(x_1, 2.5)$ lies on the parabola

put $x = x_1$, $y = 2.5$ in ②

$(2.5)^2 = 4.8(x_1)$

$\therefore x_1 = \frac{6.25}{4.8} = \frac{625}{480}$

$x_1 = 1.302m$.

∴ Required depth of the dish = 1.3m.

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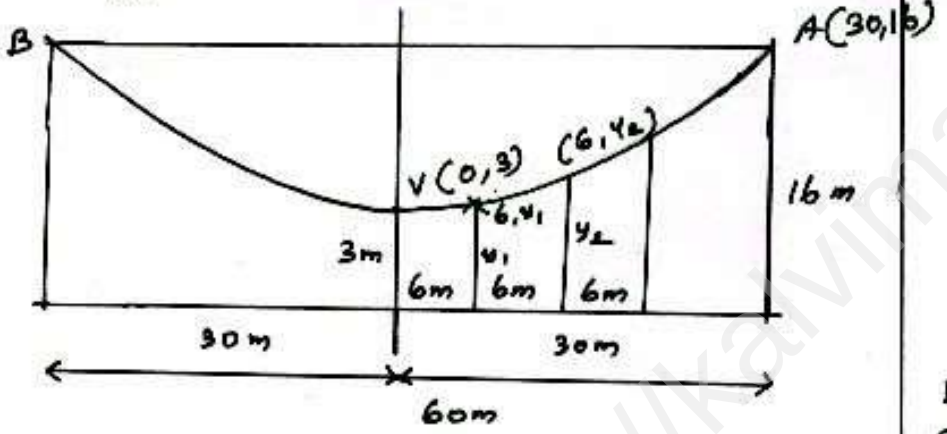
5) Parabolic cable of a 60m portion of the road bed of a suspension bridge are positioned as shown below. Vertical cables are tobe spaced every 6m along this portion of the roadbed. calculate the length of first two of these vertical cables from the verdex.



From the fig, the vertex is at V(0,3) above the x-axis

∴ The parabola is open upward

∴ The eqn of parabola is

$$x^2 = 4ay \quad \text{——} \quad (1)$$

The vertex is V(0,3)

∴ The eqn of parabola is

$$(x-0)^2 = 4a(y-3) \quad \text{——} \quad (2)$$

clearly the point A(30, 16) lies on the parabola

put $x = 30$, $y = 16$ in (2)

$$30^2 = 4a(16-3)$$

$$900 = 4a(13)$$

$$\frac{900}{52} = a$$

∴ The eqn of parabola is

$$x^2 = 4\left(\frac{900}{52}\right)(y-3) \quad \text{——} \quad (3).$$

Let y_1 be the length of the first cable.

∴ The pt $(6, y_1)$ lie on the parabola

$$36 = \frac{900}{13}(y_1 - 3)$$

$$36 \times \frac{13}{900} = y_1 - 3$$

$$225$$

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$$y_1 - 3 = \frac{117}{225}$$

$$y_1 = 3 + \frac{117}{225} = \frac{675+117}{225}$$

$$y_1 = \frac{792}{225} = 3.52\text{ m}.$$

∴ Height of the 1st vertical cable is 3.52m.

Let the height of the 2nd vertical cable from the vertex is $(12, y_2)$.

$$\therefore \quad 12^2 = \frac{900}{13}(y_2 - 3)$$

$$\frac{144 \times 13}{900} = y_2 - 3$$

$$\frac{468}{225} = y_2 - 3 \Rightarrow y_3 = 3 + \frac{468}{225}$$

$$= \frac{675+468}{225}$$

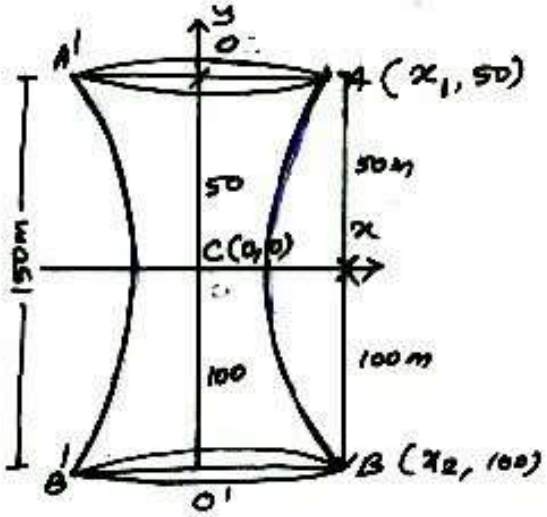
$$= \frac{1143}{225} = 5.08\text{m}$$

∴ Height of the second vertical cable is 5.08 m.

---

6) Cross section of a Nuclear cooling tower is in the shape of a hyperbola with eqn $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$

The tower is 150m tall and the distance from the top of the tower to the centre of the hyperbola is half the distance form the base of the tower to the centre of the hyperbola. Find the diameter of the top and base of the tower?



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The eqn of hyperbola is

$$\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1 \quad \text{——} \textcircled{1}.$$

Height of the tower AB = 150m.

Also given OC = 50, O'C = 100m.

Take OA = x_1

clearly the point A(x_1, 50) lies on the hyperbola.

$\therefore$ sub $x = x_1$, $y = 50$ in ①

$$\frac{x_1^2}{30^2} - \frac{50^2}{44^2} = 1$$

$$\frac{x_1^2}{30^2} = 1 + \frac{2500}{1936} = \frac{4436}{1936}$$

$$x_1^2 = \frac{4436}{1936} \times 30 \times 30$$

$$x_1 = \sqrt{\frac{4436 \times 30^2}{1936}}$$

$$x_1 = \frac{30}{44}\sqrt{4436}$$

$$x_1 = \frac{\cancel{30}^{15}}{\cancel{44}_{22}} \times \overset{33.3}{\cancel{6666}} = \frac{499.5}{11}$$

$$= 45.41$$

$\therefore$ Diameter of the top $= 2x_1$

$$= 2(45.41)$$

$$= \boxed{90.82\text{m}}$$

Take O'B = x_2

clearly the point B(x_2, 100) lies in the hyperbola.

$\therefore$ sub $x = x_2$, $y = 100$ in ①

$$\frac{x_2^2}{30^2} - \frac{100^2}{44^2} = 1$$

$$\therefore \frac{x_2^2}{30^2} = 1 + \frac{100^2}{44^2} = \frac{44^2 + 100^2}{44^2}$$

$$\frac{x_2}{30} = \sqrt{\frac{44^2 + 100^2}{44^2}} = \sqrt{\frac{1936 + 10000}{1936}}$$

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$\therefore x_2 = 30\sqrt{\frac{11936}{1936}}$

$x_2 = \frac{15}{\cancel{90}\ 44\ 22} \times 109.25 = \frac{1638.75}{22}$

$= 74.49$

$\therefore$ Diameter of the bottom $BB' = 2x_2$

$= 2(74.49)$

$= \boxed{148.98\ m}$

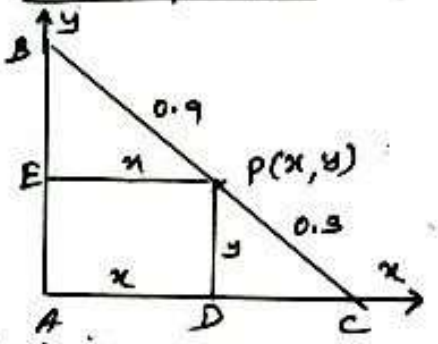
$\therefore$ Diameter of the top $= 90.82$ m

Diameter of the bottom $= 148.98$ m

---

7) A rod of length 1.2m moves with its ends always touching the coordinate axes. The locus of a point p on the rod, which is 0.3m from the end in contact with x-axes is an ellipse. Find the excentricity.

XII | CH5 | 5.5 | P16



Let BC be the rod of length 1.2m with it end always touching the axes.

Let P(x,y) be any point on the rod which is 0.3m from the point C on the x-axis.

$CP = 0.3 \quad \therefore PB = 0.9 \quad (\because BC = 1.2m)$

ΔPED and ΔPDC are similar.

$\therefore \frac{BE}{PD} = \frac{EP}{DC} = \frac{PB}{PC}$

$\frac{BE}{y} = \frac{x}{DC} = \frac{0.9}{0.3}$

$\frac{BE}{y} = \frac{0.9}{0.3} = 3$

$\therefore BE = 3y$

$\therefore AB = AE + BE$

$= y + 3y$

$= 4y$

$\frac{x}{DC} = \frac{0.9}{0.3} = 3$

$DC = \frac{x}{3}$

$\therefore AC = AD + DC$

$= x + \frac{x}{3} = \frac{4x}{3}$

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In ΔABC, $AB^2 + AC^2 = BC^2$

$$(Ay)^2 + \left(\frac{4}{3}x\right)^2 = 1.2^2$$

$$16y^2 + \frac{16x^2}{9} = 1.44$$

$$\frac{x^2}{\frac{9}{16}} + \frac{y^2}{\frac{1}{16}} = 1.44$$

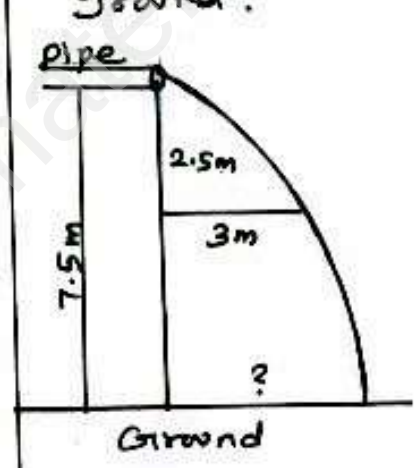
$$\frac{x^2}{\frac{9}{16}\times 1.44} + \frac{y^2}{\frac{1}{16}\times 1.44} = 1$$

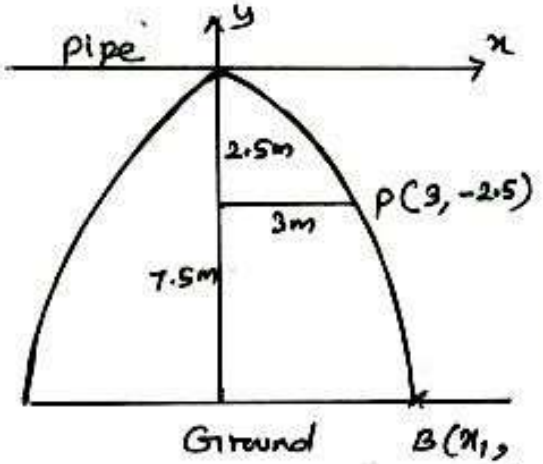
$$\therefore e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{1.44\times\frac{1}{16}}{1.44\times\frac{9}{16}}}$$

$$= \sqrt{1 - \frac{1}{9}} = \sqrt{\frac{8}{9}} = \sqrt{\frac{4\times 2}{9}}$$

$$\therefore \frac{2\sqrt{2}}{3}$$ 4

8) Assume that water is issuing from the end of horizonal pipe, 7.5m above the ground, describes parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position 2.5m below the line of the pipe, the flow of water has curved outward 3m beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?





Consider the parabolic flow of water to be opendownward.

∴ The eqn of parabola is

$$x^2 = -4ay \quad \text{——— (1)}$$

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From the given data,

The point p is on the flow of water, which is 2.5m below the line of the pipe and 3m beyond the vertical line.

∴ The point $P(3,-2.5)$ lies on the parabola

∴ Sub $x=3$, $y=-2.5$ in (1)

$$9 = -4a(-2.5)$$

$$9 = 10a \Rightarrow \boxed{a = \frac{9}{10}}$$

∴ The eqn of parabola is

$$x^2 = -4\left(\frac{9}{10}\right)y$$

$$x^2 = -\frac{18}{5}y \quad \text{——— (2)}$$

Let B be the point at which the flow of water strikes the ground.

Let x_1 be the distance betn the vertical line and the point B.

∴ The point $B(x_1, -7.5)$ lies on the parabola

put $x = x_1$, $y = -7.5$ in (2).

$$x_1^2 = -\frac{18}{5}(-7.5)$$

$$x_1^2 = 18 \times 1.5 = 27$$

∴ $x_1 = \sqrt{27} = \sqrt{9 \times 3}$

$$= 3\sqrt{3}$$

Hence the water strikes the ground $3\sqrt{3}$ m beyond the vertical line.

---

9) on lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of 4m when it is 6m away from the point of projection. Finally it reaches the ground 12m away from the starting point. Find the angle of projection.

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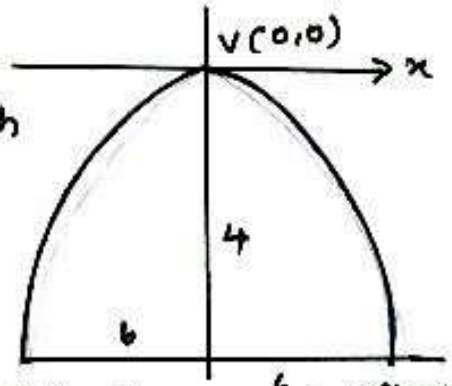

Consider the parabolic path to be open down ward.

∴ The eqn of parabola is

$x^2 = -4ay$ —— ①

From the data, the point A(6,-4) lies on the parabola

∴ $36 = -4a(-4)$

$\frac{36}{16} = a \Rightarrow a = \frac{9}{4}$

∴ The eqn of parabola is $x^2 = -4 \times \frac{9}{4} y$

$x^2 = -9y$ —— ②.

Now to find the slope at the point of projection A'(-6,-4)

To get the slope Diff ② w.r to x

$2x = -9 \frac{dy}{dx}$

∴ $\frac{dy}{dx} = \frac{2x}{-9}$

Slope at (-6,-4) $= \frac{dy}{dx}(-6,-4)$

$= \frac{2}{-9}(-6) = \frac{12}{9} = \frac{4}{3}$

∴ Slope $\frac{dy}{dx}$ = tan α = $\frac{4}{3}$

$\Rightarrow \alpha = \tan^{-1}\left(\frac{4}{3}\right)$

∴ The angle of projection is $\tan^{-1}\left(\frac{4}{3}\right)$.

---

(10) points A and B are 10km apart and it is determined from the sound of an explosion heard at these points at different times that the location of the explosion is 6km closer to A and than B. Show that the location of the explosion

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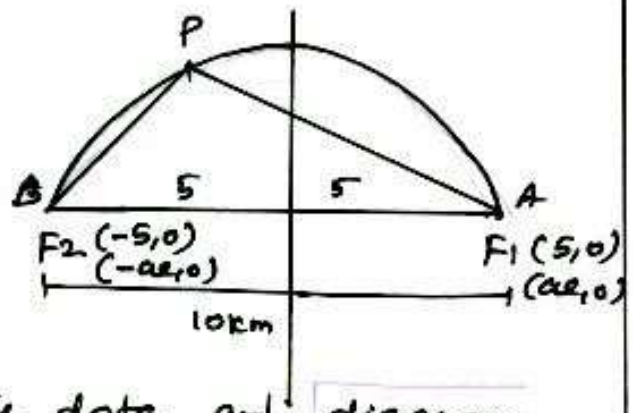
is restricted to a particular curve and find an equation of it.



From the data, and diagram

$AB = 10$ km.

Also, $F_1P - F_2P = 6$ km i.e. $2a = 6$

$\boxed{a = 3}$

clearly F_1 and F_2 are foci

and $AB = 2ae$ or $F_1F_2 = 2ae$.

$\therefore 2ae = 10$

$ae = 5$

Wkt, $b^2 = a^2e^2 - a^2$

$= 25 - 9$

$= 16$

The eqn of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\therefore$ The required eqn of ellipse is

$$\frac{x^2}{3^2} + \frac{y^2}{16} = 1$$

i.e. $$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

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Chapter 6 – Applications of Vector Algebra

* Exercise are hyperlink with page number. To get into particular exercise, please click exercise number

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CHAPTER – 6

Applications of Vector Algebra

Scalar product and vector product

Defn:- Given two vectors $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ the scalar product (or) dot product is denoted by $\bar{a}\cdot\bar{b}$ and is calculated by

$$\bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3$$

and vector product is denoted by

$$\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Note: $\bar{a}\cdot\bar{b}$ is scalar and $\bar{a}\times\bar{b}$ is vector.

1) An angle betn two vectors $\bar{a}$ and $\bar{b}$ is $\theta = \cos^{-1}\left(\frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}\right)$

2) $\bar{a}$ and $\bar{b}$ are said to be parallel if the angle betn them is 0 or π

3) $\bar{a}$ and $\bar{b}$ are said to be perpendicular if the angle betn them is $\pi/2$ or $\frac{3\pi}{2}$.

Property:-

1) Let $\bar{a}$ and $\bar{b}$ be any two vectors Then $\bar{a}\cdot\bar{b} = 0$ iff $\bar{a}$ and $\bar{b}$ are perpendicular to each other.

2) $\bar{a}\times\bar{b} = 0$ iff $\bar{a}$ and $\bar{b}$ are parallel to each other.

3) $\bar{a}, \bar{b}, \bar{c}$ are any three vectors then

a) $\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}$

b) $(\bar{a}+\bar{b})\cdot\bar{c} = \bar{a}\cdot\bar{c} + \bar{b}\cdot\bar{c}$

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CHAPTER-6

Applications of Vector Algebra

scalar product and vector product.

Defn:- Given two vectors $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ the scalar product (or) dot product is denoted by $\bar{a}\cdot\bar{b}$ and is calculated by

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a) $\bar{a}\cdot\vec{b} = \bar{b}\cdot\bar{a}$

b) $(\bar{a}+\bar{b})\cdot\vec{c} = \bar{a}\cdot\bar{c} + \bar{b}\cdot\vec{c}$

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c) $(\alpha\vec{a})\cdot\vec{b} = \alpha(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(\alpha\vec{b})$

d) $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

e) $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c} + \vec{b}\times\vec{c}$

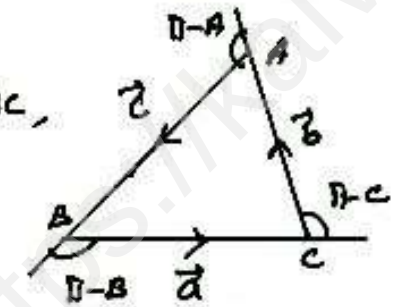
f) $(\alpha\vec{a})\times\vec{b} = \alpha(\vec{a}\times\vec{b})$
$= \vec{a}\times(\alpha\vec{b})$

Ex 6.2 With usual notations, in any triangle ABC, prove by vector method.

(i) $a = b\cos C + c\cos B$.

In any triangle ABC,

$\vec{BC} = \vec{a}$
$\vec{CA} = \vec{b}$
$\vec{AB} = \vec{c}$



Then $|\vec{BC}| = a$
$|\vec{CA}| = b$
$|\vec{AB}| = c$

Now $\vec{BC} + \vec{CA} + \vec{AB} = 0$

$\vec{BC} = -(\vec{CA} + \vec{AB})$

$\vec{BC}\cdot\vec{BC} = -(\vec{CA}+\vec{AB})\cdot\vec{BC}$ (or)

$= -\vec{CA}\cdot\vec{BC} - \vec{AB}\cdot\vec{BC}$

$= -|\vec{CA}||\vec{BC}|\cos(\pi - C) - |\vec{AB}||\vec{BC}|\cos(\pi - B)$

$a^2 = ba\cos C + ca\cos B$

$\therefore a^2 = ab\cos C + ac\cos B$

$a = b\cos C + c\cos B$

Illy we can prove that.

$b = c\cos A + a\cos C$ and

$c = a\cos B + b\cos A$.

Ex 6.1 With usual notations in any triangle ABC, prove that

(i) $a^2 = b^2 + c^2 - 2bc\cos A$.

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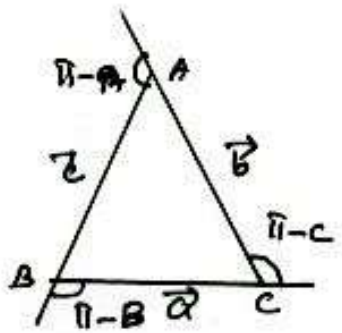
In any triangle ABC,



$\vec{BC} = \vec{a}$

$\vec{CA} = \vec{b}$

$\vec{AB} = \vec{c}$

$|\vec{BC}| = a, |\vec{CA}| = b$

$|\vec{AB}| = c$.

and $\vec{BC} + \vec{CA} + \vec{AB} = 0$

$\vec{BC} = -\vec{CA} - \vec{AB}$

$\vec{BC} \cdot \vec{BC} = -(\vec{CA} + \vec{AB}) \cdot [-(\vec{CA} + \vec{AB})]$

$= (\vec{CA} + \vec{AB})^2$

$= |\vec{CA}|^2 + |\vec{AB}|^2 + 2\vec{CA} \cdot \vec{AB}$

$= b^2 + c^2 + 2\cos(\pi - A)$

(i) $a^2 = b^2 + c^2 - 2\cos A$.

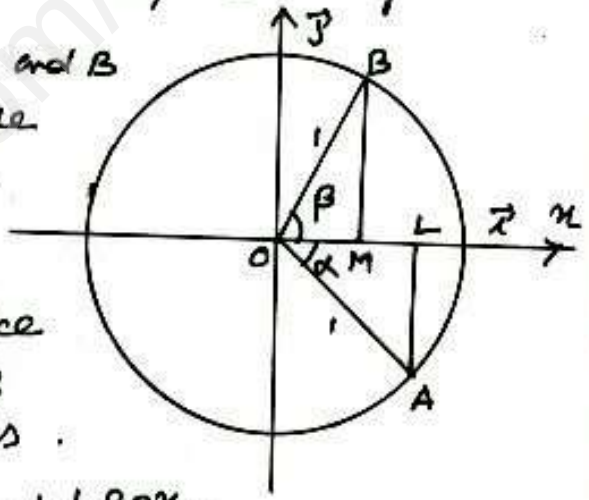
(ii) we can prove that

$b^2 = c^2 + a^2 - 2\cos B$

$c^2 = a^2 + b^2 - 2\cos C$

6.3 By vector method, prove that $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Take a point A and B on the unit circle with centre O.



Assume that OA and OB make angles α and β with x-axis.

$\angle AOB = \angle AOx + \angle BOx$.

$= \alpha + \beta$

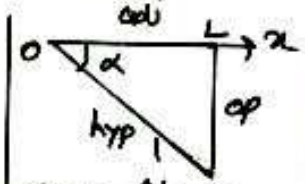
clearly the coordinates of A and B are A $(\cos\alpha, \sin\alpha)$ and B $(\cos\beta, \sin\beta)$

Take the unit vector $\vec{i}$ and $\vec{j}$ along x and y axis respectively.

From the diagram

$\vec{OA} = \vec{OL} + \vec{LA}$

$= \cos\alpha\,\vec{i} + \sin\alpha(-\vec{j})$



$\sin\alpha = \frac{AL}{OA}$

$AL = \sin\alpha$

$OL = \cos\alpha$

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$\cos\alpha\,\vec{i} - \sin\alpha\,\vec{j}$

$\overline{OB} = \overline{OM} + \overline{MB}$

$= \cos\beta\,\vec{i} + \sin\beta\,\vec{j}$

$\overline{OA}\cdot\overline{OB} = (\cos\alpha\,\vec{i} - \sin\alpha\,\vec{j})(\cos\beta\,\vec{i} + \sin\beta\,\vec{j})$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$ — (1)

But By defn

$\overline{OA}\cdot\overline{OB} = |\overline{OA}|\,|\overline{OB}|\cos(\alpha+\beta)$

$= \cos(\alpha+\beta)$ — (2)

From (1) and (2)

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

6.4 with usual notations, in any triangle ABC, prove by vector method that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

XII | CH6 | 6.1 | P4

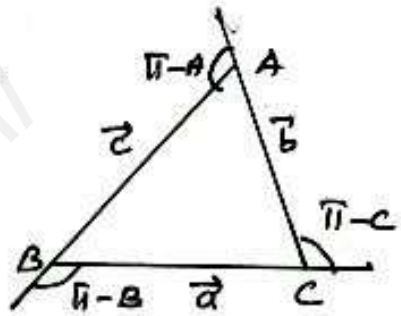
In any triangle ABC

$\overline{BC} = \vec{a}$, $\overline{CA} = \vec{b}$

$\overline{AB} = \vec{c}$.

$|\overline{BC}| = a$, $|\overline{CA}| = b$

$|\overline{AB}| = c$.



and $\overline{BC} + \overline{CA} + \overline{AB} = \overline{0}$.

$\overline{BC} \times (\overline{BC} + \overline{CA} + \overline{AB}) = \overline{0}$

$\overline{BC} \times \overline{BC} + \overline{BC} \times \overline{CA} + \overline{BC} \times \overline{AB} = \overline{0}$.

$\overline{BC} \times \overline{CA} = -\overline{BC} \times \overline{AB}$

$\overline{BC} \times \overline{CA} = \overline{AB} \times \overline{BC}$ — (1).

Illy $\overline{CA} \times (\overline{BC} + \overline{CA} + \overline{AB}) = \overline{0}$.

$\overline{CA} \times \overline{BC} + \overline{CA} \times \overline{CA} + \overline{CA} \times \overline{AB} = \overline{0}$.

$\overline{CA} \times \overline{AB} = -\overline{CA} \times \overline{BC}$

$\overline{CA} \times \overline{AB} = \overline{BC} \times \overline{CA}$ — (2).

From (1) and (2).

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$\bar{AB} \times \bar{BC} = \bar{BC} \times \bar{CA} = \bar{CA} \times \bar{AB}$

So, $|\bar{AB} \times \bar{BC}| = |\bar{BC} \times \bar{CA}| = |\bar{CA} \times \bar{AB}|$

$|\bar{AB}||\bar{BC}| \cos(\pi - B) = |\bar{BC}||\bar{CA}| \cos(\pi - C)$

$= |\bar{CA}||\bar{AB}| \cos(\pi - A)$

$ca \cos B = ab \cos C = bc \cos A$

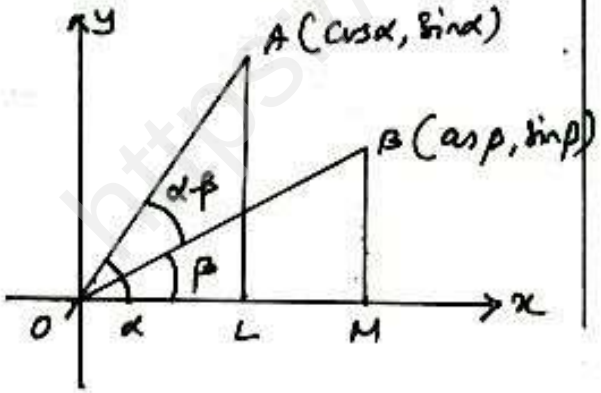
Dividing by abc, we get.

$$\frac{ac \cos B}{abc} = \frac{ab \cos C}{abc} = \frac{bc \cos A}{abc}$$

$$\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}.$$

6.5 Prove by vector method

$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$



Take the point A and B on the unit-circle with centre O.

OA and OB make angles α and β with x-axis.

The coordinates of A and B are A $(\cos\alpha, \sin\alpha)$ and B $(\cos\beta, \sin\beta)$

Take the unit vector $\bar{i}$ and $\bar{j}$ along x and y-axis respectively.

$\bar{OA} = \vec{OL} + \bar{LA}$

$= \cos\alpha\, \bar{i} + \sin\alpha\, \bar{j}$

$\bar{OB} = \bar{OM} + \bar{MB}$

$= \cos\beta\, \bar{i} + \sin\beta\, \bar{j}$

By defn:

$\bar{OB} \times \bar{OA} = |\bar{OB}||\bar{OA}| \sin(\alpha - \beta)\, \bar{k}$

$= \sin(\alpha - \beta)\, \bar{k}$ ——— (1)

$$\bar{OB} \times \bar{OA} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$= \bar{k}(\sin\alpha \cos\beta - \cos\alpha \sin\beta)$ ——— (2)

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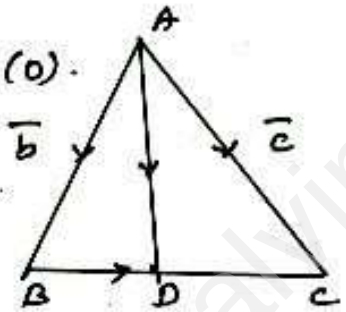
From ① and ②

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Ex 6.6 (Apollonius's Theorem)

If D is mid point of the side BC of a triangle ABC, show by vector method that

$|\bar{AB}|^2 + |\bar{AC}|^2 = 2\left(|\bar{AD}|^2 + |\bar{BD}|^2\right)$



Let A be the origin (O).

The position vector of B is $\bar{b}$.

The position vector of C is $\bar{c}$.

D is mid point of BC.

$\therefore$ position vector of D is $\frac{\bar{b}+\bar{c}}{2}$

Now $|\bar{AD}|^2 = \bar{AD}\cdot\bar{AD}$

$= \left(\frac{\bar{b}+\bar{c}}{2}\right)\cdot\left(\frac{\bar{b}+\bar{c}}{2}\right)$

$= \frac{1}{4}(\bar{b}+\bar{c})^2$

$= \frac{1}{4}\left[|\bar{b}|^2 + |\bar{c}|^2 + 2\bar{b}\cdot\bar{c}\right]$ ——— ①

$|\bar{BD}|^2 = \bar{BD}\cdot\bar{BD}$

$= \left(\frac{\bar{c}-\bar{b}}{2}\right)\cdot\left(\frac{\bar{c}-\bar{b}}{2}\right)$

$= \frac{1}{4}(\bar{c}-\bar{b})^2$

$= \frac{1}{4}\left[|\bar{c}|^2 + |\bar{b}|^2 - 2\bar{b}\bar{c}\right]$

But

$\bar{BD} = \bar{AD} - \bar{AB}$

$= \frac{\bar{b}+\bar{c}}{2} - \bar{b}$

$= \frac{\bar{b}+\bar{c}-2\bar{b}}{2}$

$= \frac{\bar{c}-\bar{b}}{2}$.

↳ ②.

① + ② ⇒ $|\bar{AD}|^2 + |\bar{BD}|^2$

$= \frac{1}{4}\left[|\bar{b}|^2 + |\bar{c}|^2 + 2\bar{b}\cdot\bar{c} + |\bar{c}|^2 + |\bar{b}|^2 - 2\bar{b}\bar{c}\right]$

$= \frac{1}{4}\left[2\left(|\bar{b}|^2 + |\bar{c}|^2\right)\right]$

$= \frac{1}{2}\left(|\bar{b}|^2 + |\bar{c}|^2\right)$

$2\left[|\bar{AD}|^2 + |\bar{BD}|^2\right] = |\bar{b}|^2 + |\bar{c}|^2$

$= |\bar{AB}|^2 + |\bar{AC}|^2$

——— × ———

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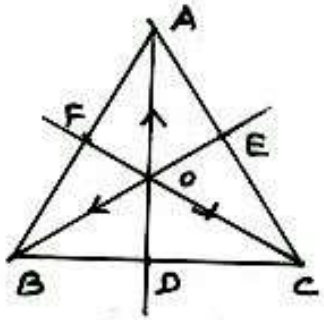
Ex 6.7 prove by vector method that the perpendiculars (altitudes) from the vertices to the opposite sides of a triangle are concurrent.



Let ABC be any triangle.
Let AD and BE be its altitude intersecting at O.
Let O be the origin.
The position vector of A, B, C are $\bar{a}$, $\bar{b}$ and $\bar{c}$ respectively.
To prove altitudes of a triangle are concurrent.
It is sufficient to prove that CO is ⊥r to AB.

Now $\overline{OA} = \bar{a}$, $\overline{OB} = \bar{b}$, $\overline{OC} = \bar{c}$

$\overline{AD}$ is ⊥r to $\overline{BC}$

i.e. $\overline{OA}$ is ⊥r to $\overline{BC}$

$\Rightarrow \overline{OA} \cdot \overline{BC} = 0$

$\overline{OA} \cdot (\overline{OC} - \overline{OB}) = 0$

$\overline{OA} \cdot \overline{OC} - \overline{OA} \cdot \overline{OB} = 0$.

$\bar{a} \cdot \bar{c} - \bar{a} \cdot \bar{b} = 0$ —— (1)

$\overline{BE}$ is ⊥r $\overline{CA}$ (2)

i.e. $\overline{BE}$ ⊥r $\overline{CA} \Rightarrow \overline{OB} \cdot \overline{CA} = 0$

i.e. $\bar{b} \cdot (\overline{OA} - \overline{OC}) = 0$

$\bar{b} \cdot (\bar{a} - \bar{c}) = 0$

$\bar{b} \cdot \bar{a} - \bar{b} \cdot \bar{c} = 0$ — (2)

(1) + (2) $\Rightarrow$ $\bar{a} \cdot \bar{c} - \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{b} - \bar{b} \cdot \bar{c} = 0$

$\bar{a} \cdot \bar{c} - \bar{b} \cdot \bar{c} = 0$

$\bar{c} \cdot (\bar{a} - \bar{b}) = 0$

$\overline{OC} \cdot (\overline{OA} - \overline{OB}) = 0$

$\overline{OC} \cdot (\overline{BA}) = 0$

$\Rightarrow \overline{OC}$ ⊥r $\overline{BA}$.

$\Rightarrow \overline{CF}$ is ⊥r to $\overline{AB}$.

Hence three altitudes are concurrent.

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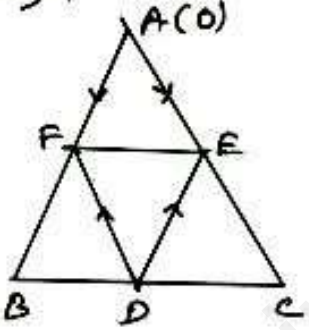
6.8 In triangle ABC, the points D, E, F are the mid points of the sides BC, CA, and AB respectively. Using vector method, show that the area of ΔDEF is equal to $\frac{1}{4}$ (area of ABC).



In ΔABC, D, E, F are mid point of BC, CA and AB respectively.

Consider A as the origin.

$\therefore$ position vector of D is $\frac{\bar{AB}+\bar{AC}}{2}$

position vector of E $= \frac{\bar{AC}}{2}$

position vector of F $= \frac{\bar{AB}}{2}$.

Now Area of $\Delta ABC = \frac{1}{2}\left|\bar{AB}\times\bar{AC}\right|$

Area of $\Delta DEF = \frac{1}{2}\left|\bar{DE}\times\bar{DF}\right|$

$$= \frac{1}{2}\left|\frac{1}{2}(\bar{BA})\times\frac{1}{2}(\bar{CA})\right|$$

$$= \frac{1}{2}\left|-\frac{1}{2}(\bar{AB})\times-\frac{1}{2}(\bar{AC})\right|$$

$$= \frac{1}{2}\left|\frac{1}{4}(\bar{AB}\times\bar{AC})\right|$$

$$= \frac{1}{4}\left|\frac{1}{2}(\bar{AB}\times\bar{AC})\right|$$

$$= \frac{1}{4}\text{ Area of ABC.}$$

Applications of dot and cross product

1) Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

Then magnitude of $\bar{a}$ is

$$|\bar{a}| = \sqrt{a_1^2+a_2^2+a_3^2}.$$

2) If α, β, γ are the angles made by a vector $\vec{a}$ with positive x, y, z are respectively, Then $\cos\alpha, \cos\beta, \cos\gamma$ are known as the direction cosine of $\bar{a}$.

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$\therefore \cos\alpha = \frac{\bar{a}_1}{|\bar{a}|}$, $\cos\beta = \frac{\bar{a}_2}{|\bar{a}|}$

$\cos\gamma = \frac{\bar{a}_3}{|\bar{a}|}$.

Note that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

3) The unit vector in the direction of $\bar{a}$ is

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{a_1}{|\bar{a}|}\bar{i} + \frac{a_2}{|\bar{a}|}\bar{j} + \frac{a_3}{|\bar{a}|}\bar{k}$$

Multiplication of Vectors:-

Consider two vectors $\bar{a}$ and $\bar{b}$ let α be the angle betn two vectors $\bar{a}$ and $\bar{b}$. Then the scalar product $\bar{a}\cdot\bar{b}$ is defined as $\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\alpha$

$$\boxed{\bar{a}\cdot\bar{b} = ab\cos\alpha}$$

properties of Scalar product.

1) $\bar{a}\cdot\bar{b}$ is scalar quantity and hence it can be positive (α is acute) or negative (α is obtuse)

2) $\bar{a}\cdot\bar{b} = 0 \Rightarrow \bar{a} = 0$ or $\bar{b} = 0$ or $\bar{a} \perp_r \bar{b}$.

3) $\alpha = \cos^{-1}\left[\frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}\right]$

4) $\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}$ (commutative property)

5) $\bar{a}\cdot(\bar{b}+\bar{c}) = \bar{a}\cdot\bar{b} + \bar{a}\cdot\bar{c}$.

6) $\bar{a}\cdot\bar{a} = |\bar{a}|^2 = a^2 = \bar{a}^2$

7) $\bar{i}\cdot\bar{i} = \bar{j}\cdot\bar{j} = \bar{k}\cdot\bar{k} = 1$ and $\bar{i}\cdot\bar{j} = \bar{j}\cdot\bar{k} = \bar{k}\cdot\bar{i} = 0$.

8) If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ then $\bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Vector product:-

Consider any two vectors $\bar{a}$ and $\bar{b}$. Then the product $\bar{a}\times\bar{b} = |\bar{a}||\bar{b}|\sin\alpha\,\hat{n}$.

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where θ is angle betn $\bar{a}$ and $\bar{b}$ and $\hat{n}$ is a unit vector $\perp$r to $\bar{a}$ and $\bar{b}$.

$$|\bar{a}\times\bar{b}| = |\bar{a}||\bar{b}|\sin\theta .$$

Properties of Vector product :-

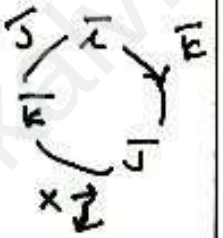
1) $\bar{a}\times\bar{b} = 0$ if $\bar{a}=0$, or $\bar{b}=0$ or $\bar{a}$ and $\bar{b}$ are collinear.

2) $\bar{a}\times\bar{b} = -\bar{b}\times\bar{a}$.

3) $\bar{a}\times\bar{a} = 0$.

4) $\bar{i}\times\bar{i} = \bar{j}\times\bar{j} = \bar{k}\times\bar{k} = \bar{0}$

$$\begin{array}{l|l} \bar{i}\times\bar{j} = \bar{k} & \bar{j}\times\bar{i} = -\bar{k} \\ \bar{j}\times\bar{k} = \bar{i} & \bar{k}\times\bar{j} = -\bar{i} \\ \bar{k}\times\bar{i} = \bar{j} & \bar{i}\times\bar{k} = -\bar{j} \end{array}$$



5) unit vector $\perp$r to $\bar{a}$ and $\bar{b}$

$$\hat{n} = \pm \frac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|}$$

6) If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

then $\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

7) The magnitude of vector product is equal to the area of the $\parallel^{gm}$ formed with $\bar{a}$ and $\bar{b}$ as adjacent sides

8) Area of Δ.

1) Vector area $= \frac{1}{2}(\bar{AB}\times\bar{AC})$

2) Area of $\Delta = \frac{1}{2}|\bar{AB}\times\bar{AC}|$

3) Area of triangle ABC, if the position vector of A is $\bar{a}$, position vector of B is $\bar{b}$ and position vector of C is $\bar{c}$

$$= \frac{1}{2}\left|\bar{a}\times\bar{b} + \bar{b}\times\bar{c} + \bar{c}\times\bar{a}\right|$$

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Application of dot and Cross product in physics.

Defn: If $\vec{d}$ is the displacement vector of a particle moved from a point to another point after applying a constant force $\vec{F}$ on the particle, then the **work done** by the force on the particle is $\boxed{W = \vec{F} \cdot \vec{d}}$

Ex 6.9 A particle acted upon by a constant forces $2\vec{i}+5\vec{j}+6\vec{k}$ and $-\vec{i}-2\vec{j}-\vec{k}$ is displaced from the point $(4,-3,-2)$ to the point $(6,1,-3)$. Find the total work done by the forces.

Given $\vec{F_1} = 2\vec{i}+5\vec{j}+6\vec{k}$ and

$\vec{F_2} = -\vec{i}-2\vec{j}-\vec{k}$.

$\therefore \vec{F} = \vec{F_1}+\vec{F_2} = \vec{i}+3\vec{j}+5\vec{k}$

Let A and B be the points $(4,-3,-2)$ and $(6,1,-3)$ respectively.

position vector of $A = \vec{OA} = 4\vec{i}-3\vec{j}-2\vec{k}$

position vector of $B = \vec{OB} = 6\vec{i}+\vec{j}-3\vec{k}$

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Displacement $\vec{AB} = \vec{d} = \vec{OB} - \vec{OA}$

$= 2\vec{i}+4\vec{j}-\vec{k}$

Work done $W = \vec{F} \cdot \vec{d}$

$= (\vec{i}+3\vec{j}+5\vec{k}) \cdot (2\vec{i}+4\vec{j}-\vec{k})$

$= 2+12-5 = 14-5 = \boxed{9 \text{ unit}}$

Ex 6.10 A particle is acted upon by the forces $3\vec{i}-2\vec{j}+2\vec{k}$ and $2\vec{i}+\vec{j}-\vec{k}$ is displaced from the point $(1,3,-1)$ to the point $(4,-1,\lambda)$. If the work done by the forces is 16 units, find the values of λ.

Given $\vec{F_1} = 3\vec{i}-2\vec{j}+2\vec{k}$

$\vec{F_2} = 2\vec{i}+\vec{j}-\vec{k}$

$\therefore$ Resultant force $\vec{F} = \vec{F_1}+\vec{F_2}$

$= 5\vec{i}-\vec{j}+\vec{k}$

Let A and B be the points $(1,3,-1)$ and $(4,-1,\lambda)$

$\therefore$ position vector of $A = \vec{OA} = \vec{i}+3\vec{j}-\vec{k}$

position vector of $B = \vec{OB} = 4\vec{i}-\vec{j}+\lambda\vec{k}$

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$\therefore$ Displacement vector $\bar{d} = \overline{AB}$

$= \overline{OB} - \overline{OA}$

$= 3\bar{i} - 4\bar{j} + (\lambda+1)\bar{k}$

Given work done by the force = 16 units

i.e. $\bar{F} \cdot \bar{d} = 16$ units

i.e. $(5\bar{i} - \bar{j} + \bar{k}) \cdot (3\bar{i} - 4\bar{j} + (\lambda+1)\bar{k}) = 16$

$15 + 4 + \lambda + 1 = 16$

$\lambda + 20 = 16$

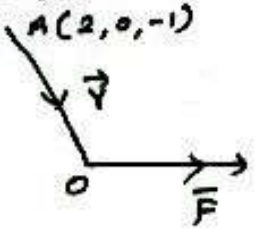
$\lambda = 16 - 20 = -4.$

$\boxed{\lambda = -4}$

Ex 6.11 Find the magnitude and the direction cosine of the torque about the point $(2,0,-1)$ of a force $2\bar{i} + \bar{j} - \bar{k}$, whose line of action passes through the origin.

Let the point be $A(2,0,-1)$

$\therefore \overline{OA} = 2\bar{i} - \bar{k}$



$\bar{r} = \overline{AO} = -(\overline{OA})$

$= -2\bar{i} + \bar{k}.$

Given force $\bar{F} = 2\bar{i} + \bar{j} - \bar{k}.$

Torque $\bar{t} = \bar{r} \times \bar{F}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -2 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$= \bar{i}(0-1) - \bar{j}(2-2) + \bar{k}(-2+0)$

$= -\bar{i} + 0\bar{j} - 2\bar{k}.$

Magnitude of torque $= |\bar{r} \times \bar{F}|$

$= |-\bar{i} - 2\bar{k}|$

$= \sqrt{(-1)^2 + (-2)^2}$

$= \sqrt{1+4} = \sqrt{5}$

Direction cosine of the torque

$= \left(-\frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}}\right)$ //

$\bar{r}$ = Through (at) – about.

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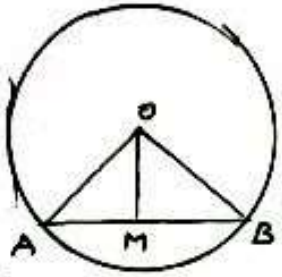
EXERCISE 6.1

1. prove by vector method that if a line is drawn from the centre of a circle to the mid pt of a chord, then the line is perpendicular to the chord.



Let AB be the chord of the circle.
Let O be the centre of the circle.
M is Mid point of AB.

$\overline{OM} = \frac{\overline{OA}+\overline{OB}}{2}$. $\overline{OA} = \overline{OB}$ (Radii)

To prove $\overline{OM} \perp r$ to $\overline{AB}$.
i.e to prove $\overline{OM} \cdot \overline{AB} = 0$.

$$\overline{OM} \cdot \overline{AB} = \frac{\overline{OA}+\overline{OB}}{2} \cdot \overline{AB}$$

$$= \frac{(\overline{OA}+\overline{OB})}{2} (\overline{OB}-\overline{OA})$$

$$= \frac{1}{2}(\overline{OB}+\overline{OA})(\overline{OB}-\overline{OA})$$

$$= \frac{1}{2}\left[\overline{OB}^2 - \overline{OA}^2\right]$$

$$= \frac{1}{2}\left(|\overline{OB}|^2 - |\overline{OA}|^2\right)$$

$$= \frac{1}{2} \times 0 \quad \because OA = OB$$

$\therefore \overline{OM} \cdot \overline{AB} = 0$.

Hence OM is perpendicular to AB.

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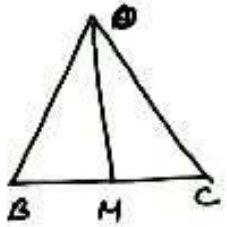
2) prove by vector method that the median to the base of an isosceles triangle is perpendicular to the base.



Let ΔOAB be an isosceles triangle with $OB = OB$.
M is mid pt of BC
To prove $\overline{OM} \perp r$ to $\overline{BC}$

$\overline{OM} = \frac{\overline{OB}+\overline{OC}}{2}$, $\overline{BC} = \overline{OC} - \overline{OB}$.

$$\overline{OM} \cdot \overline{BC} = \frac{(\overline{OB}+\overline{OC})}{2} (\overline{OC}-\overline{OB})$$

$$= \frac{1}{2}\left[(\overline{OC}+\overline{OB})(\overline{OC}-\overline{OB})\right]$$

$$= \frac{1}{2}\left[\overline{OC}^2 - \overline{OB}^2\right]$$

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$= \frac{1}{2}\left[|\bar{OC}|^2 - |\bar{OB}|^2 \right]$

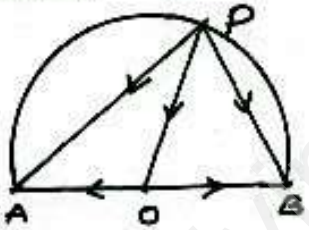
$= \frac{1}{2} \cdot 0$ $\quad \because OB = OC$

$= 0$

$\bar{OM} \cdot \bar{BC} = 0 \Rightarrow OM$ is $\perp_r$ to BC

---

3) prove by vector method that an angle in a semi-circle is a right angle.



Let AB be the diameter of the circle with centre O.

Let P be any point on the semi circle.

To prove $\angle APB = 90^\circ$

It is enough to prove that $\bar{PA}$ and $\bar{PB}$ are perpendicular.

i.e to prove $\bar{PA} \cdot \bar{PB} = 0$.

we have $OA = OB$ (radii) $= OP$

$\bar{PA} = \bar{PO} + \bar{OA}$

$\bar{PB} = \bar{PO} + \bar{OB}$

$\bar{PA} \cdot \bar{PB} = (\bar{PO} + \bar{OA})(\bar{PO} + \bar{OB})$ $\quad (\because \bar{OB} = -\bar{OA})$

$= (\bar{PO} + \bar{OA})(\bar{PO} - \bar{OA})$

$= (\bar{PO})^2 - (\bar{OA})^2$

$= PO^2 - OA^2$

$= OA^2 - OA^2$ $\quad \because OP = OA = OB$ (radii)

$= 0$

$\bar{PA} \cdot \bar{PB} = 0 \Rightarrow \bar{PA}$ $\perp_r$ to $\bar{PB}$.

$\Rightarrow APB = 90^\circ$

Hence the result.

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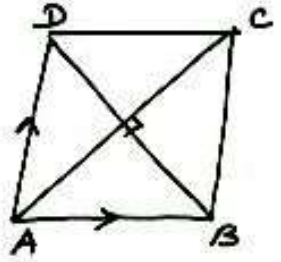
4) prove by vector method that the diagonals of a rhombus bisect each other at right angles.

Let ABCD be a rhombus.



To prove $\bar{AC} \cdot \bar{BD} = 0$.

In rhombus

$AB = BC = CD = AD$.

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$\overline{AC} = \overline{AB} + \overline{BC}$

$\overline{BD} = \overline{BC} + \overline{CD}$

$= \overline{BC} - \overline{AB}$ $\because$ ~~BC~~ $CD = AB$ But opposite in direction.

$\overline{AC} \cdot \overline{BD} = (\overline{AB} + \overline{BC}) \cdot (\overline{BC} - \overline{AB})$

$= (\overline{BC} + \overline{AB}) \cdot (\overline{BC} - \overline{AB})$

$= \overline{BC}^2 - \overline{AB}^2$

$= BC^2 - AB^2$

$= AB^2 - AB^2$ $\because AB = BC$

$= 0$

$\overline{AC} \cdot \overline{BD} = 0$

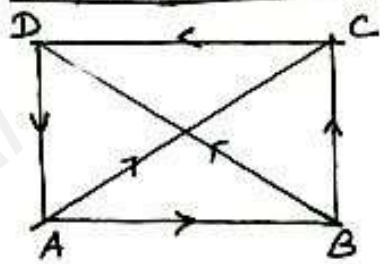
$\Rightarrow$ $\overline{AC} \perp_r$ to $\overline{BD}$.

Hence the result.

---

5) Using vector method, prove that if the diagonals of a parallelogram are equal, then it is a rectangle.

X11 | CH6 | 6.1 | P15



Let ABCD be a parallelogram.

To prove ABCD be a rectangle.

Give that $AC = BD$.

$\overline{AC} = \overline{AB} + \overline{BC}$

$\overline{BD} = \overline{BC} + \overline{CD}$

$= \overline{BC} - \overline{AB}$ $\because$ ($BC = AB$ But opposite in direction)

But $\overline{AC}^2 = \overline{BD}^2$

$(\overline{AB} + \overline{BC})^2 = (\overline{BC} - \overline{AB})^2$

$\overline{AB}^2 + \overline{BC}^2 + 2\overline{AB} \cdot \overline{BC} = \overline{BC}^2 + \overline{AB}^2 - 2\overline{BC} \cdot \overline{AB}$

$4\overline{AB} \cdot \overline{BC} = 0$

$\Rightarrow$ $\overline{AB} \cdot \overline{BC} = 0$

$\Rightarrow$ $\overline{AB} \perp_r \overline{BC}$ $\Rightarrow \angle B = 90$.

$\therefore$ ABCD is a rectangle.

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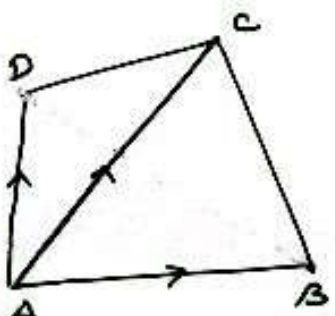
6) Prove by vector method that the area of the quadrilateral ABCD having diagonals AC and BD is $\frac{1}{2}|\bar{AC} \times \bar{BD}|$



Vector area of a quadrilateral ABCD = Vector area of $\triangle ABC$ + Vector area of ACD.

$$= \frac{1}{2}(\bar{AB} \times \bar{AC}) + \frac{1}{2}(\bar{AC} \times \bar{AD})$$

$$= -\frac{1}{2}(\bar{AC} \times \bar{AB}) + \frac{1}{2}(\bar{AC} \times \bar{AD})$$

$$= \frac{1}{2}\bar{AC} \times [-\bar{AB} + \bar{AD}]$$

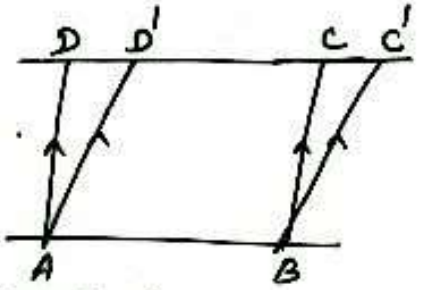
$$= \frac{1}{2}\bar{AC} \times [\bar{BA} + \bar{AD}]$$

$$= \frac{1}{2}\bar{AC} \times \bar{BD}$$

∴ area of the quadrilateral $= \frac{1}{2}\bar{AC} \times \bar{BD}$.

∴ Area of the quadrilateral ABCD $= \frac{1}{2}|\bar{AC} \times \bar{BD}|$.

7) Prove by vector method that the parallelograms on the same base and between the same parallels are equal.



Let ABCD be the given parallelogram. and ABC'D' be the new parallelogram with same base AB and between the parallel lines AB and DC.

Vector area of ABCD $= \bar{AB} \times \bar{AD}$

$$= \bar{AB} \times (\bar{AD'} + \bar{D'D})$$

$$= \bar{AB} \times \bar{AD'} + \bar{AB} \times \bar{D'D}$$

$$= AB \times \bar{AD'} + 0$$

∵ $\bar{AB}$ and $\bar{D'D}$ are parallel.

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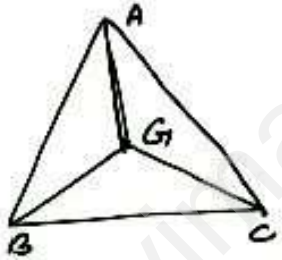
= vector Area of $ABC'D'$

∴ Area of $ABCD$ = Area of $ABC'D'$

---

8) If G is centroid of a ΔABC, prove that

area of ΔGAB = area of ΔGBC = area of $\Delta GCA = \frac{1}{3}$ area of ΔABC



Let ABC be a triangle and G is its centroid. Let O be the origin and let the position vector of A, B, C be $\bar{a}, \bar{b}, \bar{c}$ respectively.

$\overline{OA} = \bar{a}$, $\overline{OB} = \bar{b}$, $\overline{OC} = \bar{c}$

Area of $\Delta GAB = \frac{1}{2}|GA \times GB|$

$= \frac{1}{2}\left|(\overline{OA} - \overline{OG}) \times (\overline{OB} - \overline{OG})\right|$

$= \frac{1}{2}\left|\overline{OA} \times \overline{OB} - \overline{OA} \times \overline{OG} - \overline{OG} \times \overline{OB} + \overline{OG} \times \overline{OG}\right|$

$= \frac{1}{2}\left|\overline{OA} \times \overline{OB} - \overline{OA} \times \overline{OG} - \overline{OG} \times \overline{OB}\right|$

$\because \overline{OG} \times \overline{OG} = 0.$

$= \frac{1}{2}\left|\bar{a} \times \bar{b} - \bar{a} \times \frac{\bar{a}+\bar{b}+\bar{c}}{3} - \frac{\bar{a}+\bar{b}+\bar{c}}{3} \times \bar{b}\right|$

$= \frac{1}{2}\left|\bar{a} \times \bar{b} - \left[\frac{\bar{a} \times \bar{a} + \bar{a} \times \bar{b} + \bar{a} \times \bar{c}}{3}\right] - \frac{\bar{a} \times \bar{b} + \bar{b} \times \bar{b} + \bar{c} \times \bar{b}}{3}\right|$

$= \frac{1}{2}\left|\bar{a} \times \bar{b} - \frac{\bar{a} \times \bar{b} - \bar{a} \times \bar{c}}{3} - \frac{\bar{a} \times \bar{b} - \bar{c} \times \bar{b}}{3}\right|$

$= \frac{1}{2}\left|\frac{3(a \times \bar{b}) - \bar{a} \times \bar{b} - \bar{a} \times \bar{c} - \bar{a} \times \bar{b} - \bar{c} \times \bar{b}}{3}\right|$

$= \frac{1}{3} \times \frac{1}{2}\left|\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a}\right|$

Area of $\Delta GBC = \frac{1}{3} \times \frac{1}{2}\left|\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a}\right|$

and Area of $\Delta GCA = \frac{1}{3} \times \frac{1}{2}\left|\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a}\right|$ ——— (1)

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Area of $\Delta ABC = \frac{1}{2}\left|\bar{AB} \times \bar{AC}\right|$

$= \frac{1}{2}\left|(\bar{OB}-\bar{OA}) \times (\bar{OC}-\bar{OA})\right|$

$= \frac{1}{2}\left|(\bar{b}-\bar{a}) \times (\bar{c}-\bar{a})\right|$

$= \frac{1}{2}\left|\bar{b}\times\bar{c} - \bar{b}\times\bar{a} - \bar{a}\times\bar{c} + \bar{a}\times\bar{a}\right|$

$= \frac{1}{2}\left|\bar{a}\times\bar{b} + \bar{b}\times\bar{c} + \bar{c}\times\bar{a}\right|$ — ②

From ① and ②

Area of ΔGAB = Area of ΔGBC = Area of $\Delta GCA = \frac{1}{3}$ area of ΔABC

---

9) Using vector method prove that

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

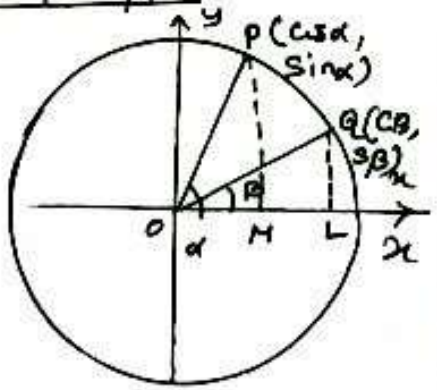
Take the points P and Q on the unit circle with centre O.

Assume that OP and OQ make angles α and β with x-axis respectively.

XII | CH6 | 6.1 | P18

$\angle POx = \alpha$, $\angle QOx = \beta$

$\therefore \angle POQ = \alpha - \beta$.



The coordinates of P and Q are $P(\cos\alpha, \sin\alpha)$ and $Q(\cos\beta, \sin\beta)$.

Let the unit vectors along x-axis and y-axis be $\bar{i}$ and $\bar{j}$ respectively.

From the fig,

$OM = \cos\alpha$ | $MP = \sin\alpha$

$\bar{OM} = \cos\alpha\,\bar{i}$ | $\bar{MP} = \sin\alpha\,\bar{j}$

$OL = \cos\beta$ | $LQ = \sin\beta$

$\bar{OL} = \cos\beta\,\bar{i}$ | $\bar{LQ} = \sin\beta\,\bar{j}$.

$\bar{OP} = \bar{OM} + \bar{MP}$

$= \cos\alpha\,\bar{i} + \sin\alpha\,\bar{j}$

$\bar{OQ} = \bar{OL} + \bar{LQ}$

$= \cos\beta\,\bar{i} + \sin\beta\,\bar{j}$.

$\bar{OP}\cdot\bar{OQ} = |\bar{OP}||\bar{OQ}|\cos(\alpha-\beta)$

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$|x| \times \cos(\alpha-\beta) = \cos(\alpha-\beta)$ ——— ①

$\overline{OP} \cdot \overline{OQ} = (\cos\alpha\,\bar{i} + \sin\alpha\,\bar{j}) \cdot (\cos\beta\,\bar{i} + \sin\beta\,\bar{j})$

$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$ ——— ②

From ① and ②

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

---

(10) prove by vector method that

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

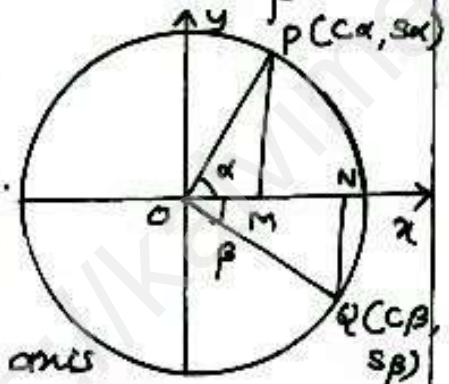
Take the points P and Q on the unit circle with centre O.

Assume that OP and OQ make angles α and β with x-axis respectively.

$\angle POX = \alpha$, $\angle QOX = \beta$, $\angle POQ = \alpha+\beta$.

Coordinates of P and Q are

$P(\cos\alpha, \sin\alpha)$ $Q(\cos\beta, -\sin\beta)$



From the fig,

$OM = \cos\alpha$ | $ON = \cos\beta$

$\overline{OM} = \cos\alpha\,\bar{i}$ | $\overline{ON} = \cos\beta\,\bar{i}$

$MP = \sin\alpha$ | $NQ = -\sin\beta$

$\overline{MP} = \sin\alpha\,\bar{j}$ | $\overline{NQ} = +\sin\beta(-\bar{j})$

$= -\sin\beta\,\bar{j}$.

$\overline{OQ} = \overline{ON} + \overline{NQ}$

$= \cos\beta\,\bar{i} + (-\sin\beta\,\bar{j})$

$= \cos\beta\,\bar{i} - \sin\beta\,\bar{j}$

$\overline{OP} = \overline{OM} + \overline{MP}$

$= \cos\alpha\,\bar{i} + \sin\alpha\,\bar{j}$

$\overline{OQ} \times \overline{OP} = |\overline{OQ}|\,|\overline{OP}| \sin(\alpha+\beta)\,\bar{k}$

$= |x| \times \sin(\alpha+\beta)\,\bar{k}$

$= \sin(\alpha+\beta)\,\bar{k}$ ——— ①

$$\overline{OQ} \times \overline{OP} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos\beta & -\sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

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$= \bar{k}(\cos\beta \sin\alpha + \cos\alpha \sin\beta)$

$= \bar{k}(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$ ——— (2)

from (1) and (2)

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

---

11) A particle acted on by constant forces $8\bar{i}+2\bar{j}-6\bar{k}$ and $6\bar{i}+2\bar{j}-2\bar{k}$ is displaced from the point (1, 2, 3) to the point (5, 4, 1). Find the work done by the forces.

Given $\bar{F}_1 = 8\bar{i}+2\bar{j}-6\bar{k}$

$\bar{F}_2 = 6\bar{i}+2\bar{j}-2\bar{k}$

Resultant force $\bar{F} = \bar{F}_1+\bar{F}_2 = 14\bar{i}+4\bar{j}-8\bar{k}$

Let the points be A(1,2,3), and B(5,4,1)

∴ position vector of $A = \overline{OA} = \bar{i}+2\bar{j}+3\bar{k}$

position vector of $B = \overline{OB} = 5\bar{i}+4\bar{j}+\bar{k}$

Displacement vector $\overline{AB} = \bar{d} = \overline{OB}-\overline{OA}$

$= 4\bar{i}+2\bar{j}-2\bar{k}$

Total work done $= \bar{F}\cdot\bar{d}$

$= (14\bar{i}+4\bar{j}-8\bar{k})\cdot(4\bar{i}+2\bar{j}-2\bar{k})$

$= 56+8+16 = 80$ units

Note:- The vector having magnitude λ and along the direction of the vector $\bar{a}$ is $\frac{\lambda\bar{a}}{|\bar{a}|}$.

---

12) Forces of magnitude $5\sqrt{2}$ and $10\sqrt{2}$ units acting in the direction $3\bar{i}+4\bar{j}+5\bar{k}$ and $10\bar{i}+6\bar{j}-8\bar{k}$ respectively, act on a particle which is displaced from the point with position vector $4\bar{i}-3\bar{j}-2\bar{k}$ to the point with position vector $6\bar{i}+\bar{j}-3\bar{k}$ find the work done by the forces.

Let $\bar{F}_1$ and $\bar{F}_2$ are forces with magnitude $5\sqrt{2}$ and $10\sqrt{2}$ and acting into direction $3\bar{i}+4\bar{j}+5\bar{k}$ and $10\bar{i}+6\bar{j}-8\bar{k}$ respectively.

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$$\bar{F}_1 = 5\sqrt{2}\,\frac{(3\bar{i}+4\bar{j}+5\bar{k})}{|3\bar{i}+4\bar{j}+5\bar{k}|}$$

$$= 5\sqrt{2}\,\frac{(3\bar{i}+4\bar{j}+5\bar{k})}{\sqrt{9+16+25}} = \frac{5\sqrt{2}(3\bar{i}+4\bar{j}+5\bar{k})}{\sqrt{50}}$$

$$= \frac{5\sqrt{2}(3\bar{i}+4\bar{j}+5\bar{k})}{5\sqrt{2}} = 3\bar{i}+4\bar{j}+5\bar{k}$$

$$\bar{F}_2 = 10\sqrt{2}\,\frac{(10\bar{i}+6\bar{j}-8\bar{k})}{|10\bar{i}+6\bar{j}-8\bar{k}|} = \frac{10\sqrt{2}(10\bar{i}+6\bar{j}-8\bar{k})}{\sqrt{100+36+64}}$$

$$= \frac{10\sqrt{2}(10\bar{i}+6\bar{j}-8\bar{k})}{\sqrt{200}} = \frac{10\sqrt{2}(10\bar{i}+6\bar{j}-8\bar{k})}{10\sqrt{2}}$$

$$= 10\bar{i}+6\bar{j}-8\bar{k}$$

Resultant force $\bar{F} = \bar{F}_1 + \bar{F}_2$

$$\bar{F} = 13\bar{i}+10\bar{j}-3\bar{k}$$

position vector of $A = \overline{OA} = 4\bar{i}-3\bar{j}-2\bar{k}$

position Vector of $B = \overline{OB} = 6\bar{i}+\bar{j}-3\bar{k}$

XII/CH6/6.1/P21

Displacement vector $\overline{AB} = \bar{d}$

$$= \overline{OB} - \overline{OA}$$

$$= 2\bar{i}+4\bar{j}-\bar{k}$$

Total work done $= \bar{F}\cdot\bar{d}$

$$= (13\bar{i}+10\bar{j}-3\bar{k})\cdot(2\bar{i}+4\bar{j}-\bar{k})$$

$$= 26+40+3 = 69 \text{ units}$$

13) Find the magnitude and direction cosine of the torque of a force represented by $3\bar{i}+4\bar{j}-5\bar{k}$ about the point with position vector $2\bar{i}-3\bar{j}+4\bar{k}$ acting through a point whose position vector is $4\bar{i}+2\bar{j}-3\bar{k}$

Given $\bar{F} = 3\bar{i}+4\bar{j}-5\bar{k}$

Let $\overline{OP} = 2\bar{i}-3\bar{j}+4\bar{k}$

$\overline{OQ} = 4\bar{i}+2\bar{j}-3\bar{k}$

$\therefore \bar{r}$ = Through − about [(or) at − about]

$$= \overline{OQ} - \overline{OP}$$

$$= 2\bar{i}+5\bar{j}-7\bar{k}.$$

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Torque of force $\bar{t} = \bar{r} \times \bar{F}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 5 & -7 \\ 3 & 4 & -5 \end{vmatrix}$$

$$= \bar{i}(-25+28) - \bar{j}(-10+21) + \bar{k}(8-15)$$

$$= 3\bar{i} - 11\bar{j} - 7\bar{k}.$$

Magnitude of torque $= |3\bar{i} - 11\bar{j} - 7\bar{k}|$

$$= \sqrt{9+121+49} = \sqrt{179}$$

Direction cosine of torque

$$\left(\frac{3}{\sqrt{179}}, \frac{-11}{\sqrt{179}}, \frac{-7}{\sqrt{179}} \right)$$

---

14) Find the torque of the resultant of the three forces represented by $-3\bar{i} + 6\bar{j} - 3\bar{k}$, $4\bar{i} - 10\bar{j} + 12\bar{k}$ and $4\bar{i} + 7\bar{j}$ acting at the point with position vector $8\bar{i} - 6\bar{j} - 4\bar{k}$, about the point with position vector $18\bar{i} + 3\bar{j} - 9\bar{k}$.

XII | CH 6 | 6.1 | P22

Given $\bar{F_1} = -3\bar{i} + 6\bar{j} - 3\bar{k}$

$\bar{F_2} = 4\bar{i} - 10\bar{j} + 12\bar{k}$

$\bar{F_3} = 4\bar{i} + 7\bar{j} + 0\bar{k}$.

$\therefore$ Resultant force $\bar{F} = \bar{F_1} + \bar{F_2} + \bar{F_3}$

$= 5\bar{i} + 3\bar{j} + 9\bar{k}$.

let $\overline{OP} = 8\bar{i} - 6\bar{j} - 4\bar{k}$

$\overline{OQ} = 18\bar{i} + 3\bar{j} - 9\bar{k}$.

$\bar{r}$ = at - about

$= \overline{OP} - \overline{OQ}$

$= -10\bar{i} - 9\bar{j} + 5\bar{k}$

$\therefore$ Torque of the force $\bar{t} = \bar{r} \times \bar{F}$

$$\bar{t} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -10 & -9 & 5 \\ 5 & 3 & 9 \end{vmatrix}$$

$$= \bar{i}(-81-15) - \bar{j}(-90-25) + \bar{k}(-30+45)$$

$$= -96\bar{i} + 115\bar{j} + 15\bar{k}$$



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Scalar Triple product:-

$\bar{a} \cdot (\bar{b} \times \bar{c}) = [\bar{a}\ \bar{b}\ \bar{c}]$ is scalar triple product of vectors $\bar{a}, \bar{b}$ and $\bar{c}$ if $\bar{a} = (a_1, a_2, a_3)$ $\bar{b} = (b_1, b_2, b_3)$ and $\bar{c} = (c_1, c_2, c_3)$ then

$$[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{Volume}$$

of parallelopiped whose adjacent sides are along $\bar{a}, \bar{b}$ and $\bar{c}$.

properties:-

(i) $[\bar{a}\ \bar{b}\ \bar{c}] = [\bar{b}\ \bar{c}\ \bar{a}] = [\bar{c}\ \bar{a}\ \bar{b}]$

(ii) $[\bar{a}\ \bar{b}\ \bar{c}] = -[\bar{b}\ \bar{a}\ \bar{c}]$

(iii) If the vectors are coplanar, then $[\bar{a}\ \bar{b}\ \bar{c}] = 0$.

(iv) $[\bar{a}, \bar{b}, \bar{c}+\bar{d}] = [\bar{a}\ \bar{b}\ \bar{c}] + [\bar{a}\ \bar{b}\ \bar{d}]$

Theorem:- Three vectors $\bar{a}, \bar{b}, \bar{c}$ are coplanar if and only if, there exists scalars, $r, s, t \in R$ such that atleast one of them is non zero and $r\bar{a} + s\bar{b} + t\bar{c} = \bar{0}$.

Theorem:- If $\bar{a}, \bar{b}, \bar{c}$ and $\bar{p}, \bar{q}, \bar{r}$ are any two system of three vectors, and if $\bar{p} = x_1\bar{a} + y_1\bar{b} + z_1\bar{c}$ $\bar{q} = x_2\bar{a} + y_2\bar{b} + z_2\bar{c}$ and $\bar{r} = x_3\bar{a} + y_3\bar{b} + z_3\bar{c}$, then

$$[\bar{p}\ \bar{q}\ \bar{r}] = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} [\bar{a}\ \bar{b}\ \bar{c}]$$

Ex 6.12 If $\bar{a} = -3\bar{i} - \bar{j} + 5\bar{k}$, $\bar{b} = \bar{i} - 2\bar{j} + \bar{k}$, $\bar{c} = 4\bar{j} - 5\bar{k}$, find $\bar{a} \cdot (\bar{b} \times \bar{c})$

The given vectors are

$\bar{a} = -3\bar{i} - \bar{j} + 5\bar{k}$

$\bar{b} = \bar{i} - 2\bar{j} + \bar{k}$

$\bar{c} = 4\bar{j} - 5\bar{k}$.

$$\bar{a} \cdot (\bar{b} \times \bar{c}) = \begin{vmatrix} -3 & -1 & 5 \\ 1 & -2 & 1 \\ 0 & 4 & -5 \end{vmatrix}$$

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$= -3(10-4)+1(-5-0)+5(4+0)$

$= -3(6)-5+20$

$= -18-5+20$

$= -23+20 = \boxed{-3}$

Eg 6.13 Find the volume of parallelopiped whose edges are given by the vectors $2\bar{i}-3\bar{j}+4\bar{k}$, $\bar{i}+2\bar{j}-\bar{k}$, $3\bar{i}-\bar{j}+2\bar{k}$.

WKT Volume of parallelopiped

$= [\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$

$= 2(4-1)+3(2+3)+4(-1-6)$

$= 2(3)+3(5)+4(-7)$

$= 6+15-28$

$= 21-28 = -7$

∴ Volume of parallelopiped $= |-7|$

$= 7$ cu. units

6.14 Show that the vectors $\bar{i}+2\bar{j}-3\bar{k}$, $2\bar{i}-\bar{j}+2\bar{k}$ and $3\bar{i}+\bar{j}-\bar{k}$ are coplanar.

Given $\bar{a} = \bar{i}+2\bar{j}-3\bar{k}$

$\bar{b} = 2\bar{i}-\bar{j}+2\bar{k}$

$\bar{c} = 3\bar{i}+\bar{j}-\bar{k}$.

To prove $\bar{a}, \bar{b}, \bar{c}$ are coplanar.

i.e to prove $[\bar{a}\ \bar{b}\ \bar{c}] = 0$.

$[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix}$.

$= 1(1-2)-2(-2-6)-3(2+3)$

$= 1(-1)-2(-8)-3(5)$

$= -1+16-15$

$= 16-16 = 0$

∴ The given vectors are coplanar.

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Eg 6.15 If $2\bar{i}-\bar{j}+3\bar{k}$, $3\bar{i}+2\bar{j}+\bar{k}$, $\bar{i}+m\bar{j}+4\bar{k}$ are coplanar, find the value of m.

Let the given vectors be

$\bar{a} = 2\bar{i}-\bar{j}+3\bar{k}$, $\bar{b} = 3\bar{i}+2\bar{j}+\bar{k}$

$\bar{c} = \bar{i}+m\bar{j}+4\bar{k}$.

Given that $\bar{a}, \bar{b}, \bar{c}$ are coplanar.

$\Rightarrow [\bar{a}\ \bar{b}\ \bar{c}] = 0$.

$\Rightarrow \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & m & 4 \end{vmatrix} = 0$

$\Rightarrow 2(8-m)+1(12-1)+3(3m-2) = 0$.

$\Rightarrow 16-2m+11+9m-6 = 0$

$21+7m = 0$

$7m = -21$

$m = -\frac{21}{7} = -3$

$\therefore \boxed{m = -3}$

Eg 6.16 Show that the four points (6, -7, 0), (16, -19, -4), (0, 3, -6) (2, -5, 10) lie on a same plane.

Let the given points be

A(6, -7, 0), B(16, -19, -4),

C(0, 3, -6), D(2, -5, 10)

To prove that the four points A, B, C, D lie on a plane.

$\Rightarrow$ to prove $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ are coplanar.

Now $\overline{AB} = \overline{OB} - \overline{OA}$

$= (16\bar{i}-19\bar{j}-4\bar{k}) - (6\bar{i}-7\bar{j}+0\bar{k})$

$= 10\bar{i}-12\bar{j}-4\bar{k}$.

$\overline{BC} = \overline{OC} - \overline{OB}$

$= (0\bar{i}+3\bar{j}-6\bar{k}) - (16\bar{i}-19\bar{j}-4\bar{k})$

$= -16\bar{i}+22\bar{j}-2\bar{k}$

$\overline{CD} = \overline{OD} - \overline{OC}$

$= (2\bar{i}-5\bar{j}+10\bar{k}) - (0\bar{i}+3\bar{j}-6\bar{k})$

$= 2\bar{i}-8\bar{j}+16\bar{k}$.

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$$[\bar{AB}, \bar{BC}, \bar{CD}] = \begin{vmatrix} 10 & -12 & -4 \\ -16 & 22 & -2 \\ 2 & -8 & 16 \end{vmatrix}$$

$= 10(352 - 16) + 12(-256 + 4) - 4(128 - 44)$

$= 10(336) + 12(-252) - 4(84)$

$= 3360 - 3024 - 336$

$= 3360 - 3360$

$= 0$

$\bar{AB}, \bar{BC}, \bar{CD}$ are coplanar.
Hence the given points A, B, C, D lie on the same plane.

6.17 If the vectors $\bar{a}, \bar{b}, \bar{c}$ are coplanar, then prove that the vector $\bar{a}+\bar{b}$, $\bar{b}+\bar{c}$, $\bar{c}+\bar{a}$ are coplanar.

Given $\bar{a}, \bar{b}, \bar{c}$ are coplanar

$\therefore [\bar{a}, \bar{b}, \bar{c}] = 0$.

Now $[\bar{a}+\bar{b}, \bar{b}+\bar{c}, \bar{c}+\bar{a}]$

$= [\bar{a}, \bar{b}+\bar{c}, \bar{c}+\bar{a}] + [\bar{b}, \bar{b}+\bar{c}, \bar{c}+\bar{a}]$

$= [\bar{a}, \bar{b}, \bar{c}+\bar{a}] + [\bar{a}, \bar{c}, \bar{c}+\bar{a}] + [\bar{b}, \bar{b}, \bar{c}+\bar{a}] + [\bar{b}, \bar{c}, \bar{c}+\bar{a}]$

$= [\bar{a}, \bar{b}, \bar{c}] + [\bar{a}, \bar{b}, \bar{a}] + [\bar{a}, \bar{c}, \bar{c}] + [\bar{a}, \bar{c}, \bar{a}] + [\bar{b}, \bar{b}, \bar{c}] + [\bar{b}, \bar{b}, \bar{a}] + [\bar{b}, \bar{c}, \bar{c}] + [\bar{b}, \bar{c}, \bar{a}]$

$= [\bar{a}\ \bar{b}\ \bar{c}] + 0 + 0 + 0 + 0 + 0 + 0 + 0 + [\bar{a}\ \bar{b}\ \bar{c}]$

$= 2[\bar{a}\ \bar{b}\ \bar{c}] = 0$. $\because [\bar{a}\ \bar{b}\ \bar{c}] = 0$

$\therefore \bar{a}+\bar{b}, \bar{b}+\bar{c}, \bar{c}+\bar{a}$ are coplanar

---

Eg 6.18 If $\bar{a}, \bar{b}, \bar{c}$ are three vectors, prove that

$[\bar{a}+\bar{c}, \bar{a}+\bar{b}, \bar{a}+\bar{b}+\bar{c}] = [\bar{a}, \bar{b}, \bar{c}]$

Let $\bar{p} = \bar{a}+\bar{c} = 1\cdot\bar{a} + 0\bar{b} + 1\cdot\bar{c}$

$\bar{q} = \bar{a}+\bar{b} = 1\cdot\bar{a} + 1\cdot\bar{b} + 0\cdot\bar{c}$

$\bar{r} = \bar{a}+\bar{b}+\bar{c} = 1\cdot\bar{a} + 1\cdot\bar{b} + 1\cdot\bar{c}$

$$[\bar{p}, \bar{q}, \bar{r}] = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} [\bar{a}\ \bar{b}\ \bar{c}]$$

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ii $[\bar{a}+\bar{c}, \bar{a}+\bar{b}, \bar{a}+\bar{b}+\bar{c}] = [1(1-0)+0+1(1-1)]\,[\bar{a},\bar{b},\bar{c}]$

$[\bar{a}+\bar{c}, \bar{a}+\bar{b}, \bar{a}+\bar{b}+\bar{c}] = [\bar{a},\bar{b},\bar{c}]$

EXERCISE 6.2

1) If $\bar{a} = \bar{i}-2\bar{j}+3\bar{k}$, $\bar{b} = 2\bar{i}+\bar{j}-2\bar{k}$
$\bar{c} = 3\bar{i}+2\bar{j}+\bar{k}$, find $\bar{a}\cdot(\bar{b}\times\bar{c})$

Given $\bar{a} = \bar{i}-2\bar{j}+3\bar{k}$
$\bar{b} = 2\bar{i}+\bar{j}-2\bar{k}$
$\bar{c} = 3\bar{i}+2\bar{j}+\bar{k}$

$\bar{a}\cdot(\bar{b}\times\bar{c}) = [\bar{a},\bar{b},\bar{c}]$

$= \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$

$= 1(1+4)+2(2+6)+3(4-3)$
$= 5+16+3$
$= 24$

---

2) Find the volume of parallelepiped whose coterminous edges are represented by the vectors $-6\bar{i}+14\bar{j}+10\bar{k}$, $14\bar{i}-10\bar{j}-6\bar{k}$ and $2\bar{i}+4\bar{j}-2\bar{k}$.

Let $\bar{a} = -6\bar{i}+14\bar{j}+10\bar{k}$
$\bar{b} = 14\bar{i}-10\bar{j}-6\bar{k}$
$\bar{c} = 2\bar{i}+4\bar{j}-2\bar{k}$

Volume of parallelepiped is $[\bar{a},\bar{b},\bar{c}]$

$= \begin{vmatrix} -6 & 14 & 10 \\ 14 & -10 & -6 \\ 2 & 4 & -2 \end{vmatrix}$

$= -6(20+24)-14(-28+12)+10(56+20)$
$= -6(44)-14(-16)+10(76)$
$= -264+224+760$
$= 720$ cu. units //

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3) The volume of the parallelepiped whose coterminous edges are $7\bar{i}+\lambda\bar{j}-3\bar{k}$, $\bar{i}+2\bar{j}-\bar{k}$, $-3\bar{i}+7\bar{j}+5\bar{k}$ is 90 cu units. find the value of λ.

Let $\bar{a} = 7\bar{i}+\lambda\bar{j}-3\bar{k}$

$\bar{b} = \bar{i}+2\bar{j}-\bar{k}$

$\bar{c} = -3\bar{i}+7\bar{j}+5\bar{k}$

Given that volume of parallelepiped is 90 cu. units.

$\therefore [\bar{a}\ \bar{b}\ \bar{c}] = 90$ cu. units

$$\begin{vmatrix} 7 & \lambda & -3 \\ 1 & 2 & -1 \\ -3 & 7 & 5 \end{vmatrix} = 90$$

$7(10+7) - \lambda(5-3) - 3(7+6) = 90$

$7(17) - \lambda(2) - 3(13) = 90$

$119 - 2\lambda - 39 = 90$

$80 - 2\lambda = 90$

$-2\lambda = 90-80 = 10$

$\lambda = \frac{10}{-2} = -5$

$\therefore \boxed{\lambda = -5}$

4) If $\bar{a}, \bar{b}, \bar{c}$ are three non-coplanar vectors represented by concurrent edges of a parallelepiped of volume 4 cubic units, find the value of

$(\bar{a}+\bar{b})\cdot(\bar{b}\times\bar{c}) + (\bar{b}+\bar{c})\cdot(\bar{c}\times\bar{a}) + (\bar{c}+\bar{a})\cdot(\bar{a}\times\bar{b})$

Given volume of parallelepiped = 4 cu. units

$\therefore [\bar{a}\ \bar{b}\ \bar{c}] = \pm 4$

$(\bar{a}+\bar{b})\cdot(\bar{b}\times\bar{c}) = \bar{a}\cdot(\bar{b}\times\bar{c}) + \bar{b}\cdot(\bar{b}\times\bar{c})$

$= [\bar{a}\ \bar{b}\ \bar{c}] + 0$

$= [\bar{a}, \bar{b}, \bar{c}]$ ——— (1)

$(\bar{b}+\bar{c})\cdot(\bar{c}\times\bar{a}) = (\bar{b}+\bar{c})\cdot\bar{c} + (\bar{b}+\bar{c})\cdot\bar{a}$

$= 0 + [\bar{a}\ \bar{b}\ \bar{c}]$

$= [\bar{a}, \bar{b}, \bar{c}]$ ——— (2)

$(\bar{c}+\bar{a})\cdot(\bar{a}\times\bar{b}) = \bar{c}\cdot(\bar{a}\times\bar{b}) + \bar{a}\cdot(\bar{a}\times\bar{b})$

$= [\bar{a}, \bar{b}, \bar{c}]$ ——— (3)

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$(\bar{a}+\bar{b})\cdot(\bar{b}\times\bar{c}) + (\bar{b}+\bar{c})\cdot(\bar{c}\times\bar{a}) + (\bar{c}+\bar{a})\cdot(\bar{a}\times\bar{b}) = 3[\bar{a},\bar{b},\bar{c}]$

$= 3(\pm 4)$

$= \pm 12$

---

5) Find the altitude of a parallelepiped determined by the vectors $\bar{a} = -2\bar{i}+5\bar{j}+3\bar{k}$, $\bar{b} = \bar{i}+3\bar{j}-2\bar{k}$, $\bar{c} = -3\bar{i}+\bar{j}+4\bar{k}$, if the base is taken as the parallelogram determined by $\bar{b}$ and $\bar{c}$.

WKT Volume = base (Area) × height

i.e. $[\bar{a},\bar{b},\bar{c}] = |\bar{b}\times\bar{c}| \times$ Height ——— ①

$$[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} -2 & 5 & 3 \\ 1 & 3 & -2 \\ -3 & 1 & 4 \end{vmatrix}$$

$= -2(12+2) - 5(4-6) + 3(1+9)$

$= -2(14) - 5(-2) + 3(10)$

$= -28 + 10 + 30 = 12$

$$\bar{b}\times\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 3 & -2 \\ -3 & 1 & 4 \end{vmatrix}$$

$= \bar{i}(12+2) - \bar{j}(4-6) + \bar{k}(1+9)$

$= 14\bar{i} + 2\bar{j} + 10\bar{k}$

$|\bar{b}\times\bar{c}| = \sqrt{196+4+100} = \sqrt{300} = 10\sqrt{3}$

From ① $12 = 10\sqrt{3} \times$ Height

$\therefore$ Height $= \frac{12}{10\sqrt{3}} = \frac{6}{5\sqrt{3}}$

$= \frac{3\times 2}{5\sqrt{3}} = \frac{2\sqrt{3}}{5}$ units

---

6) Determine whether the three vectors are $2\bar{i}+3\bar{j}+\bar{k}$, $\bar{i}-2\bar{j}+2\bar{k}$, $3\bar{i}+\bar{j}+3\bar{k}$ are coplanar.

Let $\bar{a} = 2\bar{i}+3\bar{j}+\bar{k}$

$\bar{b} = \bar{i}-2\bar{j}+2\bar{k}$

$\bar{c} = 3\bar{i}+\bar{j}+3\bar{k}$

$$[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & 1 & 3 \end{vmatrix}$$

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$= 2(-6-2) - 3(3-6) + 1(1+6)$

$= 2(-8) - 3(-3) + 1(7)$

$= -16 + 9 + 7$

$= -16 + 16 = 0$

∴ The given vectors are coplanar

---

7) Let $\bar{a} = \bar{i} + \bar{j} + \bar{k}$, $\bar{b} = \bar{i}$ and $\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$. If $c_1 = 1$ and $c_2 = 2$, find c_3 such that $\bar{a}, \bar{b}$ and $\bar{c}$ are coplanar

Given $\bar{a} = \bar{i} + \bar{j} + \bar{k}$

$\bar{b} = \bar{i}$

$\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$ are coplanar.

But $c_1 = 1$, $c_2 = 2$

∴ $\bar{c} = \bar{i} + 2\bar{j} + c_3\bar{k}$.

$$[\bar{a}\ \bar{b}\ \bar{c}] = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & c_3 \end{vmatrix} = 0$$

$= 1(0) - 1(c_3 - 0) + 1(2 - 0) = 0$

$-c_3 + 2 = 0 \Rightarrow \boxed{c_3 = 2}$

8) If $\bar{a} = \bar{i} - \bar{k}$, $\bar{b} = x\bar{i} + \bar{j} + (1-x)\bar{k}$ $\bar{c} = y\bar{i} + x\bar{j} + (1+x-y)\bar{k}$, show that $[\bar{a}\ \bar{b}\ \bar{c}]$ depends on neither x nor y.

Given $\bar{a} = \bar{i} + 0\bar{j} - \bar{k}$

$\bar{b} = x\bar{i} + \bar{j} + (1-x)\bar{k}$

$\bar{c} = y\bar{i} + x\bar{j} + (1+x-y)\bar{k}$

$$[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix}$$

$= 1(1+x-y-x(1-x)) + 0 - 1(x^2 - y)$

$= 1 + x - y - x + x^2 - x^2 + y$

$= 1$

$[\bar{a}\ \bar{b}\ \bar{c}] = 1 \Rightarrow [\bar{a}, \bar{b}, \bar{c}]$ depends on neither x nor y

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9) If the vectors $a\bar{i}+a\bar{j}+c\bar{k}$, $\bar{i}+\bar{k}$ and $c\bar{i}+c\bar{j}+b\bar{k}$ are coplanar, prove that c is the geometric mean of a and b.

Let $\bar{p} = a\bar{i}+a\bar{j}+c\bar{k}$

$\bar{q} = \bar{i}+\bar{k}$

$\bar{r} = c\bar{i}+c\bar{j}+b\bar{k}$

Given that $\bar{p}, \bar{q}, \bar{r}$ are coplanar.

i $[\bar{p}, \bar{q}, \bar{r}] = 0$

ii $\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$

$a(0-c) - a(b-c) + c(c-0) = 0$

$-ac + ac + ab + c^2 = 0$

$c^2 = ab$

$\therefore$ c is geometric mean of a and b.

---

10) Let $\bar{a}, \bar{b}, \bar{c}$ be three non-zero vectors such that $\bar{c}$ is unit vector $\perp r$ to both $\bar{a}$ and $\bar{b}$. If the angle betn $\bar{a}$ and $\bar{b}$ is $\pi/6$ show that $[\bar{a}\ \bar{b}\ \bar{c}]^2 = \frac{1}{4}|\bar{a}|^2|\bar{b}|^2$

Given $\bar{c}$ is $\perp r$ to both $\bar{a}$ and $\bar{b}$.

so $\bar{c}$ is $\perp l$ to $\bar{b}\times\bar{c}$.

$\therefore [\bar{a}\ \bar{b}\ \bar{c}] = \bar{a}\cdot(\bar{b}\times\bar{c})$

$|[\bar{a}, \bar{b}, \bar{c}]| = |\bar{a}|\,|\bar{b}\times\bar{c}|$

$= |\bar{a}|\,|\bar{b}||\bar{c}| \sin\pi/6 \qquad |\bar{c}| = 1$

$= |\bar{a}|\,|\bar{b}|\left(\frac{1}{2}\right)$

Squaring on both sides

$|[\bar{a}\ \bar{b}\ \bar{c}]|^2 = |\bar{a}|^2|\bar{b}|^2 \cdot \frac{1}{4}$

$= \frac{1}{4}|\bar{a}|^2|\bar{b}|^2$

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Vector triple product

1) For any three vectors $\bar{a}, \bar{b}, \bar{c}$ the vector

$$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$$

2) For any three vectors $\bar{a}, \bar{b}, \bar{c}$ we have

$$\bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b}) = \bar{0}$$

3) Lagrange's identity:-

For any four vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}$

$$(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) = \begin{vmatrix} \bar{a} \cdot \bar{c} & \bar{a} \cdot \bar{d} \\ \bar{b} \cdot \bar{c} & \bar{b} \cdot \bar{d} \end{vmatrix}$$

Eg 6.19 prove that

$$[\bar{a} \times \bar{b}, \bar{b} \times \bar{c}, \bar{c} \times \bar{a}] = [\bar{a}, \bar{b}, \bar{c}]^2$$

$$[\bar{a} \times \bar{b}, \bar{b} \times \bar{c}, \bar{c} \times \bar{a}]$$

$$= (\bar{a} \times \bar{b}) \cdot [(\bar{b} \times \bar{c}) \times (\bar{c} \times \bar{a})]$$

$$= (\bar{a} \times \bar{b}) \cdot [((\bar{b} \times \bar{c}) \cdot \bar{a})\bar{c} - ((\bar{b} \times \bar{c}) \cdot \bar{c})\bar{a}]$$

$$= (\bar{a} \times \bar{b}) \cdot [(\bar{a} \cdot (\bar{b} \times \bar{c}))\bar{c} - (\bar{c} \cdot (\bar{b} \times \bar{c}))\bar{a}]$$

$$= (\bar{a} \times \bar{b}) \cdot [[\bar{a}\ \bar{b}\ \bar{c}]\bar{c} - 0] \quad \because [\bar{c}\ \bar{b}\ \bar{c}] = 0$$

$$= [(\bar{a} \times \bar{b}) \cdot \bar{c}] \cdot [\bar{a}\ \bar{b}\ \bar{c}]$$

$$= [\bar{a}\ \bar{b}\ \bar{c}] \cdot [\bar{a}\ \bar{b}\ \bar{c}]$$

$$= [\bar{a}\ \bar{b}\ \bar{c}]^2$$

$$\therefore [\bar{a} \times \bar{b}, \bar{b} \times \bar{c}, \bar{c} \times \bar{a}] = [\bar{a}, \bar{b}, \bar{c}]^2$$

6.20 prove that $(\bar{a} \cdot (\bar{b} \times \bar{c}))\bar{a} = (\bar{a} \times \bar{b}) \times (\bar{a} \times \bar{c})$

RHS $(\bar{a} \times \bar{b}) \times (\bar{a} \times \bar{c})$

$$= ((\bar{a} \times \bar{b}) \cdot \bar{c})\bar{a} - ((\bar{a} \times \bar{b}) \cdot \bar{a})\bar{c}$$

$$= (\bar{a} \cdot (\bar{b} \times \bar{c}))\bar{a} - 0$$

$$= (\bar{a} \cdot (\bar{b} \times \bar{c}))\bar{a}$$

$$= \text{LHS}$$

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6.21 For any four vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}$, we have

$$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{d}$$
$$= [\bar{a}\ \bar{c}\ \bar{d}]\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\bar{a}$$

In LHS, Take $\bar{a}\times\bar{b} = \bar{p}$

$$\bar{p}\times(\bar{c}\times\bar{d}) = (\bar{p}\cdot\bar{d})\bar{c} - (\bar{p}\cdot\bar{c})\bar{d}$$
$$= [(\bar{a}\times\bar{b})\cdot\bar{d}]\bar{c} - [(\bar{a}\times\bar{b})\cdot\bar{c}]\bar{d}$$
$$= [\bar{a},\bar{b},\bar{d}]\bar{c} - [\bar{a},\bar{b},\bar{c}]\bar{d} \quad \text{—— (1)}$$

Take $\bar{c}\times\bar{d} = \bar{q}$

$$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = (\bar{a}\times\bar{b})\times\bar{q}$$
$$= (\bar{a}\cdot\bar{q})\bar{b} - (\bar{b}\cdot\bar{q})\bar{a}$$
$$= (\bar{a}\cdot(\bar{c}\times\bar{d}))\bar{b} - (\bar{b}\cdot(\bar{c}\times\bar{d}))\bar{a}$$
$$= [\bar{a},\bar{c},\bar{d}]\bar{b} - [\bar{b},\bar{c},\bar{d}]\bar{a} \quad \text{—— (2)}$$

From (1) and (2)

$$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{d}$$
$$= [\bar{a}\ \bar{c}\ \bar{d}]\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\bar{a}$$

6.22 If $\bar{a} = -2\bar{i}+3\bar{j}-2\bar{k}$, $\bar{b} = 3\bar{i}-\bar{j}+3\bar{k}$, $\bar{c} = 2\bar{i}-5\bar{j}+\bar{k}$, find $(\bar{a}\times\bar{b})\times\bar{c}$ and $\bar{a}\times(\bar{b}\times\bar{c})$. state whether they are equal.

Given $\bar{a} = -2\bar{i}+3\bar{j}-2\bar{k}$
$\bar{b} = 3\bar{i}-\bar{j}+3\bar{k}$
$\bar{c} = 2\bar{i}-5\bar{j}+\bar{k}$

$$\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -2 & 3 & -2 \\ 3 & -1 & 3 \end{vmatrix}$$
$$= \bar{i}(9-2) - \bar{j}(-6+6) + \bar{k}(2-9)$$
$$= 7\bar{i}+0\bar{j}-7\bar{k}$$

$$(\bar{a}\times\bar{b})\times\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 7 & 0 & -7 \\ 2 & -5 & 1 \end{vmatrix}$$
$$= \bar{i}(0-35) - \bar{j}(7+14) + \bar{k}(-35-0)$$
$$= -35\bar{i}-21\bar{j}-35\bar{k} \quad \text{—— (1)}$$

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$$\bar{b}\times\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & -1 & 3 \\ 2 & -5 & 1 \end{vmatrix}$$

$= \bar{i}(-1+15) - \bar{j}(3-6) + \bar{k}(-15+2)$

$= 14\bar{i} + 3\bar{j} - 13\bar{k}$

$$\bar{a}\times(\bar{b}\times\bar{c}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -2 & 3 & -2 \\ 14 & 3 & -13 \end{vmatrix}$$

$= \bar{i}(-39+6) - \bar{j}(26+28) + \bar{k}(-6-42)$

$= -33\bar{i} - 54\bar{j} - 48\bar{k}$ ——— (2)

From (1) and (2), we get

$(\bar{a}\times\bar{b})\times\bar{c} \neq \bar{a}\times(\bar{b}\times\bar{c})$

Ex 6.23 If $\bar{a} = \bar{i}-\bar{j}$, $\bar{b} = \bar{i}-\bar{j}-4\bar{k}$, $\bar{c} = 3\bar{j}-\bar{k}$ and $\bar{d} = 2\bar{i}+5\bar{j}+\bar{k}$ verify

a) $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a},\bar{b},\bar{d}]\bar{c} - [\bar{a},\bar{b},\bar{c}]\bar{d}$

b) $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a}\ \bar{c}\ \bar{d}]\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\bar{a}$

Given $\bar{a} = \bar{i}-\bar{j}$

$\bar{b} = \bar{i}-\bar{j}-4\bar{k}$

$\bar{c} = 3\bar{j}-\bar{k}$

$\bar{d} = 2\bar{i}+5\bar{j}+\bar{k}$

$$\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -1 & 0 \\ 1 & -1 & -4 \end{vmatrix}$$

$= \bar{i}(4-0) - \bar{j}(-4+0) + \bar{k}(-1+1)$

$= 4\bar{i} + 4\bar{j} + 0\bar{k}$.

$$\bar{c}\times\bar{d} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix}$$

$= \bar{i}(3+5) - \bar{j}(0+2) + \bar{k}(0-6)$

$= 8\bar{i} - 2\bar{j} - 6\bar{k}$

$$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 4 & 4 & 0 \\ 8 & -2 & -6 \end{vmatrix}$$

$= \bar{i}(-24+0) - \bar{j}(-24-0) + \bar{k}(-8-32)$

$= -24\bar{i} + 24\bar{j} - 40\bar{k}$. ——— (1)

$$[\bar{a}\ \bar{b}\ \bar{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 2 & 5 & 1 \end{vmatrix}$$

$= 1(-1+20) + 1(1+8) + 0(5+2)$

$= 19 + 9 + 0 = 28$

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$[\bar{a}\ \bar{b}\ \bar{d}]\,\bar{c} = 28(3\bar{j}-\bar{k})$

$= 84\bar{j} - 28\bar{k}$

$[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 0 & 3 & -1 \end{vmatrix}$

$= 1(1+12)+1(-1+0)+0(\ )$

$= 13-1 = 12$

$[\bar{a}\ \bar{b}\ \bar{c}]\,\bar{d} = 12(2\bar{i}+5\bar{j}+\bar{k})$

$= 24\bar{i}+60\bar{j}+12\bar{k}$

$[\bar{a}\ \bar{b}\ \bar{d}]\,\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\,\bar{d} = 84\bar{j}-28\bar{k} - (24\bar{i}+60\bar{j}+12\bar{k})$

$= -24\bar{i}+24\bar{j}-40\bar{k}$ —— (2)

From (1) and (2)

$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a}\ \bar{b}\ \bar{d}]\,\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\,\bar{d}$

b) $[\bar{a}\ \bar{c}\ \bar{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix}$

$= 1(3+5)+1(0+2)+0(\ )$

$= 8+2 = 10$

$[\bar{a}\ \bar{c}\ \bar{d}]\,\bar{b} = 10(\bar{i}-\bar{j}-4\bar{k})$

$= 10\bar{i}-10\bar{j}-40\bar{k}$

$[\bar{b}\ \bar{c}\ \bar{d}] = \begin{vmatrix} 1 & -1 & -4 \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix}$

$= 1(3+5)+1(0+2)-4(0-6)$

$= 8+2+24 = 34$

$[\bar{b}\ \bar{c}\ \bar{d}]\,\bar{a} = 34(\bar{i}-\bar{j})$

$= 34\bar{i}-34\bar{j}$

$[\bar{a}\ \bar{c}\ \bar{d}]\,\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\,\bar{a} = 10\bar{i}-10\bar{j}-40\bar{k} - (34\bar{i}-34\bar{j})$

$= -24\bar{i}+24\bar{j}-40\bar{k}$ — (3)

From (1) and (3) we get

$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a}\ \bar{c}\ \bar{d}]\,\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\,\bar{a}$.

——∝——

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EXERCISE 6.3

1. If $\bar{a} = \bar{i} - 2\bar{j} + 3\bar{k}$, $\bar{b} = 2\bar{i} + \bar{j} - 2\bar{k}$ $\bar{c} = 3\bar{i} + 2\bar{j} + \bar{k}$. find

(i) $(\bar{a} \times \bar{b}) \times \bar{c}$ (ii) $\bar{a} \times (\bar{b} \times \bar{c})$

Given $\bar{a} = \bar{i} - 2\bar{j} + 3\bar{k}$

$\bar{b} = 2\bar{i} + \bar{j} - 2\bar{k}$

$\bar{c} = 3\bar{i} + 2\bar{j} + \bar{k}$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -2 & 3 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= \bar{i}(4-3) - \bar{j}(-2-6) + \bar{k}(1+4)$$

$$= \bar{i} + 8\bar{j} + 5\bar{k}$$

$$(\bar{a} \times \bar{b}) \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 8 & 5 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= \bar{i}(8-10) - \bar{j}(1-15) + \bar{k}(2-24)$$

$$= -2\bar{i} + 14\bar{j} - 22\bar{k}$$

(ii) $\bar{a} \times (\bar{b} \times \bar{c})$

$$\bar{b} \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= \bar{i}(1+4) - \bar{j}(2+6) + \bar{k}(4-3)$$

$$= 5\bar{i} - 8\bar{j} + \bar{k}$$

$$\bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -2 & 3 \\ 5 & -8 & 1 \end{vmatrix}$$

$$= \bar{i}(-2+24) - \bar{j}(1-15) + \bar{k}(-8+10)$$

$$= 22\bar{i} + 14\bar{j} + 2\bar{k}$$

---

2) For any vector $\bar{a}$, prove that

$\bar{i} \times (\bar{a} \times \bar{i}) + \bar{j} \times (\bar{a} \times \bar{j}) + \bar{k} \times (\bar{a} \times \bar{k}) = 2\bar{a}$.

Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$$\bar{i} \times (\bar{a} \times \bar{i}) = (\bar{i} \cdot \bar{i})\bar{a} - (\bar{i} \cdot \bar{a})\bar{i}$$

$$= \bar{a} - a_1\bar{i} \quad \text{—— (1)}$$

Similarly $\bar{j} \times (\bar{a} \times \bar{j}) = \bar{a} - a_2\bar{j}$ —— (2)

$\bar{k} \times (\bar{a} \times \bar{k}) = \bar{a} - a_3\bar{k}$ — (3)

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① + ② + ③ ⇒

$\bar{i} \times (\bar{a} \times \bar{i}) + \bar{j} \times (\bar{a} \times \bar{j}) + \bar{k} \times (\bar{a} \times \bar{k})$

$= 3\bar{a} - (a_1\bar{i} + a_2\bar{j} + a_3\bar{k})$

$= 3\bar{a} - \bar{a}$

$= 2\bar{a}$

---

3) prove that $[\bar{a}-\bar{b}, \bar{b}-\bar{c}, \bar{c}-\bar{a}] = 0$

$[\bar{a}-\bar{b}, \bar{b}-\bar{c}, \bar{c}-\bar{a}]$

$= (\bar{a}-\bar{b}) \cdot [(\bar{b}-\bar{c}) \times (\bar{c}-\bar{a})]$

$= (\bar{a}-\bar{b}) \cdot [\bar{b}\times\bar{c} - \bar{b}\times\bar{a} - \bar{c}\times\bar{c} + \bar{c}\times\bar{a}]$

$= (\bar{a}-\bar{b}) \cdot [\bar{b}\times\bar{c} - \bar{b}\times\bar{a} + \bar{c}\times\bar{a}] \quad \because \bar{c}\times\bar{c} = 0$

$= \bar{a}\cdot(\bar{b}\times\bar{c}) - \bar{a}\cdot(\bar{b}\times\bar{a}) + \bar{a}\cdot(\bar{c}\times\bar{a})$

$- \bar{b}\cdot(\bar{b}\times\bar{c}) + \bar{b}\cdot(\bar{b}\times\bar{a}) - \bar{b}\cdot(\bar{c}\times\bar{a})$

$= [\bar{a}, \bar{b}, \bar{c}] - 0 + 0 - 0 + 0 - [\bar{a}\,\bar{b}\,\bar{c}]$

$= 0$

Hence proved.

4) If $\bar{a} = 2\bar{i} + 3\bar{j} - \bar{k}$, $\bar{b} = 3\bar{i} + 5\bar{j} + 2\bar{k}$, $\bar{c} = -\bar{i} - 2\bar{j} + 3\bar{k}$

Verify that

a) $(\bar{a}\times\bar{b})\times\bar{c} = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{b}\cdot\bar{c})\bar{a}$

b) $\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$.

Given $\bar{a} = 2\bar{i} + 3\bar{j} - \bar{k}$

$\bar{b} = 3\bar{i} + 5\bar{j} + 2\bar{k}$

$\bar{c} = -\bar{i} - 2\bar{j} + 3\bar{k}$

a) $\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 3 & -1 \\ 3 & 5 & 2 \end{vmatrix}$

$= \bar{i}(6+5) - \bar{j}(4+3) + \bar{k}(10-9)$

$= 11\bar{i} - 7\bar{j} + \bar{k}$

$(\bar{a}\times\bar{b})\times\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 11 & -7 & 1 \\ -1 & -2 & 3 \end{vmatrix}$

$= \bar{i}(-21+2) - \bar{j}(33+1) + \bar{k}(-22-7)$

$= -19\bar{i} - 34\bar{j} - 29\bar{k}$ ——— ①

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$\bar{a}\cdot\bar{b} = (2\bar{i}+3\bar{j}-\bar{k})\cdot(-\bar{i}-2\bar{j}+3\bar{k})$

$= -2-6-3 = -11$

$(\bar{a}\cdot\bar{c})\bar{b} = -11(3\bar{i}+5\bar{j}+2\bar{k})$

$= -33\bar{i}-55\bar{j}-22\bar{k}$

$\bar{b}\cdot\bar{c} = (3\bar{i}+5\bar{j}+2\bar{k})\cdot(-\bar{i}-2\bar{j}+3\bar{k})$

$= -3-10+6 = -7$

$(\bar{b}\cdot\bar{c})\bar{a} = -7(2\bar{i}+3\bar{j}-\bar{k})$

$= -14\bar{i}-21\bar{j}+7\bar{k}$

$(\bar{a}\cdot\bar{c})\bar{b}-(\bar{b}\cdot\bar{c})\bar{a} = -33\bar{i}-55\bar{j}-22\bar{k} - (-14\bar{i}-21\bar{j}+7\bar{k})$

$= -19\bar{i}-34\bar{j}-29\bar{k}$ ——— ②

From ① and ②.

$(\bar{a}\times\bar{b})\times\bar{c} = (\bar{a}\cdot\bar{c})\bar{b}-(\bar{b}\cdot\bar{c})\bar{a}$

verified.

b) $(\bar{a}\cdot\bar{c})\bar{b} = -33\bar{i}-55\bar{j}-22\bar{k}$

$\bar{a}\cdot\bar{b} = (2\bar{i}+3\bar{j}-\bar{k})\cdot(3\bar{i}+5\bar{j}+2\bar{k})$

$= 6+15-2 = 19$

$(\bar{a}\cdot\bar{b})\bar{c} = 19(-\bar{i}-2\bar{j}+3\bar{k})$

$= -19\bar{i}-38\bar{j}+57\bar{k}$

$(\bar{a}\cdot\bar{c})\bar{b}-(\bar{a}\cdot\bar{b})\bar{c} = (-33\bar{i}-55\bar{j}-22\bar{k}) - (-19\bar{i}-38\bar{j}+57\bar{k})$

$= -14\bar{i}-17\bar{j}-79\bar{k}$ ——— ③

$\bar{b}\times\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & 5 & 2 \\ -1 & -2 & 3 \end{vmatrix}$

$= \bar{i}(15+4)-\bar{j}(9+2)+\bar{k}(-6+5)$

$= 19\bar{i}-11\bar{j}-\bar{k}$

$\bar{a}\times(\bar{b}\times\bar{c}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 3 & -1 \\ 19 & -11 & -1 \end{vmatrix}$

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$= \bar{i}(-3-11) - \bar{j}(-2+19) + \bar{k}(-22-57)$

$= -14\bar{i} - 17\bar{j} - 79\bar{k}$ —— (4)

From (3) and (4)

$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$

verified.

---

5) $\bar{a} = 2\bar{i} + 3\bar{j} - \bar{k}$, $\bar{b} = -\bar{i} + 2\bar{j} - 4\bar{k}$

$\bar{c} = \bar{i} + \bar{j} + \bar{k}$ Then find the value of $(\bar{a} \times \bar{b}) \cdot (\bar{a} \times \bar{c})$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 3 & -1 \\ -1 & 2 & -4 \end{vmatrix}$$

$= \bar{i}(-12+2) - \bar{j}(-8-1) + \bar{k}(4+3)$

$= -10\bar{i} + 9\bar{j} + 7\bar{k}$

$$\bar{a} \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix}$$

$= \bar{i}(3+1) - \bar{j}(2+1) + \bar{k}(2-3)$

$= 4\bar{i} - 3\bar{j} - \bar{k}$.

$(\bar{a} \times \bar{b}) \cdot (\bar{a} \times \bar{c})$

$= (-10\bar{i} + 9\bar{j} + 7\bar{k}) \cdot (4\bar{i} - 3\bar{j} - \bar{k})$

$= -40 - 27 - 7 = \boxed{-74}$

---

6) If $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are coplanar vectors, show that

$(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) = 0$.

$(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) = [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{d}$

Given that $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are coplanar.

$\therefore [\bar{a}\ \bar{b}\ \bar{d}] = 0$ and $[\bar{a}\ \bar{b}\ \bar{c}] = 0$.

$\therefore (\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) = 0\cdot\bar{c} - 0\cdot\bar{d}$

$= 0$.

Hence proved.

---

7)

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7) If $\bar{a} = \bar{i} + 2\bar{j} + 3\bar{k}$, $\bar{b} = 2\bar{i} - \bar{j} + \bar{k}$ $\bar{c} = 3\bar{i} + 2\bar{j} + \bar{k}$ and $\bar{a} \times (\bar{b} \times \bar{c}) = l\bar{a} + m\bar{b} + n\bar{c}$ find the value of l, m, n.

Given $\bar{a} = \bar{i} + 2\bar{j} + 3\bar{k}$, $\bar{b} = 2\bar{i} - \bar{j} + \bar{k}$

$\bar{c} = 3\bar{i} + 2\bar{j} + \bar{k}$.

$$\bar{b} \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & -1 & 1 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= \bar{i}(-1-2) - \bar{j}(2-3) + \bar{k}(4+3)$$

$$= -3\bar{i} + \bar{j} + 7\bar{k}$$

$$\bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 2 & 3 \\ -3 & 1 & 7 \end{vmatrix}$$

$$= \bar{i}(14-3) - \bar{j}(7+9) + \bar{k}(1+6)$$

$$= 11\bar{i} - 16\bar{j} + 7\bar{k} \quad \text{——— (1)}$$

$$l\bar{a} + m\bar{b} + n\bar{c} = l(\bar{i} + 2\bar{j} + 3\bar{k}) + m(2\bar{i} - \bar{j} + \bar{k}) + n(3\bar{i} + 2\bar{j} + \bar{k})$$

$$= l\bar{i} + 2l\bar{j} + 3l\bar{k} + 2m\bar{i} - m\bar{j} + m\bar{k} + 3n\bar{i} + 2n\bar{j} + n\bar{k}$$

$$= (l + 2m + 3n)\bar{i} + (2l - m + 2n)\bar{j} + (3l + m + n)\bar{k} \quad \text{——— (2)}$$

Given $\bar{a} \times (\bar{b} \times \bar{c}) = l\bar{a} + m\bar{b} + n\bar{c}$

i.e. $11\bar{i} - 16\bar{j} + 7\bar{k} = (l + 2m + 3n)\bar{i} + (2l - m + 2n)\bar{j} + (3l + m + n)\bar{k}$

$\Rightarrow$ $l + 2m + 3n = 11$ ——— (3)

$2l - m + 2n = -16$ ——— (4)

$3l + m + n = 7$ ——— (5)

(4) + (5) $\Rightarrow$ $5l + 3n = -9$ ——— (6)

(4) $\times 2$ $\Rightarrow$ $4l - 2m + 4n = -32$ ——— (7)

(3) + (7) $\Rightarrow$ $5l + 7n = -21$ ——— (8)

$5l + 3n = -9$ ——— (6)

(8) − (6) $\Rightarrow$ $4n = -12$

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$n = \frac{-12}{4} = 3 \quad \therefore \boxed{n = -3}$

Sub $n = -3$ in (6)

$5l + 3(-3) = -9$

$5l - 9 = -9$

$5l = -9 + 9 = 0$

$\boxed{l = 0}$

Sub $l = 0$, $n = -3$ in (1)

$0 + 2m + 3(-3) = 11$

$2m - 9 = 11$

$2m = 20 \Rightarrow \boxed{m = 10}$

$\therefore \boxed{l = 0}, \boxed{m = 10}, \boxed{n = -3}$

---

8) If $\hat{a}, \hat{b}, \hat{c}$ are three unit vectors such that $\hat{b}$ and $\hat{c}$ are non-parallel and $\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b}$. find the angle betn $\hat{a}$ and $\hat{c}$.

Given $\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b}$

i $(\hat{a} \cdot \hat{c})\hat{b} - (\hat{a} \cdot \hat{b})\hat{c} = \frac{1}{2}\hat{b} - 0\hat{c}$

$\Rightarrow \hat{a} \cdot \hat{c} = \frac{1}{2}$

i $|\hat{a}||\hat{c}| \cos\theta = \frac{1}{2}$

$\Rightarrow 1.1. \cos\theta = \frac{1}{2}$

$\cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \pi/3$

$\therefore$ angle betn $\hat{a}$ and $\hat{c}$ is $\theta = \pi/3$

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Application of vectors to 3-Dimensional Geometry.

Note (1) The vector equation of a straight line passing through a fixed point with position vector $\bar{a}$ and parallel to given vector $\bar{b}$ is $\boxed{\bar{r} = \bar{a} + t\bar{b}}$.

where $\bar{a} = x_1\bar{i} + y_1\bar{j} + z_1\bar{k}$

and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

2) Cartesian eqn of the st line passing through the point $A(x_1, y_1, z_1)$ and parallel to the line with direction ratios (b_1, b_2, b_3) is

$$\frac{x - x_1}{b_1} = \frac{y - y_1}{b_2} = \frac{z - z_1}{b_3}$$

Non parametric form of vector eqn of st line passing through $\bar{a}$ and ll to $\bar{b}$ is $(\bar{r} - \bar{a}) \times \bar{b} = 0$.

3) If l, m, n are direction cosine of the line, then the cartesian eqn of the line are $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$.

Straight line passing through two given points:-

Note: 1) Vector eqn of a line passing through two given points whose position vectors are $\bar{a}$ and $\bar{b}$ respectively is $\boxed{\bar{r} = \bar{a} + t(\bar{b} - \bar{a})}$

2) The non parametric form of vector eqn of st line passing through two given points with position vectors $\bar{a}$ and $\bar{b}$ is

$$(\bar{r} - \bar{a}) \times (\bar{r} - \bar{b}) = \bar{0}.$$

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3) The cartesian eqn of st line passing through the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Eg 6.24 A straight line passes through the points $(1, 2, -3)$ and parallel to $4\bar{i}+5\bar{j}-7\bar{k}$. Find

(i) Vector eqn in parametric form
(ii) Vector eqn in non-parametric form
(iii) Cartesian equations of st line.

Here $\bar{a} = \bar{i}+2\bar{j}-3\bar{k}$

parallel vector $\bar{b} = 4\bar{i}+5\bar{j}-7\bar{k}$

(i) parametric form of vector eqn of st line is $\bar{r} = \bar{a} + t\bar{b}$

$\therefore \bar{r} = \bar{i}+2\bar{j}-3\bar{k} + t(4\bar{i}+5\bar{j}-7\bar{k})$

(ii) Non parametric form of vector eqn of st line is

$$(\bar{r}-\bar{a}) \times \bar{b} = 0.$$

$$\therefore \left[\bar{r}-(\bar{i}+2\bar{j}-3\bar{k})\right] \times \left[4\bar{i}+5\bar{j}-7\bar{k}\right] = \bar{0}.$$

(iii) Cartesian eqn of the st line is $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$

Here $(x_1, y_1, z_1) = (1, 2, -3)$

$(b_1, b_2, b_3) = (4, 5, -7)$

$\therefore$ The required Cartesian eqn is

$$\frac{x-1}{4} = \frac{y-2}{5} = \frac{z+3}{-7}$$

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3) The cartesian eqn of st line passing through the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Eg 6.24 A straight line passes through the points $(1, 2, -3)$ and parallel to $4\bar{i}+5\bar{j}-7\bar{k}$. Find

(i) Vector eqn in parametric form
(ii) Vector eqn in non-parametric form
(iii) Cartesian equations of st line.

Here $\bar{a} = \bar{i}+2\bar{j}-3\bar{k}$

parallel vector $\bar{b} = 4\bar{i}+5\bar{j}-7\bar{k}$

(i) Parametric form of vector eqn of st line is $\bar{r} = \bar{a}+t\bar{b}$

$\therefore \bar{r} = \bar{i}+2\bar{j}-3\bar{k}+t(4\bar{i}+5\bar{j}-7\bar{k})$

(ii) Non parametric form of vector eqn of st line is

$(\bar{r}-\bar{a}) \times \bar{b} = 0$.

$\therefore [\bar{r}-(\bar{i}+2\bar{j}-3\bar{k})] \times [4\bar{i}+5\bar{j}-7\bar{k}] = \bar{0}$.

(iii) Cartesian eqn of the st line is $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$

Here $(x_1, y_1, z_1) = (1, 2, -3)$

$(b_1, b_2, b_3) = (4, 5, -7)$

$\therefore$ The required Cartesian eqn is

$$\frac{x-1}{4} = \frac{y-2}{5} = \frac{z+3}{-7}$$

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Ex 6.25 The vector eqn in parametric form of a line is $\bar{r} = (3\bar{i} - 2\bar{j} + 6\bar{k}) + t(2\bar{i} - \bar{j} + 3\bar{k})$

Find a) the direction cosines of the st line

b) Vector eqn in non-parametric form of the line.

c) Cartesian eqn of the line.

Given

parametric form of vector eqn of st line is

$\bar{r} = (3\bar{i} - 2\bar{j} + 6\bar{k}) + t(2\bar{i} - \bar{j} + 3\bar{k})$

Compare this eqn with

$\bar{r} = \bar{a} + t\bar{b}$.

Here $\bar{a} = 3\bar{i} - 2\bar{j} + 6\bar{k}$

parallel vector $\bar{b} = 2\bar{i} - \bar{j} + 3\bar{k}$.

∴ Direction ratios of the given line is $(2, -1, 3)$.

∴ Direction cosine of the given line is

$$\left(\frac{2}{\sqrt{4+1+9}}, \frac{-1}{\sqrt{4+1+9}}, \frac{3}{\sqrt{4+1+9}}\right)$$

$$\left(\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$$

(ii) non parametric form of vector eqn is $(\bar{r} - \bar{a}) \times \bar{b} = 0$

$$\left[\bar{r} - (3\bar{i} - 2\bar{j} + 6\bar{k})\right] \times \left[2\bar{i} - \bar{j} + 3\bar{k}\right] = \bar{0}.$$

(iii) Cartesian eqn of st line is

$$\frac{x - x_1}{b_1} = \frac{y - y_1}{b_2} = \frac{z - z_1}{b_3}$$

Here $(x_1, y_1, z_1) = (3, -2, 6)$

$(b_1\ b_2\ b_3) = (2, -1, 3)$

∴ The required cartesian eqn is

$$\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z-6}{3}$$

———— x ————

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6.26 Find the vector eqn in parametric form and Cartesian eqns of the line passing through $(-4,2,-3)$ and is parallel to the line

$$-\frac{x-2}{4} = \frac{y+3}{-2} = \frac{2z-6}{3}$$

Here $\bar{a} = -4\bar{i}+2\bar{j}-3\bar{k}$

The given st line is

$$-\frac{(x+2)}{4} = \frac{y+3}{-2} = \frac{2(z-3)}{3}$$

i.e. $$\frac{x+2}{-4} = \frac{y+3}{-2} = \frac{z-3}{3/2}$$

parallel vector to the line is

$$\bar{b} = -4\bar{i}-2\bar{j}+\frac{3}{2}\bar{k}$$

parametric form of vector eqn is

$$\bar{r} = \bar{a}+t\bar{b}$$

i.e. $$\bar{r} = (-4\bar{i}+2\bar{j}-3\bar{k}) + t\left(-4\bar{i}-2\bar{j}+\frac{3}{2}\bar{k}\right)$$

$$= (-4\bar{i}+2\bar{j}-3\bar{k}) + \frac{-t}{2}(+8\bar{i}+4\bar{j}-3\bar{k})$$

$$= (-4\bar{i}+2\bar{j}-3\bar{k}) + t_1(+8\bar{i}+4\bar{j}-3\bar{k})$$

where $t_1 = -t/2$.

Cartesian eqn of the st line is

$$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$$

Here $(x_1, y_1, z_1) = (-4, 2, -3)$

$(b_1, b_2, b_3) = (8, 4, -3)$

∴ The required Cartesian eqn is

$$\frac{x+4}{8} = \frac{y-2}{4} = \frac{z+3}{-3}$$

6.28 Find the angle betn the straight line $\frac{x+3}{2} = \frac{y-1}{2} = -z$ with coordinate axes.

The eqn of given line is

$$\frac{x+3}{2} = \frac{y-1}{2} = \frac{z}{-1} \quad \text{——} (1)$$

∴ The parallel vector to the given line is $\bar{b} = 2\bar{i}+2\bar{j}-\bar{k}$.

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unit vector along $\bar{b}$ is

$$\hat{b} = \frac{\bar{b}}{|\bar{b}|} = \frac{2\bar{i}+2\bar{j}-\bar{k}}{\sqrt{4+4+1}}$$

$$= \frac{1}{3}(2\bar{i}+2\bar{j}-\bar{k})$$

The vector $\hat{b}$ makes angles α, β, γ with the coordinate axes.

∴ Direction cosine of $\hat{b}$ are

$$\cos\alpha = \frac{2}{3}, \quad \cos\beta = \frac{2}{3}, \quad \cos\gamma = -\frac{1}{3}$$

∴ The angle made by the given line ① with coordinate axes are

$$\alpha = \cos^{-1}\left(\frac{2}{3}\right)$$

$$\beta = \cos^{-1}\left(\frac{2}{3}\right)$$

$$\gamma = \cos^{-1}\left(-\frac{1}{3}\right)$$

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Eg 6.27 Find the vector equation in parametric form and cartesian equations of a st line passing through the points $(-5,7,-4)$ and $(13,-5,2)$. Find the point where the st line crosses the xy-plane.

Given $\bar{a} = -5\bar{i} + 7\bar{j} - 4\bar{k}$

$\bar{b} = 13\bar{i} - 5\bar{j} + 2\bar{k}$

$\bar{b} - \bar{a} = 18\bar{i} - 12\bar{j} + 6\bar{k}$

$= 3\bar{i} - 2\bar{j} + \bar{k}$

∴ parametric form of vector eqn of the line is

$\bar{r} = \bar{a} + t(\bar{b} - \bar{a})$

$\bar{r} = (-5\bar{i} + 7\bar{j} - 4\bar{k}) + t(3\bar{i} - 2\bar{j} + \bar{k})$

Cartesian eqn of the st line is

$\frac{x+5}{3} = \frac{y-7}{-2} = \frac{z+4}{1}$

let $\frac{x+5}{3} = \frac{y-7}{-2} = \frac{z+4}{1} = t$ (say)

any point on this line is

$(3t-5, -2t+7, t-4)$

If the line cuts the xy plane

∴ z coordinate of point of intersection is zero.

i.e. $t - 4 = 0 \Rightarrow \boxed{t = 4}$

Hence the st line cuts the xy-plane at $(7, -1, 0)$

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Angle betn two st lines:-

Angle betn two st lines $\bar{r} = \bar{a} + t\bar{b}$ and $\bar{r} = \bar{c} + s\bar{d}$ is same as that of the angle between $\bar{b}$ and $\bar{d}$.

So $\cos\theta = \left(\frac{|\bar{b}\cdot\bar{d}|}{|\bar{b}||\bar{d}|} \right)$

Note:

1) Two given lines $\bar{r} = \bar{a} + t\bar{b}$ and $\bar{r} = \bar{c} + s\bar{d}$ are parallel iff $\bar{b} = \lambda\bar{d}$ for some λ.

2) Two given lines $\bar{r} = \bar{a} + t\bar{b}$ and $\bar{r} = \bar{c} + s\bar{d}$ are perpendicular iff $\bar{b}\cdot\bar{d} = 0$.

Cartesian form

If two lines are given in Cartesian form as

$$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3} \text{ and}$$

$$\frac{x-x_2}{d_1} = \frac{y-y_2}{d_2} = \frac{z-z_1}{d_3}$$

Then the angle betn the given lines is

$$\theta = \cos^{-1}\left[\frac{|b_1d_1 + b_2d_2 + b_3d_3|}{\sqrt{b_1^2+b_2^2+b_3^2}\ \sqrt{d_1^2+d_2^2+d_3^2}} \right]$$

Note: 1) The two given lines with direction ratios (b_1, b_2, b_3) and (d_1, d_2, d_3) are parallel iff $\frac{b_1}{d_1} = \frac{b_2}{d_2} = \frac{b_3}{d_3}$

2) Two lines with direction ratios (b_1, b_2, b_3) and (d_1, d_2, d_3) are perpendicular iff $b_1d_1 + b_2d_2 + b_3d_3 = 0$.

3) If the direction cosines of two given st lines are (l_1, m_1, n_1) and (l_2, m_2, n_2) then the angle betn the two st lines $\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$

4) Direction ratios of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ and its direction cosines are

$$\frac{x_2-x_1}{|\bar{PQ}|}, \frac{y_2-y_1}{|PQ|}, \frac{z_2-z_1}{|PQ|}$$

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6.29 Find the acute angle betn the lines $\bar{r} = (\bar{i}+2\bar{j}+4\bar{k}) + t(2\bar{i}+2\bar{j}+\bar{k})$ and the straight line passing through the points (5, 1, 4) and (9, 2, 12).

Given line is $\bar{r} = (\bar{i}+2\bar{j}+4\bar{k}) + t(2\bar{i}+2\bar{j}+\bar{k})$

parallel vector to this line is

$\bar{b} = 2\bar{i}+2\bar{j}+\bar{k}$.

Direction ratios of line joining the two given points A(5, 1, 4) and B(9, 2, 12) are

$(9-5, 2-1, 12-4) = (4, 1, 8)$

∴ The parallel vector to the line AB is $\bar{d} = 4\bar{i}+\bar{j}+8\bar{k}$.

∴ The angle betn two given st lines

$$\theta = \cos^{-1}\left[\frac{|\bar{b}\cdot\bar{d}|}{|\bar{b}||\bar{d}|}\right]$$

$$\theta = \cos^{-1}\left[\frac{|(2\bar{i}+2\bar{j}+\bar{k})\cdot(4\bar{i}+\bar{j}+8\bar{k})|}{\sqrt{4+4+1}\ \sqrt{16+1+64}}\right]$$

$$\theta = \cos^{-1}\left[\frac{8+2+8}{\sqrt{9}\ \sqrt{81}}\right]$$

$$= \cos^{-1}\left[\frac{18}{3\times 9}\right] = \cos^{-1}\left[\frac{2}{3}\right]$$

Ex 6.30 Find the acute angle betn the st lines $\frac{x-4}{2} = \frac{y}{1} = \frac{z+1}{-2}$ and $\frac{x-1}{4} = \frac{y+1}{1} = \frac{z-2}{2}$ and state whether they are parallel or perpendicular.

Given lines are

$\frac{x-4}{2} = \frac{y}{1} = \frac{z+1}{-2}$ and

$\frac{x-1}{4} = \frac{y+1}{-4} = \frac{z-2}{2}$

parallel vectors to the given lines are

$\bar{b} = 2\bar{i}+\bar{j}-2\bar{k}$ and

$\bar{d} = 4\bar{i}-4\bar{j}+2\bar{k}$

Let θ be the angle betn the lines

$$\cos\theta = \left[\frac{|\bar{b}\cdot\bar{d}|}{|\bar{b}||\bar{d}|}\right]$$

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$$\cos\theta = \left(\frac{2(4)+1(-4)-2(2)}{\sqrt{4+1+4}\ \sqrt{16+16+4}}\right)$$

$$= \left(\frac{8-4-4}{\sqrt{9}\ \sqrt{36}}\right) = \frac{0}{3\times 6} = 0$$

$\cos\theta = 0 \Rightarrow \theta = \pi/2$

∴ The two st lines are perpendicular.

Ex 6.31 Show that the st line passing through the points A(6,7,5) and B(8,10,6) is perpendicular to the st line passing through the points C(10,2,-5) and D(8,3,-4)

The direction ratios of the line joining the points A(6,7,5) and B(8,10,6) are

$8-6,\ 10-7,\ 6-5$

i.e. $(2,\ 3,\ 1)$

∴ The parallel vector to the line AB is $\bar{b} = 2\bar{i}+3\bar{j}+\bar{k}$.

The direction ratios of the line joining the points C(10,2,-5) and D(8,3,-4) are $(8-10,\ 3-2,\ -4+5)$

i.e. $(-2,\ 1,\ 1)$

∴ The parallel vector to the line CD is $\bar{d} = -2\bar{i}+\bar{j}+\bar{k}$.

Let θ be the angle betn the lines.

$$\cos\theta = \left[\frac{|\bar{b}\cdot\bar{d}|}{|\bar{b}||\bar{d}|}\right] = \left[\frac{-4+3+1}{\sqrt{4+9+1}\ \sqrt{4+1+1}}\right]$$

$$= \left(\frac{0}{\sqrt{14}\sqrt{6}}\right) = 0$$

$\cos\theta = 0 \Rightarrow \theta = \pi/2$

Hence the given lines are perpendicular.

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Ex 6.32 Show that the lines

$\frac{x-1}{4} = \frac{2-y}{6} = \frac{z-4}{12}$ and

$\frac{x-3}{-2} = \frac{y-3}{3} = \frac{5-z}{6}$ are parallel.

The given line is

$$\frac{x-1}{4} = \frac{2-y}{6} = \frac{z-4}{12}$$

$$\frac{x-1}{4} = \frac{-(y-2)}{6} = \frac{z-4}{12}$$

i.e. $\frac{x-1}{4} = \frac{y-2}{-6} = \frac{z-4}{12}$

The parallel vector to this line is

$\bar{b} = 4\bar{i} - 6\bar{j} + 12\bar{k}$.

and the given line is

$$\frac{x-3}{-2} = \frac{y-3}{3} = \frac{5-z}{6}$$

i.e. $\frac{x-3}{-2} = \frac{y-3}{3} = \frac{-(z-5)}{6}$

i.e. $\frac{x-3}{-2} = \frac{y-3}{3} = \frac{z-5}{-6}$

$\therefore$ The parallel vector to this line is $\bar{d} = -2\bar{i} + 3\bar{j} - 6\bar{k}$

Now, $\bar{b} = 4\bar{i} - 6\bar{j} + 12\bar{k}$

$= -2(-2\bar{i} + 3\bar{j} - 6\bar{k})$

$= -2\bar{d}$

$\bar{b} = -2\bar{d} \Rightarrow \bar{b}$ and $\bar{d}$ are parallel.

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EXERCISE 6.4

1) Find the non-parametric form of vector eqn and Cartesian eqns of the st line passing through the point with PV $4\bar{i}+3\bar{j}-7\bar{k}$ and parallel to the vector $2\bar{i}-6\bar{j}+7\bar{k}$.

Here $\bar{a} = 4\bar{i}+3\bar{j}-7\bar{k}$

parallel vector $\bar{b} = 2\bar{i}-6\bar{j}+7\bar{k}$

a) parametric form of vector eqn $\bar{r} = \bar{a}+t\bar{b}$

i.e. $\bar{r} = (4\bar{i}+3\bar{j}-7\bar{k})+t(2\bar{i}-6\bar{j}+7\bar{k})$

b) Cartesian eqn of the st line is $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$

Here $(x_1, y_1, z_1) = (4, 3, -7)$

$(b_1, b_2, b_3) = (2, -6, 7)$

∴ The required Cartesian eqn of st line is

$$\frac{x-4}{2} = \frac{y-3}{-6} = \frac{z+7}{7} \quad //$$

2) Find the parametric form of vector and Cartesian eqns of the st line passing through the point $(-2, 3, 4)$ and parallel to the st line $\frac{x-1}{-4} = \frac{y+3}{5} = \frac{8-z}{6}$

The given st line is $\frac{x-1}{-4} = \frac{y+3}{5} = \frac{z-8}{-6}$

parallel vector to the line is $\bar{b} = -4\bar{i}+5\bar{j}-6\bar{k}$.

Here $\bar{a} = -2\bar{i}+3\bar{j}+4\bar{k}$

∴ a) parametric form of vector eqn is $\bar{r} = \bar{a}+t\bar{b}$

i.e. $\bar{r} = (-2\bar{i}+3\bar{j}+4\bar{k})+t(-4\bar{i}+5\bar{j}-6\bar{k})$

Non parametric form of vector eqn is $(\bar{r}-\bar{a}) \times \bar{b} = 0$

i.e. $(\bar{r}-(4\bar{i}+3\bar{j}-7\bar{k})) \times (2\bar{i}-6\bar{j}+7\bar{k}) = 0$

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Cartesian eqn of st line is

$$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$$

Here $(x_1, y_1, z_1) = (-2, 3, 4)$

$(b_1, b_2 b_3) = (-4, 5, -6)$

∴ The required Cartesian eqn is $\frac{x+2}{-4} = \frac{y-3}{5} = \frac{z-4}{-6}$

---

4) Find the direction cosines of the st line passing through the points $(5, 6, 7)$ and $(7, 9, 13)$. Also find the parametric form of vector eqn and Cartesian equations of the st line passing through two given pts.

Here $\bar{a} = 5\bar{i} + 6\bar{j} + 7\bar{k}$

$\bar{b} = 7\bar{i} + 9\bar{j} + 13\bar{k}$

a) parametric form of vector eqn is $\bar{r} = \bar{a} + t(b - \bar{a})$

$\Rightarrow \bar{r} = (5\bar{i} + 6\bar{j} + 7\bar{k}) + t(2\bar{i} + 3\bar{j} + 6\bar{k})$

Cartesian eqn of the st line passing through two given pts is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Here $(x_1, y_1, z_1) = (5, 6, 7)$

$(x_2 y_2 z_2) = (7, 9, 13)$

∴ The eqn is

$$\frac{x-5}{7-5} = \frac{y-6}{9-6} = \frac{z-7}{13-7}$$

$\Rightarrow \frac{x-5}{2} = \frac{y-6}{3} = \frac{z-7}{6}$

c) Direction ratios are $(2, 3, 6)$

∴ Direction cosines are

$$\left(\frac{2}{\sqrt{2^2+3^2+6^2}}, \frac{3}{\sqrt{2^2+3^2+6^2}}, \frac{6}{\sqrt{2^2+3^2+6^2}}\right)$$

$\Rightarrow \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$ //

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3) Find the point where the st line passes through (6,7,4) and (8,4,9) cuts the xz and yz planes.

Eqn of the line passing through two points (6,7,4) and (8,4,9) is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

i $\frac{x-6}{2} = \frac{y-7}{-3} = \frac{z-4}{5}$

Take $\frac{x-6}{2} = \frac{y-7}{-3} = \frac{z-4}{5} = t$ (say)

Any point on this line is

$(2t+6, -3t+7, 5t+4)$

(i) If the line cuts xz plane, the y-coordinate is zero,

i $-3t+7 = 0$

$-3t = -7 \Rightarrow t = 7/3$

∴ The required point is

$\left(2\left(\frac{7}{3}\right)+6, -3\left(\frac{7}{3}\right)+7, 5\left(\frac{7}{3}\right)+4\right)$

$\left(\frac{14}{3}+6, -7+7, \frac{35}{3}+4\right)$

$\left(\frac{32}{4}, 0, \frac{47}{3}\right)$

(ii) If the line cuts yz plane, x coordinate is zero.

i $2t+6 = 0 \Rightarrow 2t = -6$

$\boxed{t = -3}$

∴ The required point is

$(0, 16, -11)$

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5) Find the acute angle betn the following lines

(i) $\bar{r} = (4\bar{i} - \bar{j}) + t(\bar{i} + 2\bar{j} - 2\bar{k})$,

$\bar{r} = (\bar{i} - 2\bar{j} + 4\bar{k}) + s(-\bar{i} - 2\bar{j} + 2\bar{k})$

The parallel vectors to the given lines are

$\bar{b} = \bar{i} + 2\bar{j} - 2\bar{k}$ and

$\bar{d} = -\bar{i} - 2\bar{j} + 2\bar{k}$.

Let α be the angle betn the given lines

$\therefore \cos\alpha = \left(\frac{|\bar{b}\cdot\bar{d}|}{|\bar{b}||\bar{d}|}\right)$ — (1)

$\bar{b}\cdot\bar{d} = (\bar{i} + 2\bar{j} - 2\bar{k})\cdot(-\bar{i} - 2\bar{j} + 2\bar{k})$

$= -1 - 4 - 4 = -9$

$|\bar{b}| = \sqrt{1+4+4} = \sqrt{9} = 3$

$|\bar{d}| = \sqrt{1+4+4} = \sqrt{9} = 3$

$\therefore \cos\alpha = \left(\frac{|-9|}{3\times 3}\right) = \frac{9}{9} = 1$

$\cos\alpha = 1 \Rightarrow \alpha = 0$.

$\therefore$ The angle betn given lines is 0.

(ii) $\frac{x+4}{3} = \frac{y-7}{4} = \frac{z+5}{5}$, $\bar{r} = 4\bar{k} + t(2\bar{i} + \bar{j} + \bar{k})$

parallel vectors to the given lines are

$\bar{b} = 3\bar{i} + 4\bar{j} + 5\bar{k}$ and

$\bar{d} = 2\bar{i} + \bar{j} + \bar{k}$.

Let α be the angle betn the given lines

$\therefore \cos\alpha = \left(\frac{|\bar{b}\cdot\bar{d}|}{|\bar{b}||\bar{d}|}\right)$

$\bar{b}\cdot\bar{d} = (3\bar{i} + 4\bar{j} + 5\bar{k})\cdot(2\bar{i} + \bar{j} + \bar{k})$

$= 6 + 4 + 5 = 15$

$|\bar{b}| = \sqrt{9+16+25} = \sqrt{50} = 5\sqrt{2}$

$|\bar{d}| = \sqrt{4+1+1} = \sqrt{6}$

$\therefore \cos\alpha = \frac{|15|}{5\sqrt{2}\sqrt{6}} = \frac{15}{5\sqrt{2}\sqrt{6}} = \frac{3}{\sqrt{2}\sqrt{6}}$

$= \frac{\sqrt{3}\sqrt{3}}{\sqrt{2}\sqrt{3}\sqrt{2}}$

$= \frac{\sqrt{3}}{2}$

$\cos\alpha = \sqrt{3}/2 \Rightarrow \alpha = \pi/6$.

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(iii) $2x = 3y = -z$ and $6x = -y = -4z$

8) The given eqns are

$$\frac{x}{1/2} = \frac{y}{1/3} = \frac{z}{-1} \quad \text{and} \quad \frac{x}{1/6} = \frac{y}{-1} = \frac{z}{-1/4}$$

∴ The parallel vectors to the given lines are

$$\bar{b} = \frac{1}{2}\bar{i} + \frac{1}{3}\bar{j} - \bar{k} \quad \text{and}$$

$$\bar{d} = \frac{1}{6}\bar{i} - \bar{j} - \frac{1}{4}\bar{k}.$$

Let θ be the angle betn the lines.

$$\therefore \cos\theta = \left(\frac{|\bar{b}\cdot\bar{d}|}{|\bar{b}||\bar{d}|}\right)$$

$$\bar{b}\cdot\bar{d} = \left(\frac{1}{2}\bar{i} + \frac{1}{3}\bar{j} - \bar{k}\right)\cdot\left(\frac{1}{6}\bar{i} - \bar{j} - \frac{1}{4}\bar{k}\right)$$

$$= \frac{1}{12} - \frac{1}{3} + \frac{1}{4} = \frac{1-4+3}{12} = 0$$

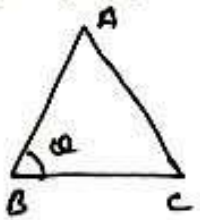
~~$\bar{b}\cdot\bar{d}$~~

$$\therefore \cos\theta = \left(\frac{0}{|\bar{b}||\bar{d}|}\right) = 0$$

$$\Rightarrow \theta = \pi/2$$

∴ The given lines are $\perp^r$.

6) The vertices of ΔABC are $A(7,2,1)$, $B(6,0,3)$ and $C(4,2,4)$. Find $\angle ABC$.



Given vertices are $A(7,2,1)$, $B(6,0,3)$ and $C(4,2,4)$.

The eqn of line joining the points AB is

$$\frac{x-7}{6-7} = \frac{y-2}{0-2} = \frac{z-1}{3-1}$$

i.e. $$\frac{x-7}{-1} = \frac{y-2}{-2} = \frac{z-1}{2} \quad \text{—— (1)}$$

The eqn of line joining the points BC is $\frac{x-6}{4-6} = \frac{y-0}{2-0} = \frac{z-3}{4-3}$

i.e. $$\frac{x-6}{-2} = \frac{y}{2} = \frac{z-3}{1} \quad \text{—— (2)}$$

∴ The parallel vectors to the given lines are

$$\bar{b} = -\bar{i} - 2\bar{j} + 2\bar{k} \quad \text{and} \quad \bar{d} = -2\bar{i} + 2\bar{j} + \bar{k}$$

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Let θ be the angle betn the lines

$$\therefore \cos\theta = \left(\frac{|\bar{b}\cdot\bar{d}|}{|\bar{b}||\bar{d}|}\right)$$

$$\bar{b}\cdot\bar{d} = (-\bar{i}-2\bar{j}+2\bar{k})\cdot(-2\bar{i}+2\bar{j}+\bar{k})$$

$$= 2-4+2 = 0.$$

$$\therefore \cos\theta = \frac{0}{|b||\bar{d}|} = 0$$

$$\cos\theta = 0 \Rightarrow \theta = \pi/2$$

$\therefore$ ~~The given~~ $\angle ABC = 90^\circ$

---

7) If the st line joining the points $(2,1,4)$ and $(a-1,4,-1)$ is parallel to the line joining the points $(0,2,b-1)$ and $(5,3,-2)$. find a and b.

The eqn of line joing the pts $(2,1,4)$ and $(a-1,4,-1)$ is

$$\frac{x-2}{a-1-2} = \frac{y-1}{4-1} = \frac{z-4}{-1-4}$$

i.e. $$\frac{x-2}{a-3} = \frac{y-1}{3} = \frac{z-4}{-5} \quad \text{---- (1)}$$

The eqn of line joing the points $(0,2,b-1)$ and $(5,3,-2)$. is

$$\frac{x-0}{5-0} = \frac{y-2}{3-2} = \frac{z-(b-1)}{-2-(b-1)}$$

i.e. $$\frac{x}{5} = \frac{y-2}{1} = \frac{z-b+1}{-2-b+1}$$

$$\frac{x}{5} = \frac{y-2}{1} = \frac{z-b+1}{-b-1} \quad \text{---- (2)}$$

Given that lines (1) and (2) are parallel

$$\Rightarrow \frac{a_1}{d_1} = \frac{b_1}{d_1} = \frac{b_3}{d_3}$$

i.e. $$\frac{a-3}{5} = \frac{3}{1} = \frac{-5}{-b-1}$$

$$\frac{a-3}{5} = 3 \qquad \frac{-b}{-(b+1)} = 3$$

$$a-3 = 15 \qquad b = 3(b+1)$$

$$a = 18 \qquad b = 3b+3$$

$$3b-b = -3$$

$$2b = -3$$

$$b = -3/2$$

$\therefore a = 18$

$b = -3/2$

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8) If the st lines $\frac{x-5}{5m+2} = \frac{2-y}{5} = \frac{1-z}{-1}$ and $x = \frac{2y+1}{4m} = \frac{1-z}{-3}$ are perpendicular, find 'm'.

Given lines are

$$\frac{x-5}{5m+2} = \frac{y-2}{-5} = \frac{z-1}{1} \quad \text{and}$$

$$x = \frac{y+\frac{1}{2}}{2m} = \frac{z-1}{3}$$

∴ parallel vectors to the given lines are $\bar{b} = (5m+2)\bar{i} - 5\bar{j} + \bar{k}$ and

$$\bar{d} = \bar{i} + 2m\bar{j} + 3\bar{k}$$

Given that they are ⊥r.

∴ $\bar{b} \cdot \bar{d} = 0$

i $(5m+2) - 10m + 3 = 0$

$-5m + 5 = 0$

$-5m = -5 \quad -5m = -5$

$m = 1$

9) Show that the points (2,3,4), (-1,4,5) and (8,1,2) are collinear.

Let the given pts be A(2,3,4), B(-1,4,5) and C(8,1,2).

To prove A, B, C are collinear.

i to prove the point C lies on the line joining the points A and B.

The eqn of line joining the pts A(2,3,4) and B(-1,4,5) is

$$\frac{x-2}{-1-2} = \frac{y-3}{4-3} = \frac{z-4}{5-4}$$

i $\frac{x-2}{-3} = \frac{y-3}{1} = \frac{z-4}{1}$ — (1).

Sub the pt C(8,1,2) in (1)

$$\frac{8-2}{-3} = \frac{1-3}{1} = \frac{2-4}{1}$$

$$\frac{6}{-3} = -2 = -2$$

i $-2, = -2 = -3$

∴ The pt C(8,1,2) lies on the line AB.

∴ The given pts are collinear //

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point of intersection of two st lines:-

If the eqn of st lines are given in vector form, write them in cartesian form and proceed to get the pt of intersection.

Eg 6.33 Find the pt of intersection of the lines

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and

$\frac{x-4}{5} = \frac{y-1}{2} = z$.

The given lines are

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and

$\frac{x-4}{5} = \frac{y-1}{2} = z$.

Let $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = t$ (say)

$\Rightarrow \frac{x-1}{2} = t$ | $\frac{y-2}{3} = t$ | $\frac{z-3}{4} = t$

$x-1 = 2t$ | $y-2 = 3t$ | $z-3 = 4t$

$x = 2t+1$ | $y = 3t+2$ | $z = 4t+3$

$\therefore$ any point on this line is

$(2t+1, 3t+2, 4t+3)$

and let $\frac{x-4}{5} = \frac{y-1}{2} = z = s$ (say)

$\frac{x-4}{5} = s$ | $\frac{y-1}{2} = s$ | $z = s$

$x-4 = 5s$ | $y-1 = 2s$

$x = 5s+4$ | $y = 2s+1$

$\therefore$ any point on this line is

$(5s+4, 2s+1, s)$.

Since the given lines are intersecting for some s and t.

$(2t+1, 3t+2, 4t+3) = (5s+4, 2s+1, s)$

$\Rightarrow 2t+1 = 5s+4$ | $3t+2 = 2s+1$

$2t-5s = 3$ — (1) | $3t-2s = 1-2$

$3t-2s = -1$ — (2)

(1) $\times 3 \Rightarrow 6t-15s = 9$ — (3)

(2) $\times 2 \Rightarrow \overset{(-)}{6t} \overset{(+)}{-4s} = \overset{(+)}{-2}$ — (4)

(3) − (4) $\Rightarrow -11s = 11$

$s = -1$

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Sub $s = -1$ in ①

$2t - 5(-1) = 3$

$2t + 5 = 3$

$2t = 3 - 5$

$2t = -2$

$t = -1$

Hence point of intersection is either put $s = -1$ or $t = -1$

$(2(-1)+1, 3(-1)+2, 4(-1)+3)$

(i) $(-2+1, -3+2, -4+3)$

(ii) $(-1, -1, -1)$ //

Coplanar:- Two lines are said to be coplanar if they lie in the same plane.

If two line are either parallel or intersection, then they are coplanar.

Two lines in space are called skew lines if they are not parallel and do not intersecting.

If two lines are skew lines, then they are non coplaner.

* The shortest distance between two parallel lines $\bar{r} = \bar{a} + t\bar{b}$ and $\bar{r} = \bar{c} + s\bar{d}$ is given by

$$d = \frac{|(\bar{c} - \bar{a}) \times \bar{b}|}{|\bar{b}|}, \text{ where } |\bar{b}| \neq 0.$$

* The shortest between the two skew lines $\bar{r} = \bar{a} + s\bar{b}$ and $\bar{r} = \bar{c} + t\bar{d}$ is given by $d = \frac{|(\bar{c} - \bar{a}) \cdot (\bar{b} \times \bar{d})|}{|\bar{b} \times \bar{d}|}$

* The condition for $\bar{r} = \bar{a} + t\bar{b}$ and $\bar{r} = \bar{c} + s\bar{d}$ to intersect each other (ie coplanar) if

$$(\bar{c} - \bar{a}) \cdot (\bar{b} \times \bar{d}) = 0.$$

* If two lines $\frac{x - x_1}{b_1} = \frac{y - y_1}{b_2} = \frac{z - z_1}{b_3}$ and $\frac{x - x_2}{d_1} = \frac{y - y_2}{d_2} = \frac{z - z_2}{d_3}$ intersect each other, then we have

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$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

Eg 6.34 find the paremetric form of vector eqn of a st line passing through the point of intersection of the st lines $\bar{r} = (\bar{i}+3\bar{j}-\bar{k}) + t(2\bar{i}+3\bar{j}+2\bar{k})$ and $\frac{x-2}{1} = \frac{y-4}{2} = \frac{z+3}{4}$, and perpendicular to both the st lines.

The cartesian eqn of the st line $\bar{r} = (\bar{i}+3\bar{j}-\bar{k}) + t(2\bar{i}+3\bar{j}+2\bar{k})$ is $\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+1}{2} = t$ (say)

∴ any point on this line is of the form $(2t+1, 3t+3, 2t-1)$

Let $\frac{x-2}{1} = \frac{y-4}{2} = \frac{z+3}{4} = s$ (say)

$\frac{x-2}{1} = s$ | $\frac{y-4}{2} = s$ | $\frac{z+3}{4} = s$

$x = s+2$ | $y = 2s+4$ | $z = 4s-3$.

∴ any point on this line is of the form $(s+2, 2s+4, 4s-3)$.

Since the lines are intersecting

$(2t+1, 3t+3, 2t-1) = (s+2, 2s+4, 4s-3)$

$\Rightarrow$ $2t+1 = s+2$ | $3t+3 = 2s+4$

$2t-s = 1$ — (1) | $3t-2s = 1$ — (2)

(1) × 2 $\Rightarrow$ $4t-2s = 2$ — (3)

$3t-2s = 1$ — (4)

(3) − (2) $\Rightarrow$ $t = 1$

sub $t=1$ in (1) $2(1)-s = 1$

$2-s = 1$

$2-1 = s$

$\boxed{s = 1}$

∴ The point of intersection is put $t=1$ or $s=1$

∴ The pt of intersection is $(3, 6, 1)$

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∴ The parallel vectors to the given lines are

$\bar{b} = 2\bar{i} + 3\bar{j} + 2\bar{k}$ and

$\bar{d} = \bar{i} + 2\bar{j} + 4\bar{k}$

The vector $\bar{b} \times \bar{d}$ is ⊥r to both the st lines.

$$\bar{b} \times \bar{d} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 3 & 2 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= \bar{i}(12-4) - \bar{j}(8-2) + \bar{k}(4-3)$$

$$= 8\bar{i} - 6\bar{j} + \bar{k}.$$

∴ The required st line passing through (3, 6, 1) and ⊥r to both the st lines is the line passing through (3, 6, 1) and ∥l to $8\bar{i} - 6\bar{j} + \bar{k}$.

∴ The required eqn of st line is $\bar{r} = (3\bar{i} + 6\bar{j} + \bar{k}) + t(8\bar{i} - 6\bar{j} + \bar{k})$

Eg 6.35 Determine the pair of straight lines $\bar{r} = (2\bar{i} + 6\bar{j} + 3\bar{k}) + t(2\bar{i} + 3\bar{j} + 4\bar{k})$, $\bar{r} = (2\bar{j} - 3\bar{k}) + s(\bar{i} + 2\bar{j} + 3\bar{k})$ are parallel. find the shortest distance betn them.

Compare the given lines with $\bar{r} = \bar{a} + t\bar{b}$ and $\bar{r} = \bar{c} + s\bar{d}$

Here $\bar{a} = 2\bar{i} + 6\bar{j} + 3\bar{k}$

$\bar{b} = 2\bar{i} + 3\bar{j} + 4\bar{k}$

$\bar{c} = 2\bar{j} - 3\bar{k}$

$\bar{d} = \bar{i} + 2\bar{j} + 3\bar{k}$

clearly $\bar{b}$ is not a multiple of $\bar{d}$

∴ $\bar{b}$ and $\bar{d}$ are not parallel vectors.

The shortest ~~betn~~ distance betn two given st lines is

$$d = \left| \frac{(\bar{c} - \bar{a}) \cdot (\bar{b} \times \bar{d})}{|\bar{b} \times \bar{d}|} \right| \quad \text{——} (1)$$

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$\bar{c} - \bar{a} = -2\bar{i} - 4\bar{j} - 6\bar{k}$

$$\bar{b} \times \bar{d} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix}$$

$= \bar{i}(9-8) - \bar{j}(6-4) + \bar{k}(4-3)$

$= \bar{i} - 2\bar{j} + \bar{k}$

$(\bar{c} - \bar{a}) \cdot (\bar{b} \times \bar{d}) = -2 + 8 - 6 = 0$

$\therefore$ ① $\Rightarrow$ $d = \dfrac{0}{|\bar{b} \times \bar{d}|} = 0$

Distance betn the two given st lines is zero.

Hence the given st lines intersect each other.

Skew lines:- Two lines in space are called skew lines if they are not parallel and do not intersect.

6.36 find the shortest distance betn the two given st lines

$\bar{r} = (2\bar{i} + 3\bar{j} + 4\bar{k}) + t(-2\bar{i} + \bar{j} - 2\bar{k})$

and $\dfrac{x-3}{2} = \dfrac{y}{-1} = \dfrac{z+2}{2}$.

The parallel vectors to the given lines are

$\bar{b} = -2\bar{i} + \bar{j} - 2\bar{k}$ and

$\bar{d} = 2\bar{i} - \bar{j} + 2\bar{k}$

$\bar{b} = -1(2\bar{i} - \bar{j} + 2\bar{k}) = -1\,\bar{d}$

clearly $\bar{b}$ is parallel to $\bar{d}$.

Hence the given st lines are parallel.

Shortest betn two ll lines is

$$d = \frac{|(\bar{c} - \bar{a}) \times \bar{b}|}{|\bar{b}|}$$

Here $\bar{a} = 2\bar{i} + 3\bar{j} + 4\bar{k}$

$\bar{b} = -2\bar{i} + \bar{j} - 2\bar{k}$

$\bar{c} = 3\bar{i} + 0\bar{j} - 2\bar{k}$

$\bar{d} = 2\bar{i} - \bar{j} - 2\bar{k}$

$\bar{c} - \bar{a} = \bar{i} - 3\bar{j} - 6\bar{k}$

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Now $(\bar{c}-\bar{a})\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -3 & -6 \\ -2 & 1 & -2 \end{vmatrix}$

$= \bar{i}(6+6) - \bar{j}(-2-12) + \bar{k}(1-6)$

$= 12\bar{i} + 14\bar{j} - 5\bar{k}$

$|(\bar{c}-\bar{a})\times\bar{b}| = \sqrt{12^2+14^2+5^2}$

$= \sqrt{144+196+25} = \sqrt{365}$

$|\bar{b}| = \sqrt{4+1+4} = \sqrt{9} = 3.$

$\therefore d = \dfrac{|(c-a)\times\bar{b}|}{|\bar{b}|} = \dfrac{\sqrt{365}}{3}$ //

EXERCISE 6.5

Find the parametric form of vector eqn and cartesian eqns of a st line passing through (5, 2, 8) and is ⊥r to the st lines $\bar{r} = (\bar{i}+\bar{j}+\bar{k}) + s(2\bar{i}-2\bar{j}+\bar{k})$ and $\bar{r} = (2\bar{i}-\bar{j}-3\bar{k}) + t(\bar{i}+2\bar{j}+2\bar{k})$

The given lines are

$\bar{r} = (\bar{i}+\bar{j}-\bar{k}) + s(2\bar{i}-2\bar{j}+\bar{k})$ — ①

$\bar{r} = (2\bar{i}-\bar{j}-3\bar{k}) + t(\bar{i}+2\bar{j}+2\bar{k})$ — ②

The parelled vectors to the given lines are $\bar{b} = 2\bar{i}-2\bar{j}+\bar{k}$ and $\bar{d} = \bar{i}+2\bar{j}+2\bar{k}$

$\bar{b}\times\bar{d}$ is ⊥r to both the lines ① and ②

$\bar{b}\times\bar{d} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix}$

$= \bar{i}(-4-2) - \bar{j}(4-1) + \bar{k}(4+2)$

$= -6\bar{i} - 3\bar{j} + 6\bar{k}.$

The given pt is (5, 2, 8)

The parametric vector eqn of the line passing through (5, 2, 8) and ⊥r to the lines.

That means the parametric vector eqn of the line passing

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through $(5,2,8)$ and $\perp^l$ to

$\bar{b} \times \bar{d} = -6\bar{i} - 3\bar{j} + 6\bar{k}$

∴ The required vector is

$\bar{r} = (5\bar{i} + 2\bar{j} + 8\bar{k}) + t(-6\bar{i} - 3\bar{j} + 6\bar{k})$

$= (5\bar{i} + 2\bar{j} + 8\bar{k}) + t(-3)(2\bar{i} + \bar{j} - 2\bar{k})$

$= (5\bar{i} + 2\bar{j} + 8\bar{k}) + s(2\bar{i} + \bar{j} - 2\bar{k})$

The cartesian eqn of the line is

$$\frac{x-5}{2} = \frac{y-2}{1} = \frac{z-8}{-2}$$

---

2) Show that the lines $\bar{r} = (6\bar{i} + \bar{j} + 2\bar{k}) + s(\bar{i} + 2\bar{j} - 3\bar{k})$ and $\bar{r} = (3\bar{i} + 2\bar{j} - 2\bar{k}) + t(2\bar{i} + 4\bar{j} - 5\bar{k})$ are skew lines and hence find the shortest distance betn them.

Compare the given lines with $\bar{r} = \bar{a} + s\bar{b}$ and $\bar{r} = \bar{c} + t\bar{d}$

Here $\bar{a} = 6\bar{i} + \bar{j} + 2\bar{k}$

$\bar{b} = \bar{i} + 2\bar{j} - 3\bar{k}$

$\bar{c} = 3\bar{i} + 2\bar{j} - 2\bar{k}$

$\bar{d} = 2\bar{i} + 4\bar{j} - 5\bar{k}$

The condition for skew lines

$(\bar{c} - \bar{a}) \cdot (\bar{b} \times \bar{d}) \neq 0$.

$\bar{c} - \bar{a} = -3\bar{i} + \bar{j} - 4\bar{k}$

$$\bar{b} \times \bar{d} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}$$

$= \bar{i}(-10+12) - \bar{j}(-5+6) + \bar{k}(4-4)$

$= 2\bar{i} - \bar{j} + 0\bar{k}$.

$(\bar{c} - \bar{a}) \cdot (\bar{b} \times \bar{d}) = (-3\bar{i} + \bar{j} - 4\bar{k}) \cdot (2\bar{i} - \bar{j} + 0\bar{k})$

$= -6 - 1 + 0 = -7 \neq 0$

∴ The given lines are skew lines.

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The shortest betn the two lines is $\dfrac{|(\bar{c}-\bar{a})\cdot(\bar{b}\times\bar{d})|}{|\bar{b}\times\bar{d}|}$

$= \dfrac{|-7|}{\sqrt{4+1}} = 7/\sqrt{5}$ units.

3) If the two lines $\dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-1}{4}$ and $\dfrac{x-3}{1} = \dfrac{y-m}{2} = z$ intersect at a point find m.

The given lines are

$\dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-1}{4}$ and

$\dfrac{x-3}{1} = \dfrac{y-m}{2} = z$.

The condition for the lines intersect is

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0 \quad \text{——— (1)}$$

Here $(x_1, y_1, z_1) = (1, -1, 1)$

$(x_2\ y_2\ z_2) = (3, m, 0)$

$(b_1\ b_2\ b_3) = (2, 3, 4)$

$(d_1\ d_2\ d_3) = (1, 2, 1)$

$\therefore$ (1) $\Rightarrow$ $\begin{vmatrix} 2 & m+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0.$

$2(3-8) - (m+1)(2-4) - 1(4-3) = 0$

$2(-5) - (m+1)(-2) - 1(1) = 0$

$-10 + 2m + 2 - 1 = 0$

$2m - 9 = 0$

$2m = 9 \Rightarrow m = 9/2$

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4) Show that the lines

$\frac{x-3}{3} = \frac{y-3}{-1}$, $z-1=0$ and

$\frac{x-6}{2} = \frac{z-1}{3}$, $y-2=0$ intersect

Also find the point of intersection

Given lines are

$\frac{x-3}{3} = \frac{y-3}{-1}$, $z-1=0 \Rightarrow z=1$

$\frac{x-6}{2} = \frac{z-1}{3}$ & $y-2=0 \Rightarrow y=2$

Here $(x_1, y_1, z_1) = (3, 3, 1)$

$(x_2, y_2, z_2) = (6, 2, 1)$

$(b_1, b_2, b_3) = (3, -1, 0)$

$(d_1, d_2, d_3) = (2, 0, 3)$

condition for intersecting lines is

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

i.e $\begin{vmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = 3(-3) + 1(9) + 0(+2)$

$= -9+9 = 0$

$\therefore$ given two lines are intersecting lines

Let $\frac{x-3}{3} = \frac{y-3}{-1} = \frac{z-1}{0} = t$ (say)

$\therefore$ Any point on this line is

$(3t+3, -t+3, 0t+1)$

Let $\frac{x-6}{2} = \frac{y-2}{0} = \frac{z-1}{3} = s$ (say)

$(2s+6, 2, 3s+1)$

Given that they are intersecting lines

$\therefore$ $3t+3 = 2s+6$ | $-t+3 = 2$

$3t-2s = 3$ — (1) | $-t = -1$

$t = 1$

Sub $t=1$ in (1)

$3-2s = 3$

$2s = 0$

$s = 0$

$\therefore$ The pt of intersection is $(6, 2, 1)$ //

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5) Show that the lines $x+1 = 2y = -12z$ and $x = y+2 = 6z-6$ are skew and hence find the shortest distance between them.

Given lines are

$$\frac{x+1}{1} = \frac{y}{1/2} = \frac{z}{-1/12} \quad \text{and} \quad \text{—— (1)}$$

$$\frac{x}{1} = \frac{y+2}{1} = \frac{6(z-1)}{1}$$

i.e. $$\frac{x}{1} = \frac{y+2}{1} = \frac{z-1}{1/6} \quad \text{—— (2)}$$

Here $(x_1, y_1, z_1) = (-1, 0, 0)$

$(b_1, b_2, b_3) = (1, 1/2, -1/12)$

$(x_2, y_2, z_2) = (0, -2, 1)$

$(d_1, d_2, d_3) = (1, 1, 1/6)$

Condition for the lines intersect is $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

condition for skew lines

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} \neq 0.$$

i.e. $$\begin{vmatrix} 1 & -2 & 1 \\ 1 & 1/2 & -1/12 \\ 1 & 1 & 1/6 \end{vmatrix}$$

$$= 1\left(\frac{1}{12} + \frac{1}{12}\right) + 2\left(\frac{1}{6} + \frac{1}{12}\right) + 1\left(1 - \frac{1}{2}\right)$$

$$= \frac{2}{12} + 2\left(\frac{2+1}{12}\right) + 1\left(\frac{1}{2}\right)$$

$$= \frac{1}{6} + \frac{3}{6} + \frac{1}{2} = \frac{1+3+3}{6}$$

$$= 7/6 \neq 0$$

∴ The given lines are skew lines.

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$$\bar{b}\times\bar{d} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & \frac{1}{2} & -\frac{1}{12} \\ 1 & 1 & \frac{1}{6} \end{vmatrix}$$

$$= \bar{i}\left(\frac{1}{12}+\frac{1}{12}\right) - \bar{j}\left(\frac{1}{6}+\frac{1}{12}\right) + \bar{k}\left(1-\frac{1}{2}\right)$$

$$= \frac{2}{12}\bar{i} - \bar{j}\left(\frac{2+1}{12}\right) + \bar{k}\left(\frac{1}{2}\right)$$

$$= \frac{1}{6}\bar{i} - \frac{1}{4}\bar{j} + \frac{1}{2}\bar{k}.$$

$$|\bar{b}\times\bar{d}| = \sqrt{\frac{1}{36}+\frac{1}{16}+\frac{1}{4}}$$

$$= \sqrt{\frac{4+9+36}{144}}$$

$$= \sqrt{\frac{49}{144}} = \frac{7}{12}.$$

$$\bar{c}-\bar{a} = (\bar{i}-2\bar{j}+\bar{k})$$

Shortest betn skew lines

$$= \frac{|(\bar{c}-\bar{a})\cdot(\bar{b}\times\bar{d})|}{|\bar{b}\times\bar{d}|}$$

$$= \frac{\frac{1}{6}+\frac{2}{4}+\frac{1}{2}}{7/12} = \frac{2+6+6}{12}\times\frac{12}{7}$$

$$= \frac{14}{7} = 2 \text{ units}$$

---

6) Find the parametric form of a vector eqn of the st line passing through $(-1,2,1)$ and parallel to the st line $\bar{r} = (2\bar{i}+3\bar{j}-\bar{k}) + t(\bar{i}-2\bar{j}+\bar{k})$ and hence find the shortest distance betn the lines.

Given pt is $\bar{a} = -\bar{i}+2\bar{j}+\bar{k}$

parallel vectors to the given lines is $\bar{b} = \bar{i}-2\bar{j}+\bar{k}$.

∴ The required vector eqn is

$$\bar{r} = \bar{a}+t\bar{b}$$

$$= (-\bar{i}+2\bar{j}+\bar{k}) + t(\bar{i}-2\bar{j}+\bar{k})$$

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Here $\bar{a} = -\bar{i} + 2\bar{j} + \bar{k}$

$\bar{b} = \bar{i} - 2\bar{j} + \bar{k}$

$\bar{c} = 2\bar{i} + 3\bar{j} - \bar{k}$.

$\bar{c} - \bar{a} = 3\bar{i} + \bar{j} - 2\bar{k}$

$$(\bar{c}-\bar{a}) \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix}$$

$= \bar{i}(1-4) - \bar{j}(3+2) + \bar{k}(-6-1)$

$= -3\bar{i} - 5\bar{j} - 7\bar{k}$.

$|(\bar{c}-\bar{a}) \times \bar{b}| = \sqrt{9+25+49}$

$= \sqrt{83}$

$|\bar{b}| = \sqrt{1+4+1} = \sqrt{6}$

∴ Shortest distance betn the 2 lines is

$d = \dfrac{|(\bar{c}-\bar{a}) \times \bar{b}|}{|\bar{b}|} = \dfrac{\sqrt{83}}{\sqrt{6}}$ units.

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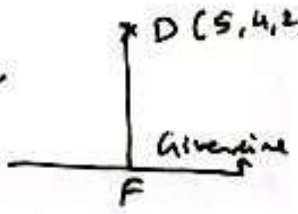
7) Find the foot of the ⊥r drawn from the point (5, 4, 2) to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1}$. Also find the eqn of the ⊥r

F is foot of the ⊥r from the point D(5,4,2) to the given line.



$\bar{b} = 2\bar{i} + 3\bar{j} - \bar{k}$

Given line is $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1}$

The ||l vector to the given line is $\bar{b} = 2\bar{i} + 3\bar{j} - \bar{k}$

Take $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1} = t$ (say)

The point F on the foot is of the form

$(2t-1, 3t+3, -t+1)$

$\therefore \quad \overrightarrow{DF} = \overrightarrow{OF} - \overrightarrow{OD}$

$= (2t-1)\bar{i} + (3t+3)\bar{j} + (-t+1)\bar{k} - (5\bar{i} + 4\bar{j} + 2\bar{k})$

$= (2t-6)\bar{i} + (3t-1)\bar{j} + (-t-1)\bar{k}$

Since $\overrightarrow{DF}$ ⊥r $\bar{b}$

$\therefore \overrightarrow{DF} \cdot \bar{b} = 0$

$\Rightarrow [(2t-6)\bar{i} + (3t-1)\bar{j} + (-t-1)\bar{k}] \cdot [2\bar{i} + 3\bar{j} - \bar{k}] = 0$

$(2t-6)2 + (3t-1)3 + (-t-1)(-1) = 0$

$4t - 12 + 9t - 3 + t + 1 = 0$

$14t - 14 = 0 \Rightarrow 14t = 14$

$\boxed{t = 1}$

$\therefore$ The foot of the line F is (1, 6, 0)

The ⊥r line passes through D(5,4,2) and F(1,6,0)

$\therefore$ The eqn of DF is

$\frac{x-5}{1-5} = \frac{y-4}{6-4} = \frac{z-2}{0-2}$ i.e.

$\frac{x-5}{-4} = \frac{y-4}{2} = \frac{z-2}{-2}$ //

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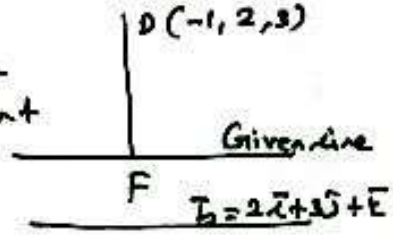
Eg 6.37 Find the coordinates of the foot of ⊥r drawn from the point $(-1,2,3)$ to the line $\bar{r} = (\bar{i} - 4\bar{j} + 3\bar{k}) + t(2\bar{i} + 3\bar{j} + \bar{k})$ Also find the shortest distance from the given point.



F is foot of the ⊥r from the point D(-1,2,3) to the given line.

The given line is

$\bar{r} = (\bar{i} - 4\bar{j} + 3\bar{k}) + t(2\bar{i} + 3\bar{j} + \bar{k})$

parallel vector to the given line is $\bar{b} = 2\bar{i} + 3\bar{j} + \bar{k}$

Cartesian form of eqn is

$$\frac{x-1}{2} = \frac{y+4}{3} = \frac{z-3}{1}$$

Take $\frac{x-1}{2} = \frac{y+4}{3} = \frac{z-3}{1} = t$ (say)

The point F on the foot is $(2t+1, 3t-4, t+3)$

$\overline{DF} = \overline{OF} - \overline{OD}$

$= [(2t+1)\bar{i} + (3t-4)\bar{j} + (t+3)\bar{k}] - [-\bar{i} + 2\bar{j} + 3\bar{k}]$

$= (2t+2)\bar{i} + (3t-6)\bar{j} + (t)\bar{k}$

Also $\overline{DF}$ ⊥r $\bar{b}$.

$\therefore \overline{DF} \cdot \bar{b} = 0$

i.e $[(2t+2)\bar{i} + (3t-6)\bar{j} + t\bar{k}] \cdot [2\bar{i} + 3\bar{j} + \bar{k}] = 0$

$(2t+2)2 + (3t-6)3 + t(1) = 0$

$4t + 4 + 9t - 18 + t = 0$

$14t - 14 = 0 \Rightarrow \boxed{t = 1}$

$\therefore$ The point F is $(3,-1,4)$

Shortest distance $\overline{DF}$ is

$(2t+2)\bar{i} + (3t-6)\bar{j} + t\bar{k}$.

$= 4\bar{i} - 3\bar{j} + \bar{k}$

$|\overline{DF}| = \sqrt{16+9+1} = \sqrt{26}$ units

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a) **Vector equation of a plane in normal form.**

The eqn of the plane at a distance p from the origin and perpendicular to the unit normal vector $\hat{d}$ is $\bar{r} \cdot \hat{d} = p$.

b) **Cartesian eqn of a plane in normal form**

Let l, m, n be the direction cosine of $\hat{d}$. Then we have

$\hat{d} = l\bar{i} + m\bar{j} + n\bar{k}$

$\therefore \bar{r} \cdot (l\bar{i} + m\bar{j} + n\bar{k}) = p$

$\therefore \boxed{lx + my + nz = p}$

Eqn of a plane perpendicular to a vector and passing through a given point

a) **Vector form of equation**

Vector form of the eqn of a plane passing through a point with position vector $\bar{a}$ and perpendicular to $\bar{n}$ is

$(\bar{r} - \bar{a}) \cdot \bar{n} = 0 \Rightarrow \bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$.

b) **Cartesian form of equation:**

If a, b, c are direction ratios of $\bar{n}$ then we have $\bar{n} = a\bar{i} + b\bar{j} + c\bar{k}$

Suppose A is (x_1, y_1, z_1) then the eqn $(\bar{r} - \bar{a}) \cdot \bar{n} = 0$ becomes

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Intercept form of the eqn of a plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

The general eqn $ax + by + cz + d = 0$ represents a plane.

Eg 6.38 Find the vector and Cartesian form of the eqns of a plane which is at a distance of 12 unit from the origin and $\perp_r$ to $6\bar{i} + 2\bar{j} - 3\bar{k}$

Let $\bar{d} = 6\bar{i} + 2\bar{j} - 3\bar{k}$

$\hat{d}$ is the unit normal vector

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then $\hat{d} = \frac{\bar{d}}{|\bar{d}|}$

$|\bar{d}| = \sqrt{36+4+9} = \sqrt{49} = 7$

$\therefore \hat{d} = \frac{1}{7}(6\bar{i}+2\bar{j}-3\bar{k})$

∴ vector eqn of a plane in normal form is $\bar{r}\cdot\hat{d} = p$

ie $\bar{r}\cdot\frac{1}{7}(6\bar{i}+2\bar{j}-3\bar{k}) = 12$

Eg 3.69 If the cartesian eqn of a plane is $3x-4y+3z=-8$ find the vector eqn of the plane in standard form.

Given cartesian eqn of the plane is $3x-4y+3z=-8$

ie $(x\bar{i}+y\bar{j}+z\bar{k})\cdot(3\bar{i}-4\bar{j}+3\bar{k}) = -8$

∴ vector eqn of the plane in standard form is

$\bar{r}\cdot(3\bar{i}-4\bar{j}+3\bar{k}) = -8$

Eg 3.40 find the direction cosine of the normal to the plane and length of the perpendicular from the origin to the plane

$\bar{r}\cdot(3\bar{i}-4\bar{j}+12\bar{k}) = 5$.

Given eqn of the plane is

$\bar{r}\cdot(3\bar{i}-4\bar{j}+12\bar{k}) = 5$

compare ~~this with $\bar{r}\cdot\hat{d}=p$~~

~~Here~~ let $\bar{d} = 3\bar{i}-4\bar{j}+12\bar{k}$

and $p = 5$.

$\hat{d} = \frac{\bar{d}}{|\bar{d}|} = \frac{3\bar{i}-4\bar{j}+12\bar{k}}{\sqrt{9+16+144}} = \frac{3\bar{i}-4\bar{j}+12\bar{k}}{\sqrt{169}}$

$= \frac{3\bar{i}-4\bar{j}+12\bar{k}}{13}$

Dividing the eqn by 13.

$\therefore \bar{r}\cdot\left(\frac{3}{13}\bar{i}-\frac{4}{13}\bar{j}+\frac{12}{13}\bar{k}\right) = \frac{5}{13}$

∴ Direction cosines of $\hat{d}$ are $\left(\frac{3}{13}, -\frac{4}{13}, \frac{12}{13}\right)$ and length of the ⊥r from the origin is $5/13$.

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Eg 6.41 Find the vector and Cartesian equations of the plane passing through the point with position vector $4\bar{i}+2\bar{j}-3\bar{k}$ and normal to vector $2\bar{i}-\bar{j}+\bar{k}$.

Here $\bar{a} = 4\bar{i}+2\bar{j}-3\bar{k}$

$\bar{n} = 2\bar{i}-\bar{j}+\bar{k}$.

The eqn of the plane passing through a point and normal to a vector is given by

$(\bar{r}-\bar{a})\cdot\bar{n} = 0$ or $\bar{r}\cdot\bar{n} = \bar{a}\cdot\bar{n}$

∴ The vector eqn of the plane is

$\bar{r}\cdot(2\bar{i}-\bar{j}+\bar{k}) = (4\bar{i}+2\bar{j}-3\bar{k})\cdot(2\bar{i}-\bar{j}+\bar{k})$

(i) $\bar{r}\cdot(2\bar{i}-\bar{j}+\bar{k}) = 8-2-3$

$\bar{r}\cdot(2\bar{i}-\bar{j}+\bar{k}) = 3$

Cartesian eqn of the plane is

$2x-y+z = 3$

6.42 A variable plane moves in such a way that the sum of the reciprocals of its intercepts on the coordinate axes is a constant. Show that the plane passes through a fixed point.

Let a, b, c be the intercepts on the x, y, z axes respectively.

Intercept form of the eqn is

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$.

Given that sum the reciprocal of the intercepts is a constant.

(i) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = k$

$\frac{1}{a}\left(\frac{1}{k}\right)+\frac{1}{b}\left(\frac{1}{k}\right)+\frac{1}{c}\left(\frac{1}{k}\right) = 1$

This shows that the plane passes through $\left(\frac{1}{k}, \frac{1}{k}, \frac{1}{k}\right)$

(i) the plane passes through the fixed point $\left(\frac{1}{k}, \frac{1}{k}, \frac{1}{k}\right)$.

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EXERCISE 6.6

1) Find a parametric form of vector eqn of a plane which is at a distance 7 unit from the origin having $(3,-4,5)$ as direction ratios of a normal to it.

Here $\bar{d} = 3\bar{i} - 4\bar{j} + 5\bar{k}$, $p = 7$

$$\hat{d} = \frac{\bar{d}}{|\bar{d}|} = \frac{3\bar{i}-4\bar{j}+5\bar{k}}{\sqrt{9+16+25}}$$

$$= \frac{3\bar{i}-4\bar{j}+5\bar{k}}{\sqrt{50}} = \frac{3\bar{i}-4\bar{j}+5\bar{k}}{5\sqrt{2}}$$

∴ The parametric form of vector eqn is $\bar{r}\cdot\hat{d} = p$.

i.e. $\bar{r}\cdot\frac{1}{5\sqrt{2}}(3\bar{i}-4\bar{j}+5\bar{k}) = 7$

---

2) Find the direction cosine of the normal to the plane $12x+3y-4z=65$. Also find the non-parametric form of vector equation of a plane and the length of the ⊥r to the plane from the origin.

Given eqn of the plane is $12x+3y-4z=65$

∴ The normal vector to the plane is

$$\bar{d} = 12\bar{i}+3\bar{j}-4\bar{k}.$$

$$\hat{d} = \frac{\bar{d}}{|\bar{d}|} = \frac{12\bar{i}+3\bar{j}-4\bar{k}}{\sqrt{144+9+16}} = \frac{12\bar{i}+3\bar{j}-4\bar{k}}{\sqrt{169}}$$

$$= \frac{12\bar{i}+3\bar{j}-4\bar{k}}{13}.$$

Non parametric form of vector eqn is

$$\bar{r}\cdot\hat{d} = p$$

i.e. $\bar{r}\cdot\left(\frac{12\bar{i}+3\bar{j}-4\bar{k}}{13}\right) = \frac{65}{13}$

$$\bar{r}\cdot\left(\frac{12\bar{i}+3\bar{j}-4\bar{k}}{13}\right) = 5.$$

Direction cosines $\left(\frac{12}{13}, \frac{3}{13}, \frac{-4}{13}\right)$.

Length of the ⊥r from the origin is 5 units.

---

3) Find the vector and cartesian eqn of the plane passing through the point with position vector $2\bar{i}+6\bar{j}+3\bar{k}$ and normal to the vector $\bar{i}+3\bar{j}+5\bar{k}$.

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Given $\bar{a} = 2\bar{i} + 6\bar{j} + 3\bar{k}$ and
$\bar{n} = \bar{i} + 3\bar{j} + 5\bar{k}$

∴ Vector eqn of the plane is
$\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$

$\Rightarrow \bar{r} \cdot (\bar{i} + 3\bar{j} + 5\bar{k}) = 2 + 18 + 15$

$\bar{r} \cdot (\bar{i} + 3\bar{j} + 5\bar{k}) = 35$

Cartesian eqn of the plane
$x + 3y + 5z = 35$

---

4) A plane passes through the point $(-1, 1, 2)$ and normal to the plane of magnitude $3\sqrt{3}$ makes equal acute angles with the coordinates axes. Find the eqn of the plane.

Given $\bar{a} = -\bar{i} + \bar{j} + 2\bar{k}$
magnitude $= 3\sqrt{3}$
normal vector makes equal angle with coordinate axes.
WKT $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
But $(\alpha = \beta = \gamma)$
$\therefore 3\cos^2\alpha = 1 \Rightarrow \cos^2\alpha = \frac{1}{3}$

$\cos\alpha = \frac{1}{\sqrt{3}}$

~~$\bar{n} = 3\sqrt{3}\left[\frac{1}{\sqrt{3}}\bar{i} + \frac{1}{\sqrt{3}}\bar{j} + \frac{1}{\sqrt{3}}\bar{k}\right]$~~

$= 3\bar{i} + 3\bar{j} + 3\bar{k}$

∴ Vector eqn of the plane is
$\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$

$\bar{r} \cdot \bar{n} = (-\bar{i} + \bar{j} + 2\bar{k}) \cdot (3\bar{i} + 3\bar{j} + 3\bar{k})$
$= -3 + 3 + 6 = 6$

$\bar{r} \cdot (3\bar{i} + 3\bar{j} + 3\bar{k}) = 6$

$\Rightarrow \bar{r} \cdot (\bar{i} + \bar{j} + \bar{k}) = 2$

Cartesian eqn of the plane is
$x + y + z = 2$

Direction cosines of $\hat{d}$ is
$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

$\therefore \hat{d} = \frac{1}{\sqrt{3}}\bar{i} + \frac{1}{\sqrt{3}}\bar{j} + \frac{1}{\sqrt{3}}\bar{k}$
$= \frac{1}{\sqrt{3}}(\bar{i} + \bar{j} + \bar{k})$

Given $\bar{d} = 3\sqrt{3}\,\hat{d} = 3\sqrt{3} \times \frac{1}{\sqrt{3}}(\bar{i} + \bar{j} + \bar{k})$
$= 3\bar{i} + 3\bar{j} + 3\bar{k}$

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1) The parametric form of vector equation of the plane passing through the given three non-collinear points is of the form is

$$\bar{r} = \bar{a} + s(\bar{b}-\bar{a}) + t(\bar{c}-\bar{a})$$

2) Non parametric form of vector equation of the plane passing through three non-collinear points is $(\bar{r}-\bar{a})\cdot[(\bar{b}-\bar{a})\times(\bar{c}-\bar{a})]=0$

(OR) $[\bar{r}-\bar{a}, \bar{b}-\bar{a}, \bar{c}-\bar{a}] = 0$

3) Cartesian eqn of the plane passing through three non-collinear points is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

---

1) parametric form of vector equation of the plane passing through a given point and parallel to two given non-parallel vectors is

$$\vec{r} = \vec{a} + t\vec{b} + s\vec{c}.$$

2) Non parametric form of vector equation of the plane passing through a given point and parallel to two given non parallel vectors is

$$(\bar{r}-\bar{a})\cdot(\bar{b}\times\bar{c}) = 0$$

3) Cartesian equation of the plane passing through a given point and parallel to two given non-parallel vectors. is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Eqn of a plane passing through two given distinct points and is parallel to a non zero vectors.

The parametric form of vector equation of the plane passing through two given distinct points A and B with position vectors $\vec{a}$ and $\vec{b}$, and parallel to a non zero vectors $\vec{c}$ is

$$\vec{r} = \vec{a} + s(\vec{b}-\vec{a}) + t\vec{c}.$$

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Non-parametric form of vector of the plane passing through two given points and parallel to a vector is

$$(\bar{r}-\bar{a})\cdot((\bar{b}-\bar{a})\times\bar{c}) = 0$$

Cartesian form of equation is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Eg 6.43 Find the non-parametric form of vector equation, and Cartesian eqn of the plane passing through the point $(0,1,-5)$ and parallel to the lines

$\bar{r} = (\bar{i}+2\bar{j}-4\bar{k}) + s(2\bar{i}+3\bar{j}+6\bar{k})$ and

$\bar{r} = (\bar{i}-3\bar{j}+5\bar{k}) + t(\bar{i}+\bar{j}-\bar{k})$

parallel vectors to the given lines are $\bar{b} = 2\bar{i}+3\bar{j}+6\bar{k}$ and

$\bar{c} = \bar{i}+\bar{j}-\bar{k}$

The required plane is passing through the point $(0,1,-5)$ with position vector $\bar{a}$ and parallel to two vectors $\bar{b}$ and $\bar{c}$.

∴ non parametric form of eqn is

$$(\bar{r}-\bar{a})\cdot(\bar{b}\times\bar{c}) = 0$$

i.e. $\bar{r}\cdot(\bar{b}\times\bar{c}) - \bar{a}\cdot(\bar{b}\times\bar{c}) = 0$ —— ①

$$\bar{b}\times\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix} = \bar{i}(-3-6) - \bar{j}(-2-6) + \bar{k}(2-3)$$

$$= -9\bar{i}+8\bar{j}-\bar{k}$$

$$\bar{a}\cdot(\bar{b}\times\bar{c}) = (0\bar{i}+\bar{j}-5\bar{k})\cdot(-9\bar{i}+8\bar{j}-\bar{k})$$

$$= 0+8+5 = 13.$$

∴ ① ⇒ $\bar{r}\cdot(-9\bar{i}+8\bar{j}-\bar{k}) = 13$

---

cartesian eqn of the plane is

$$(x\bar{i}+y\bar{j}+z\bar{k})\cdot(-9\bar{i}+8\bar{j}-\bar{k}) = 13$$

i.e. $-9x+8y-z = 13$ or

$$9x-8y+z = -13$$

$$9x-8y+z+13 = 0$$

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6.44 Find the vector parametric, vector non-parametric and cartesian form of the eqn of the plane passing through the points $(-1,2,0)$, $(2,2,-1)$ and parallel to the st line $\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$

Given line is $\frac{x-1}{1} = \frac{x(y+\frac{1}{2})}{x1} = \frac{z+1}{-1}$

parallel vector to the given line is $\bar{i}+\bar{j}-\bar{k}$.

Here $\bar{a} = -\bar{i}+2\bar{j}$

$\bar{b} = 2\bar{i}+2\bar{j}-\bar{k}$.

parallel vector is $\bar{c} = \bar{i}+\bar{j}-\bar{k}$

Vector eqn of the plane in parametric form is

$\bar{r} = \bar{a} + s(\bar{b}-\bar{a}) + t\bar{c}$.

i.e. $\bar{r} = (-\bar{i}+2\bar{j}) + s(3\bar{i}+0\bar{j}-\bar{k}) + t(\bar{i}+\bar{j}-\bar{k})$

Vector eqn of the plane in non-parametric form is

$(\bar{r}-\bar{a})\cdot((\bar{b}-\bar{a})\times\bar{c}) = 0$ ——— (1)

$\bar{b}-\bar{a} = 3\bar{i}+0\bar{j}-\bar{k}$

$$(\bar{b}-\bar{a})\times\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= \bar{i}(0+1) - \bar{j}(-3+1) + \bar{k}(3-0)$

$= \bar{i}+2\bar{j}+3\bar{k}$

(1) $\Rightarrow$

$(\bar{r}-(-\bar{i}+2\bar{j}))\cdot(\bar{i}+2\bar{j}+3\bar{k}) = 0$

$\bar{r}\cdot(\bar{i}+2\bar{j}+3\bar{k}) - (-\bar{i}+2\bar{j})\cdot(\bar{i}+2\bar{j}+3\bar{k}) = 0$

$\bar{r}\cdot(\bar{i}+2\bar{j}+3\bar{k}) - (-1+4+0) = 0$

$\bar{r}\cdot(\bar{i}+2\bar{j}+3\bar{k}) - (3) = 0$

$\bar{r}\cdot(\bar{i}+2\bar{j}+3\bar{k}) = 3$

Cartesian eqn of the plane is

$(x\bar{i}+y\bar{j}+z\bar{k})\cdot(\bar{i}+2\bar{j}+3\bar{k}) = 3$

$x+2y+3z-3 = 0$

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EXERCISE 6.7

1) Find the non-parametric form of vector eqn and cartesian eqn of the plane passing through the point $(2,3,6)$ and parallel to the st lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-3}{1}$ and $\frac{x+3}{2} = \frac{y-3}{-5} = \frac{z+1}{-3}$.

Here $\bar{a} = 2\bar{i} + 3\bar{j} + 6\bar{k}$

parallel vectors are $\bar{b} = 2\bar{i} + 3\bar{j} + \bar{k}$

and $\bar{c} = 2\bar{i} - 5\bar{j} - 3\bar{k}$

Non parametric form of vector eqn is

$(\bar{r} - \bar{a}) \cdot (\bar{b} \times \bar{c}) = 0$

or $\bar{r} \cdot (\bar{b} \times \bar{c}) = \bar{a} \cdot (\bar{b} \times \bar{c})$ ——— (1)

$$\bar{b} \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 3 & 1 \\ 2 & -5 & -3 \end{vmatrix}$$

$= \bar{i}(-9+5) - \bar{j}(-6-2) + \bar{k}(-10-6)$

$= -4\bar{i} + 8\bar{j} - 16\bar{k}$

$\therefore$ (1) $\Rightarrow \bar{r} \cdot (-4\bar{i} + 8\bar{j} - 16\bar{k})$

$= (2\bar{i} + 3\bar{j} + 6\bar{k}) \cdot (-4\bar{i} + 8\bar{j} - 16\bar{k})$

$\bar{r} \cdot (-4\bar{i} + 8\bar{j} - 16\bar{k}) = -8 + 24 - 96$

$\bar{r} \cdot (-4\bar{i} + 8\bar{j} - 16\bar{k}) = -80$

Cartesian eqn

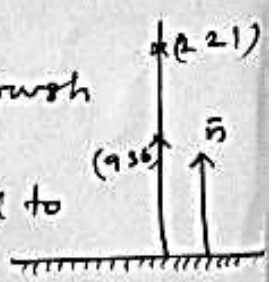
$(x\bar{i} + y\bar{j} + z\bar{k}) \cdot (-4\bar{i} + 8\bar{j} - 16\bar{k}) = -80$

$-4x + 8y - 16z + 80 = 0$

$(\div -4)$ $\boxed{x - 2y + 4z - 20 = 0}$

---

2) Find the parametric form of vector eqn, and cartesian equations of the plane passing through the points $(2,2,1)$, $(9,3,6)$ and perpendicular to the plane $2x + 6y + 6z = 9$



The required plane passes through the point $\bar{a} = 2\bar{i} + 2\bar{j} + \bar{k}$ and $\bar{b} = 9\bar{i} + 3\bar{j} + 6\bar{k}$ and parallel to the vector $\bar{c} = 2\bar{i} + 6\bar{j} + 6\bar{k}$

Non parametric form of vector eqn

$(\bar{r} - \bar{a}) \cdot ((\bar{b} - \bar{a}) \times \bar{c}) = 0$ ——— (1)

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$\bar{b}-\bar{a} = 7\bar{i}+\bar{j}+5\bar{k}$

$(\bar{b}-\bar{a}) \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix}$

$\bar{i}(6-30) - \bar{j}(42-10) + \bar{k}(42-2)$

$-24\bar{i} - 32\bar{j} + 40\bar{k}$

① $\Rightarrow \left(\bar{r} - (2\bar{i}+2\bar{j}+\bar{k})\right) \cdot (-24\bar{i}-32\bar{j}+40\bar{k}) = 0$

$\bar{r} \cdot (-24\bar{i}-32\bar{j}+40\bar{k}) - (2\bar{i}+2\bar{j}+\bar{k}) \cdot (-24\bar{i}-32\bar{j}+40\bar{k}) = 0$

$\bar{r} \cdot (-24\bar{i}-32\bar{j}+40\bar{k}) - (-48-64+40) = 0$

$\bar{r} \cdot (-24\bar{i}-32\bar{j}+40\bar{k}) - (-112+40) = 0$

$\bar{r} \cdot (-24\bar{i}-32\bar{j}+40\bar{k}) = -72$

$\div (-8)$

$\bar{r} \cdot (3\bar{i}+4\bar{j}-5\bar{k}) = 9$

Cartesian eqn is

$3x+4y-5z-9=0$.

3) Find the parametric form of vector equation and cartesian eqns of the plane passing through the points $(2,2,1)$, $(1,-2,3)$ and parallel to the straight line passing through the points $(2,1,-3)$ and $(-1,5,-8)$

Eqn of the st line passing through the points $(2,1,-3)$ and $(-1,5,-8)$ is

$$\frac{x-2}{-3} = \frac{y-1}{4} = \frac{z+3}{-5}$$

The required plane passing through the points $\bar{a} = 2\bar{i}+2\bar{j}+\bar{k}$ and $\bar{b} = \bar{i}-2\bar{j}+3\bar{k}$ and parallel to $\bar{c} = -3\bar{i}+4\bar{j}-5\bar{k}$.

Parametric form of vector eqn

$$\bar{r} = \bar{a} + s(\bar{b}-\bar{a}) + t\bar{c}$$

$\Rightarrow \bar{r} = (2\bar{i}+2\bar{j}+\bar{k}) + s(-\bar{i}-4\bar{j}+2\bar{k}) + t(-3\bar{i}+4\bar{j}-5\bar{k})$

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Cartesian eqn

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ -1 & -4 & 2 \\ -3 & 4 & -5 \end{vmatrix} = 0$$

$$(x-2)(20-8) - (y-2)(5+6) + (z-1)(-4-12) = 0$$

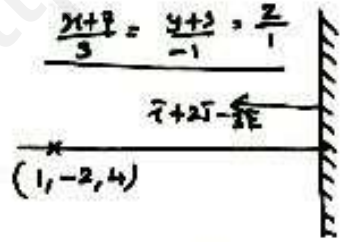
$$12(x-2) - 11(y-2) - 16(z-1) = 0$$

$$12x - 24 - 11y + 22 - 16z + 16 = 0$$

$$12x - 11y - 16z + 14 = 0$$

---

4) Find the non-parametric form of vector eqn of the plane passing through the point $(1,-2,4)$ and perpendicular to the plane $x+2y-3z=11$ and parallel to the line $\frac{x+7}{3} = \frac{y+3}{-1} = \frac{z}{1}$



The required plane passing through the point $\bar{a} = \bar{i} - 2\bar{j} + 4\bar{k}$ and parallel to the vectors $\bar{b} = \bar{i} + 2\bar{j} - 3\bar{k}$ and $\bar{c} = 3\bar{i} - \bar{j} + \bar{k}$.

Non parametric form of vector eqn

$$(\bar{r} - \bar{a}) \cdot (\bar{b} \times \bar{c}) = 0 \quad \text{---} \quad (1)$$

$$\bar{b} \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix}$$

$$= \bar{i}(2-3) - \bar{j}(1+9) + \bar{k}(-1-6)$$

$$= -\bar{i} - 10\bar{j} - 7\bar{k}$$

$$(\bar{r} - \bar{a}) \cdot (-\bar{i} - 10\bar{j} - 7\bar{k}) = 0$$

$$\bar{r} \cdot (-\bar{i} - 10\bar{j} - 7\bar{k}) - \bar{a} \cdot (-\bar{i} - 10\bar{j} - 7\bar{k}) = 0$$

$$\bar{r} \cdot (-\bar{i} - 10\bar{j} - 7\bar{k}) - (\bar{i} - 2\bar{j} + 4\bar{k}) \cdot (-\bar{i} - 10\bar{j} - 7\bar{k}) = 0$$

$$\bar{r} \cdot (-\bar{i} - 10\bar{j} - 7\bar{k}) = -1 + 20 - 28$$

$$\bar{r} \cdot (-\bar{i} - 10\bar{j} - 7\bar{k}) = -9 \quad (\text{or})$$

$$\bar{r} \cdot (\bar{i} + 10\bar{j} + 7\bar{k}) = 9$$

Cartesian eqn:

$$x + 10y + 7z - 9 = 0.$$

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5) Find the parametric form of vector eqn, and Cartesian eqns of the plane containing the line $\bar{r} = (\bar{i}-\bar{j}+3\bar{k}) + t(2\bar{i}-\bar{j}+4\bar{k})$ and perpendicular to the plane $\bar{r}\cdot(\bar{i}+2\bar{j}+\bar{k}) = 8$

Here $\bar{a} = \bar{i}-\bar{j}+3\bar{k}$ and

parallel vectors $\bar{b} = 2\bar{i}-\bar{j}+4\bar{k}$ and $\bar{c} = \bar{i}+2\bar{j}+\bar{k}$.

parametric form of vector eqn is

$$\vec{r} = \bar{a} + s\bar{b} + t\vec{c}.$$

$$\therefore \bar{r} = (\bar{i}-\bar{j}+3\bar{k}) + s(2\bar{i}-\bar{j}+4\bar{k}) + t(\bar{i}+2\bar{j}+\bar{k})$$

Cartesian eqn

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} x-1 & y+1 & z-3 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$$(x-1)(-1-8) - (y+1)(2-4) + (z-3)(4+1) = 0$$

$$-9(x-1) + 2(y+1) + 5(z-3) = 0$$

$$-9x + 9 + 2y + 2 + 5z - 15 = 0$$

$$-9x + 2y + 5z - 4 = 0.$$

$$9x - 2y - 5z + 4 = 0$$

---

6) Find the parametric vector, non-parametric vector and Cartesian form of the eqn of the plane passing through the points $(3,6,-2)$, $(-1,-2,6)$ and $(6,4,-2)$.

Here $\bar{a} = 3\bar{i}+6\bar{j}-2\bar{k}$

$\bar{b} = -\bar{i}-2\bar{j}+6\bar{k}$

$\bar{c} = 6\bar{i}+4\bar{j}-2\bar{k}$

parametric form of vector eqn

$$\bar{r} = \bar{a} + s(\bar{b}-\bar{a}) + t(\bar{c}-\bar{a}) \quad \text{---} \quad (1)$$

$$\bar{b}-\bar{a} = -4\bar{i}-8\bar{j}+8\bar{k}$$

$$\bar{c}-\bar{a} = 3\bar{i}-2\bar{j}+0\bar{k}$$

$$(1) \Rightarrow \bar{r} = (3\bar{i}+6\bar{j}-2\bar{k}) + s(-4\bar{i}-8\bar{j}+8\bar{k}) + t(3\bar{i}-2\bar{j}).$$

Non parametric form of vector eqn.

$$(\bar{b}-\bar{a}) \times (\bar{c}-\bar{a}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix}$$

$$= \bar{i}(0+16) - \bar{j}(-0-24) + \bar{k}(8+24)$$

$$= 16\bar{i}+24\bar{j}+32\bar{k}$$

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Non parametric form of vector eqn is

~~$(\bar{r}-\bar{a})\cdot((\bar{b}-\bar{a})\times\bar{c})$~~

$(\bar{r}-\bar{a})\cdot((b-\bar{a})\times(\bar{c}-\bar{a}))=0$

$(\bar{r}-\bar{a})\cdot(16\bar{i}+24\bar{j}+32\bar{k})=0$

$\bar{r}\cdot(16\bar{i}+24\bar{j}+32\bar{k})-\bar{a}\cdot(16\bar{i}+24\bar{j}+32\bar{k})=0.$

$\bar{r}\cdot(16\bar{i}+24\bar{j}+32\bar{k})-(3\bar{i}+6\bar{j}-2\bar{k})\cdot(16\bar{i}+24\bar{j}+32\bar{k})=0$

$\bar{r}\cdot(16\bar{i}+24\bar{j}+32\bar{k})-(48+144-64)=0$

$\bar{r}\cdot(16\bar{i}+24\bar{j}+32\bar{k})-(192-64)=0$

$\bar{r}\cdot(16\bar{i}+24\bar{j}+32\bar{k})-128=0$

$(\div 8),\quad \bar{r}\cdot(2\bar{i}+3\bar{j}+4\bar{k})-16=0.$

Cartesian eqn

$\bar{r}\cdot(2\bar{i}+3\bar{j}+4\bar{k})-16=0$

$2x+3y+4z-16=0.$

---

7) Find the non-parametric form of vector eqn, and cartesian eqns of the plane

$\bar{r}=(6\bar{i}-\bar{j}+\bar{k})+s(-\bar{i}+2\bar{j}+\bar{k})+t(-5\bar{i}-4\bar{j}-5\bar{k})$

Here $\bar{a}=6\bar{i}-\bar{j}+\bar{k}$

$\bar{b}=-\bar{i}+2\bar{j}+\bar{k}$

$\bar{c}=-5\bar{i}-4\bar{j}-5\bar{k}$

Non parametric form of vector eqn.

$(\bar{r}-\bar{a})\cdot(\bar{b}\times\bar{c})=0$

$$\bar{b}\times\bar{c}=\begin{vmatrix}\bar{i}&\bar{j}&\bar{k}\\-1&2&1\\-5&-4&-5\end{vmatrix}$$

$=\bar{i}(-10+4)-\bar{j}(5+5)+\bar{k}(4+10)$

$=-6\bar{i}-10\bar{j}+14\bar{k}.$

Non parametric form of eqn is

$(\bar{r}-\bar{a})\cdot(\bar{b}\times\bar{c})=0.$

$(\bar{r}-\bar{a})\cdot(-6\bar{i}-10\bar{j}+14\bar{k})=0$

$\bar{r}\cdot(-6\bar{i}-10\bar{j}+14\bar{k})-\bar{a}\cdot(-6\bar{i}-10\bar{j}+14\bar{k})=0$

$\bar{r}\cdot(-6\bar{i}-10\bar{j}+14\bar{k})-(6\bar{i}-\bar{j}+\bar{k})\cdot(-6\bar{i}-10\bar{j}+14\bar{k})=0$

$\bar{r}\cdot(-6\bar{i}-10\bar{j}+14\bar{k})-(-36+10+14)=0$

$\bar{r}\cdot(-6\bar{i}-10\bar{j}+14\bar{k})-(-12)=0$

$\bar{r}\cdot(-6\bar{i}-10\bar{j}+14\bar{k})=-12$

$(\div -2)\qquad \bar{r}\cdot(3\bar{i}+5\bar{j}-7\bar{k})=6$

Cartesian eqn:

$3x+5y-7z-6=0.$

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Condition for a line to lie in a plane :-

a) If the line $\bar{r} = \bar{a} + t\bar{b}$ lie in a plane $\bar{r} \cdot \bar{n} = d$ then $\bar{a} \cdot \bar{n} = d$ and $\bar{b} \cdot \bar{n} = 0$

b) If the line $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ lies in the plane $Ax + By + Cz + D = 0$ then $Ax_1 + By_1 + Cz_1 + D = 0$ and $aA + bB + cC = 0$

Condition for Coplanarity of two lines :-

a) Condition in vector form

The two given non-parallel lines $\bar{r} = \bar{a} + s\bar{b}$ and $\bar{r} = \bar{c} + t\bar{d}$ are coplanar. So they lie in a single plane. Let A and C be the points whose position vectors $\bar{a}$ and $\bar{c}$.

Then A and C lie on a plane. $\bar{b}$ and $\bar{d}$ are parallel to the plane. $\bar{b} \times \bar{d}$ is $\perp$r to the plane. So $\overline{AC}$ is $\perp$r to $\bar{b} \times \bar{d}$.

i.e. $(\bar{c} - \bar{a}) \cdot (\bar{b} \times \bar{d}) = 0$.

This is the required condition for coplanarity of two lines in vector form.

b) Condition in Cartesian form

Two lines $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$ and $\frac{x-x_2}{d_1} = \frac{y-y_2}{d_2} = \frac{z-z_2}{d_3}$ are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0 .$$

This is the condition for coplanarity of two lines in Cartesian form.

Equation of plane containing two non-parallel coplanar lines.

a) parametric form of vector eqn

Vector eqn in parametric form is $\bar{r} = \bar{a} + t\bar{b} + s\bar{d}$ or $\bar{r} = \bar{c} + t\bar{b} + s\bar{d}$.

b) Non parametric form of vector eqn.

Let $\bar{r} = \bar{a} + t\bar{b}$ and $\bar{r} = \bar{c} + t\bar{d}$ be two non-parallel coplanar lines. Hence the vector eqn in non-parametric form is $(\bar{r} - \bar{a}) \cdot (\bar{b} \times \bar{d}) = 0$ or $(\bar{r} - \bar{c}) \cdot (\bar{b} \times \bar{d}) = 0$.

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c) Cartesian form of equation of plane :-

In cartesian form the eqn of the plane containing the two given coplanar lines.

$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$ and

$\frac{x-x_2}{d_1} = \frac{y-y_2}{d_2} = \frac{y-y_3}{d_3}$ is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0 \quad (\text{or})$$

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

Eg 6.45 verify whether the line $\frac{x-3}{-4} = \frac{y-4}{-7} = \frac{z+3}{12}$ lies in the plane $5x-y+z=8$.

condition for the line $\frac{x-x_1}{a} = \frac{y-y_1}{b}$

$= \frac{z-z_1}{c}$ lies in the plane $Ax+By+Cz+D=0$

is $Ax_1+By_1+Cz_1+D=0$ and

$aA+bB+cC=0$

Here $(x_1, y_1, z_1) = (3, 4, -3)$

$(a, b, c) = (-4, -7, 12)$

$(A, B, C) = (5, -1, 1)$ $D = -8$.

Now $Ax_1 + By_1 + Cz_1 + D$

$= 5(3) - 1(4) + 1(-3) - 8$

$= 15 - 4 - 3 - 8$

$= 15 - 15 = 0$.

$aA+bB+cC = 5(-4) - 1(-7) + 1(12)$

$= -20 + 7 + 12$

$= 19 - 20$

$= -1$

$aA + bB + cC \neq 0$

Hence, the given line does not lie in the plane.

6.46 Show that the lines $\bar{r} = (-\bar{i} - 3\bar{j} - 5\bar{k}) + s(3\bar{i} + 5\bar{j} + 7\bar{k})$ and $\bar{r} = (2\bar{i} + 4\bar{j} + 6\bar{k}) + t(\bar{i} + 4\bar{j} + 7\bar{k})$ are coplanar. Also find the non parametric form of vector eqn of the plane containing these lines.

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Given lines are

$\bar{r} = (-\bar{i} - 3\bar{j} - 5\bar{k}) + s(3\bar{i} + 5\bar{j} + 7\bar{k})$ and

$\bar{r} = (2\bar{i} + 4\bar{j} + 6\bar{k}) + t(\bar{i} + 4\bar{j} + 7\bar{k})$

~~are coplanar. Also find.~~

Compare these two lines with

$\bar{r} = \bar{a} + t\bar{b}$ and $\bar{r} = \bar{c} + s\bar{d}$

Here $\bar{a} = -\bar{i} - 3\bar{j} - 5\bar{k}$

$\bar{b} = 3\bar{i} + 5\bar{j} + 7\bar{k}$

$\bar{c} = 2\bar{i} + 4\bar{j} + 6\bar{k}$

$\bar{d} = \bar{i} + 4\bar{j} + 7\bar{k}$.

Condition for two lines are coplanar is

$(\bar{c} - \bar{a}) \cdot (\bar{b} \times \bar{d}) = 0$.

$\bar{c} - \bar{a} = 3\bar{i} + 7\bar{j} + 11\bar{k}$

$$\bar{b} \times \bar{d} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix}$$

$= \bar{i}(35-28) - \bar{j}(21-7) + \bar{k}(12-5)$

$= 7\bar{i} - 14\bar{j} + 7\bar{k}$.

$(\bar{c} - \bar{a}) \cdot (\bar{b} \times \bar{d}) = (3\bar{i} + 7\bar{j} + 11\bar{k}) \cdot (7\bar{i} - 14\bar{j} + 7\bar{k})$

$= 21 - 98 + 77$

$= 98 - 98 = 0$

$(\bar{c} - \bar{a}) \cdot (\bar{b} \times \bar{d}) = 0$

$\therefore$ The given two lines are coplanar.

Eqn of a plane containing the two given coplanar lines is

$(\bar{r} - \bar{a}) \cdot (\bar{b} \times \bar{d}) = 0$ (or)

$(\bar{r} - \bar{c}) \cdot (\bar{b} \times \bar{d}) = 0$.

$\left(\bar{r} - (-\bar{i} - 3\bar{j} - 5\bar{k})\right) \cdot (7\bar{i} - 14\bar{j} + 7\bar{k}) = 0$

$\Rightarrow \bar{r} \cdot (7\bar{i} - 14\bar{j} + 7\bar{k}) = (-\bar{i} - 3\bar{j} - 5\bar{k}) \cdot (7\bar{i} - 14\bar{j} + 7\bar{k})$

$\bar{r} \cdot (7\bar{i} - 14\bar{j} + 7\bar{k}) = -7 + 42 - 35 = 0$

Hence the plane containing these two lines is $\bar{r} \cdot (7\bar{i} - 14\bar{j} + 7\bar{k}) = 0$.

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EXERCISE 6.8

1) Show that the st lines $\bar{r} = (5\bar{i}+7\bar{j}-3\bar{k}) + s(4\bar{i}+4\bar{j}-5\bar{k})$ and $\bar{r} = (8\bar{i}+4\bar{j}+5\bar{k}) + t(7\bar{i}+\bar{j}+3\bar{k})$ are coplanar. find the vector equation of the plane in which they lie.

Given two lines are

$\bar{r} = (5\bar{i}+7\bar{j}-3\bar{k}) + s(4\bar{i}+4\bar{j}-5\bar{k})$

and $\bar{r} = (8\bar{i}+4\bar{j}+5\bar{k}) + t(7\bar{i}+\bar{j}+3\bar{k})$

condition for two lines are coplanar is $(\bar{c}-\bar{a}) \cdot (\bar{b}\times\bar{d}) = 0$ — (1)

Here $\bar{a} = 5\bar{i}+7\bar{j}-3\bar{k}$

$\bar{b} = 4\bar{i}+4\bar{j}-5\bar{k}$

$\bar{c} = 8\bar{i}+4\bar{j}+5\bar{k}$

$\bar{d} = 7\bar{i}+\bar{j}+3\bar{k}$

$\bar{c}-\bar{a} = 3\bar{i}-3\bar{j}+8\bar{k}$

$$\bar{b}\times\bar{d} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}$$

$= \bar{i}(12+5) - \bar{j}(12+35) + \bar{k}(4-28)$

$= 17\bar{i} - 47\bar{j} - 24\bar{k}$

(1) $\Rightarrow (\bar{c}-\bar{a}) \cdot (\bar{b}\times\bar{d})$

$= (3\bar{i}-3\bar{j}+8\bar{k}) \cdot (17\bar{i}-47\bar{j}-24\bar{k})$

$= 51 + 141 - 192$

$= 192 - 192 = 0$

Hence the given two lines are coplanar.

Eqn of a plane containing the two given coplanar line is

$(\bar{r}-\bar{a}) \cdot (\bar{b}\times\bar{d}) = 0$ or

$(\bar{r}-\bar{c}) \cdot (\bar{b}\times\bar{d}) = 0$

$(\bar{r}-\bar{a}) \cdot (\bar{b}\times\bar{d}) = 0$

$\Rightarrow (\bar{r} - (5\bar{i}+7\bar{j}-3\bar{k})) \cdot (17\bar{i}-47\bar{j}-24\bar{k}) = 0$

$\bar{r} \cdot (17\bar{i}-47\bar{j}-24\bar{k}) = (5\bar{i}+7\bar{j}-3\bar{k}) \cdot (17\bar{i}-47\bar{j}-24\bar{k})$

$\bar{r} \cdot (17\bar{i}-47\bar{j}-24\bar{k}) = 85 - 329 + 72$

$\bar{r} \cdot (17\bar{i}-47\bar{j}-24\bar{k}) = 157 - 329$

$\bar{r} \cdot (17\bar{i}-47\bar{j}-24\bar{k}) = -172$ //

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2) Show that the lines

$\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{3}$ and

$\frac{x-1}{-3} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar.

Also find the plane containing these lines.

Given lines are

$\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{3}$ and

$\frac{x-1}{-3} = \frac{y-4}{2} = \frac{z-5}{1}$

Here $(x_1, y_1, z_1) = (2, 3, 4)$

$(x_2\ y_2\ z_2) = (1, 4, 5)$

$(b_1, b_2, b_3) = (1, 1, 3)$

$(d_1\ d_2\ d_3) = (-3, 2, 1)$

condition for ~~coline~~ coplanarity

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = -1(1-6) - 1(1+9) + 1(2+3)$$

$$= -1(-5) - 1(10) + 1(5)$$

$$= 5 - 10 + 5 = 0.$$

∴ The given two lines are coplanar.

Cartesian eqn of the plane containing the two given coplanar lines

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2 & y-3 & z-4 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = 0$$

$$(x-2)(1-6) - (y-3)(1+9) + (z-4)(2+3) = 0$$

$$(x-2)(-5) - (y-3)(10) + (z-4)(5) = 0$$

$$-5x + 10 - 10y + 30 + 5z - 20 = 0$$

$$-5x - 10y + 5z + 20 = 0$$

$$5x + 10y - 5z - 20 = 0$$

$(\div 5)$ $\quad x + 2y - z - 4 = 0$

---

3) If the st lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{m^2}$ and

$\frac{x-3}{1} = \frac{y-2}{m^2} = \frac{z-1}{2}$ are coplanar, find the distinct real values of m.

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2) Show that the lines

$\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{3}$ and

$\frac{x-1}{-3} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar.

Also find the plane containing these lines.

Given lines are

$\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{3}$ and

$\frac{x-1}{-3} = \frac{y-4}{2} = \frac{z-5}{1}$

Here $(x_1, y_1, z_1) = (2, 3, 4)$

$(x_2, y_2, z_2) = (1, 4, 5)$

$(b_1, b_2, b_3) = (1, 1, 3)$

$(d_1, d_2, d_3) = (-3, 2, 1)$

condition for ~~coline~~ coplanarity

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = -1(1-6) - 1(1+9) + 1(2+3)$$

$$= -1(-5) - 1(10) + 1(5)$$

$$= 5 - 10 + 5 = 0.$$

$\therefore$ The given two lines are coplanar.

Cartesian eqn of the plane containing the two given coplanar lines

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2 & y-3 & z-4 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = 0$$

$$(x-2)(1-6) - (y-3)(1+9) + (z-4)(2+3) = 0$$

$$(x-2)(-5) - (y-3)(10) + (z-4)(5) = 0$$

$$-5x + 10 - 10y + 30 + 5z - 20 = 0$$

$$-5x - 10y + 5z + 20 = 0$$

$$5x + 10y - 5z - 20 = 0$$

$(\div 5)$ $\quad x + 2y - z - 4 = 0$

---

3) If the 2 lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{m^2}$ and $\frac{x-3}{1} = \frac{y-2}{m^2} = \frac{z-1}{2}$ are coplanar, find the distinct real values of m.

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3) If the st lines ~~are~~
$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{m^2}$ and
$\frac{x-3}{1} = \frac{y-2}{m^2} = \frac{z-1}{2}$ are coplanar, find m.

Given lines are
$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{m^2}$ and
$\frac{x-3}{1} = \frac{y-2}{m^2} = \frac{z-1}{2}$

Here $(x_1, y_1, z_1) = (1, 2, 3)$
$(x_2, y_2, z_2) = (3, 2, 1)$
$(b_1, b_2, b_3) = (1, 2, m^2)$
$(d_1, d_2, d_3) = (1, m^2, 2)$

Given that the st lines are coplanar.

ie $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 2 & 0 & -2 \\ 1 & 2 & m^2 \\ 1 & m^2 & 2 \end{vmatrix} = 0$

$\Rightarrow$ ~~$2(m^2 - 2) = 0$~~

$\Rightarrow 2(4 - m^4) - 2(m^2 - 2) = 0$

$8 - 2m^4 - 2m^2 + 4 = 0$

$-2m^4 - 2m^2 + 12 = 0$

$(\div -2) \quad m^4 + m^2 - 6 = 0$

$(m^2+3)(m^2-2) = 0$

$\Rightarrow m^2 + 3 = 0$ or $m^2 - 2 = 0$

$m^2 = -3$ | $m^2 = 2 \Rightarrow m = \pm\sqrt{2}$

$m = \sqrt{-3}$

not possible | $\therefore \boxed{m = \pm\sqrt{2}}$

---

4) If the st lines $\frac{x-1}{2} = \frac{y+1}{\lambda} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{\lambda}$ are coplanar, find λ and eqn of the planes containing these two lines.

Given lines are $\frac{x-1}{2} = \frac{y+1}{\lambda} = \frac{z}{2}$

and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{\lambda}$.

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Given lines are $\frac{x-1}{2} = \frac{y+1}{\lambda} = \frac{z}{2}$

and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{\lambda}$.

Here $(x_1, y_1, z_1) = (1, -1, 0)$

$(x_2, y_2, z_2) = (-1, -1, 0)$

$(b_1, b_2, b_3) = (2, \lambda, 2)$

$(d_1, d_2, d_3) = (5, 2, \lambda)$

Given lines are coplanar

$$\Rightarrow \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -2 & 0 & 0 \\ 2 & \lambda & 2 \\ 5 & 2 & \lambda \end{vmatrix} = 0$$

$\Rightarrow -2(\lambda^2 - 4) = 0 \Rightarrow \lambda^2 - 4 = 0$

$\lambda^2 = 4$

$\lambda = \pm 2$.

<u>case(i)</u> If $\lambda = 2$

Then the eqn of the plane containing these two lines is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

i.e $\begin{vmatrix} x-1 & y+1 & z \\ 2 & 2 & 2 \\ 5 & 2 & 2 \end{vmatrix} = 0$

$(x-1)(4-4) - (y+1)(4-10) + z(4-10) = 0$

$(x-1)(0) - (y+1)(-6) + z(-6) = 0$

$6y + 6 - 6z = 0$

$(\div 6)$ $\boxed{y - z + 1 = 0}$

Case(ii) If $\lambda = -2$

Then the eqn of the plane containing these two lines is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

i.e $\begin{vmatrix} x-1 & y+1 & z \\ 2 & -2 & 2 \\ 5 & 2 & -2 \end{vmatrix} = 0$

$(x-1)(4-4) - (y+1)(-4-10) + z(4+10) = 0$

$0 + 14(y+1) + 14(z) = 0$

$14y + 14 + 14z = 0$

$(\div 14)$ $\boxed{y + z + 1 = 0}$.

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Angle betn a line and a plane:

The angle betn the line $\bar{r} = \bar{a} + t\bar{b}$ and the plane $\bar{r}\cdot\bar{n} = p$ is given by

$$\theta = \sin^{-1}\left[\frac{|\bar{b}\cdot\bar{n}|}{|\bar{b}||\bar{n}|}\right]$$

Angle betn two planes:

The angle betn two planes $\bar{r}\cdot\bar{n}_1 = p_1$ and $\bar{r}\cdot\bar{n}_2 = p_2$ is given by

$$\theta = \cos^{-1}\left[\frac{|\bar{n}_1\cdot\bar{n}_2|}{|\bar{n}_1||\bar{n}_2|}\right]$$

Angle betn the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is given by

$$\theta = \cos^{-1}\left[\frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right]$$

Angle betn a line and a plane.

In cartesian form if

$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $ax+by+cz=p$ are equations of the line and the plane then $\bar{b} = a_1\bar{i}+b_1\bar{j}+c_1\bar{k}$ and $\bar{n} = a\bar{i}+b\bar{j}+c\bar{k}$

∴ The angle betn the line and a plane is

$$\theta = \sin^{-1}\left[\frac{|aa_1+bb_1+cc_1|}{\sqrt{a^2+b^2+c^2}\sqrt{a_1^2+b_1^2+c_1^2}}\right]$$

Distence of a point from a plane

a) Vector form of eqn

The ↓r distance from a point with position vector $\bar{u}$ to the plane $\bar{r}\cdot\bar{n} = p$ is given by

$$\delta = \frac{|\bar{u}\cdot\bar{n}-p|}{|\bar{n}|}$$

b) Cartesian form of eqn

In cartesian form if $A(x_1,y_1,z_1)$ is the given point with position vector $\bar{u}$ and $ax+by+cz=p$ is the cartesian eqn of the given plane. The perpendicular distance from a point to the plane in cartesian form as

$$\delta = \left|\frac{ax_1+by_1+cz_1-p}{\sqrt{a^2+b^2+c^2}}\right| = \frac{|ax_1+by_1+cz_1-p|}{\sqrt{a^2+b^2+c^2}}$$

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Eg 6.47 Find the angle betn the planes $\vec{r}\cdot(2\vec{i}+2\vec{j}+2\vec{k}) = 11$ and $4x-2y+2z = 15$.

Normal vectors of the two given planes are

$\vec{n_1} = 2\vec{i}+2\vec{j}+2\vec{k}$ and

$\vec{n_2} = 4\vec{i}-2\vec{j}+2\vec{k}$.

If θ is angle betn two planes then

$$\theta = \cos^{-1}\left[\frac{|\vec{n_1}\cdot\vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}\right] \quad \text{—— (1)}$$

$\vec{n_1}\cdot\vec{n_2} = 2(4)+2(-2)+2(2)$

$= 8-4+4 = 8$

$|\vec{n_1}| = \sqrt{2^2+2^2+2^2} = \sqrt{4+4+4}$

$= \sqrt{12}$

$= 2\sqrt{3}$

$|\vec{n_2}| = \sqrt{16+4+4} = \sqrt{24}$

$= \sqrt{4\times 6}$

$= 2\sqrt{6}$

$= 2\sqrt{3}\sqrt{2}$

(1) $\Rightarrow \theta = \cos^{-1}\left[\frac{8}{2\sqrt{3}\,2\sqrt{3}\sqrt{2}}\right]$

$= \cos^{-1}\left[\frac{8}{4\times 3\times\sqrt{2}}\right]$

$= \cos^{-1}\left(\frac{\sqrt{2}}{3}\right)$

6.48 Find the angle betn the st line $\vec{r} = (2\vec{i}+3\vec{j}+\vec{k}) + t(\vec{i}-\vec{j}+\vec{k})$ and the plane $2x-y+z = 5$

The parallel vector to the line $\vec{b} = \vec{i}-\vec{j}+\vec{k}$ is $\vec{n}$ = normal vector to the plane $2\vec{i}-\vec{j}+\vec{k}$

angle betn the line and the plane

$$\theta = \sin^{-1}\left[\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right]$$

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$$\therefore \alpha = \sin^{-1}\left[\frac{|(\bar{i}-\bar{j}+\bar{k})\cdot(2\bar{i}-\bar{j}+\bar{k})|}{\sqrt{1+1+1}\ \sqrt{4+1+1}}\right]$$

$$= \sin^{-1}\left[\frac{2+1+1}{\sqrt{3}\sqrt{6}}\right]$$

$$= \sin^{-1}\left[\frac{4}{\sqrt{3}\sqrt{2}\sqrt{3}}\right] = \sin^{-1}\left[\frac{2\sqrt{2}}{3}\right]$$

Example 6.49 Find the distance of a point $(2,5,-3)$ from the plane $\bar{r}\cdot(6\bar{i}-3\bar{j}+2\bar{k}) = 5$.

The distance of a point and the plane is

$$\delta = \frac{|\bar{u}\cdot\bar{n}-p|}{|\bar{n}|} \quad \text{—— (1)}$$

Here $\bar{u} = 2\bar{i}+5\bar{j}-3\bar{k}$

$\bar{n} = 6\bar{i}-3\bar{j}+2\bar{k}$

$p = 5$

$\therefore$ (1) $\Rightarrow$ $\delta = \left|\frac{(12-15-6)-5}{\sqrt{36+9+4}}\right|$

$= \frac{|12-26|}{|\sqrt{49}|} = \frac{14}{7} = 2$

$\therefore$ The required distance is 2

XII | P3 | VAE 6.9

6.50 Find the distance of the point $(5,-5,10)$ from a point of intersection of a st line passing through the points $A(4,1,2)$ and $B(7,5,4)$ with the plane $x-y+z=5$.

The eqn of the line joining the points $A(4,1,2)$ and $B(7,5,4)$ is

$$\frac{x-4}{7-4} = \frac{y-1}{5-1} = \frac{z-2}{4-2} = t \text{ (say)}$$

$$\frac{x-4}{3} = \frac{y-1}{4} = \frac{z-2}{2} = t \text{ (say)} \quad \text{—— (1)}$$

Any point on this line is $(3t+4, 4t+1, 2t+2)$.

Given plane is $x-y+z=5$ —— (2)

$\frac{x-4}{3} = t \Rightarrow x = 3t+4$

$\frac{y-1}{4} = t \Rightarrow y = 4t+1$

$\frac{z-2}{2} = t \Rightarrow z = 2t+2$ of the line (1)

To find the pt of intersection and the plane (2) is

Sub the pt $(3t+4, 4t+1, 2t+2)$ in (2)

$3t+4-(4t+1)+2t+2 = 5$

$3t+4-4t-1+2t+2 = 5$

$t+5 = 5 \Rightarrow t = 0$

$\therefore$ point of intersection is $(4,1,2)$

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The distance betn the point $(4,1,2)$ and $(5,-5,-10)$ is

$$\sqrt{(5-4)^2+(-5-1)^2+(-10-2)^2}$$

$$= \sqrt{1^2+(-6)^2+(-12)^2}$$

$$= \sqrt{1+36+144} = \sqrt{181}$$

∴ The required distance is $\sqrt{181}$ units

Distance betn two parallel planes:

The distance betn the parallel planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is

$$\delta = \frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$$

Eg 6.51 Find the distance betn the parallel planes $x+2y-2z+1=0$ and $2x+4y-4z+5=0$

The given planes are

$$x+2y-2z+1=0 \quad \text{—— (1)}$$

$$2x+4y-4z+5=0$$

$$(\div 2) \quad x+2y-2z+\tfrac{5}{2}=0 \quad \text{—— (2)}.$$

Here $a=1, b=2, c=-2, d_1=1, d_2=\frac{5}{2}$

Distance betn the parallel planes (1) and (2) is

$$\delta = \frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}} = \frac{|1-\frac{5}{2}|}{\sqrt{1+4+4}}$$

$$= \frac{|-\frac{3}{2}|}{\sqrt{9}}$$

$$= \frac{\frac{3}{2}}{3} = \boxed{\frac{1}{2}}$$

Eg 6.52 Find the distance betn the planes $\vec{r}\cdot(2\vec{i}-\vec{j}-2\vec{k})=6$ and $\vec{r}\cdot(6\vec{i}-3\vec{j}-6\vec{k})=27$.

The given planes are

$$\vec{r}\cdot(2\vec{i}-\vec{j}-2\vec{k})=6 \quad \text{— (1)}$$

$$\vec{r}\cdot(6\vec{i}-3\vec{j}-6\vec{k})=27 \quad \text{— (2)}$$

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convert the given eqn into cartesian form

$2x - y - 2z = 6$ ——— (3)

$6x - 3y - 6z = 27$ ——— (4)

$(\div 3)$ $2x - y - 2z = 9$ ——— (4)

Distance betn the parallel planes (3) and (4) is

$$\delta = \frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$$

Here $a=2,\ b=-1,\ c=-2,\ d_1=6,\ d_2=9$

$$\therefore \delta = \frac{|6-9|}{\sqrt{4+1+4}} = \frac{|-3|}{\sqrt{9}} = \frac{3}{3} = 1$$

∴ The required distance = 1 unit

Eqn of line of intersection of two planes :-

a) **Vector form :-**

The vector eqn of a plane which passes through the line of intersection of the planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ is

$$(\vec{r}\cdot\vec{n}_1 - d_1) + \lambda(\vec{r}\cdot\vec{n}_2 - d_2) = 0$$

b) **Cartesian form :-**

The cartesian eqn of the plane passing through the line of intersection of the planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ is

$$(a_1x + b_1y + c_1z - d_1) + \lambda(a_2x + b_2y + c_2z - d_2) = 0.$$

Eg 6.53 Find the eqn of the plane passing through the intersection of the planes $\vec{r}\cdot(\vec{i}+\vec{j}+\vec{k}) + 1 = 0$ and $\vec{r}\cdot(2\vec{i}-3\vec{j}+5\vec{k}) = 2$ and the point $(-1, 2, 1)$.

Given planes are $\vec{r}\cdot(\vec{i}+\vec{j}+\vec{k}) + 1 = 0$ and $\vec{r}\cdot(2\vec{i}-3\vec{j}+5\vec{k}) = 2$

The vector eqn of the plane passing through the line of intersection of the planes is

$$(\vec{r}\cdot\vec{n}_1 - d_1) + \lambda(\vec{r}\cdot\vec{n}_2 - d_2) = 0$$

∴ The required eqn is

$$\left[\vec{r}\cdot(\vec{i}+\vec{j}+\vec{k}) + 1\right] + \lambda\left[\vec{r}\cdot(2\vec{i}-3\vec{j}+5\vec{k}) - 2\right] = 0$$

i.e. $[x+y+z+1] + \lambda[2x-3y+5z-2] = 0$ ——— (1)

Given that the plane passes through the pt $(-1, 2, 1)$

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$\therefore$

$(*) \Rightarrow (-1+2+1+1)+\lambda(-2-6+5-2)=0$

$3+\lambda(-5)=0$

$5\lambda=3 \Rightarrow \boxed{\lambda=3/5}$

$\therefore$ The required eqn (1)

$\Rightarrow (x+y+z+1)+\frac{3}{5}(2x-3y+5z-2)=0$

~~$3x+3y+3z+$~~

$\frac{5x+5y+5z+5+6x-9y+15z-6}{5}=0$

$\boxed{11x-4y+20z-1=0}$

Eg 6.54

Find the eqn of the plane passing through the intersection of the planes $2x+3y-z+7=0$ and $x+y-2z+5=0$ and is $\perp^r$ to the plane $x+y-3z-5=0$

Intersection of the planes are

$2x+3y-z+7=0$ —— (1) and

$x+y-2z+5=0$ —— (2)

The eqn of the plane passing through the line of intersection of the planes (1) and (2) is

$(2x+3y-z+7)+\lambda(x+y-2z+5)=0$

~~$a(a+\lambda)x$~~

$2x+3y-z+7+\lambda x+\lambda y-2\lambda z+5\lambda=0$

$(2+\lambda)x+(3+\lambda)y-(1+2\lambda)z+7+5\lambda=0$ —— (3)

Given that (3) and the plane $x+y-3z-5=0$ are $\perp^r$.

$\therefore (2+\lambda)x+(3+\lambda)y-(1+2\lambda)z+(7+5\lambda)\cdot(x+y-3z-5)=0$.

$2+\lambda+3+\lambda-(1+2\lambda)(-3)=0$

$5+2\lambda+3+6\lambda=0$

$8\lambda+8=0 \Rightarrow 8\lambda=-8 \Rightarrow \boxed{\lambda=-1}$

$\therefore$ The required plane is (3) $\Rightarrow$

$(2-1)x+(3-1)y-(1-2)z+7+5(-1)=0$

$x+2y+z+2=0$

—— ∧ ——

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EXERCISE 6.9

1) Find the eqn of the plane passing through the line of intersection of the planes $\vec{r}\cdot(2\vec{i}-7\vec{j}+4\vec{k}) = 3$ and $3x-5y+4z+11=0$ and the point $(-2,1,3)$.

Given planes are

$\vec{r}\cdot(2\vec{i}-7\vec{j}+4\vec{k}) = 3$

i.e. $2x-7y+4z-3=0$ —— (1) and

$3x-5y+4z+11=0$ —— (2)

The eqn of the plane passing through the line of intersection of the planes (1) and (2) is

$(2x-7y+4z-3)+\lambda(3x-5y+4z+11)=0$ —— (3)

This plane passes through the point $(-2,1,3)$.

$\therefore (2(-2)-7(1)+4(3)-3)+\lambda[3(-2)-5(1)+4(3)+11]=0$

$(-4-7+12-3)+\lambda(-6-5+12+11)=0$

$(12-12)+\lambda(12)=0$

$-2+12\lambda=0$

$12\lambda = 2 \Rightarrow \lambda = 2/12 = 1/6$

Sub $\lambda = 1/6$ in (3)

$(2x-7y+4z-3)+\frac{1}{6}(3x-5y+4z+11)=0$

$\frac{12x-42y+24z-18+3x-5y+4z+11}{6}=0$

$15x-47y+28z-7=0$

which is required eqn of the plane

2) Find the eqn of the plane passing through the line of intersection of the planes $x+2y+3z=2$ and $x-y+z=3$ and at a distance $\frac{2}{\sqrt{3}}$ from the point $(3,1,-1)$.

Given planes are

$x+2y+3z-2=0$ —— (1) and

$x-y+z-3=0$ —— (2)

The eqn of the plane passing through the line of intersection of the planes is

$(x+2y+3z-2)+\lambda(x-y+z-3)=0$

$(1+\lambda)x+(2-\lambda)y+(3+\lambda)z-2-3\lambda=0$ —— (3)

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Given that the distance of the point $(3, 1, -1)$ and the plane (3) is $\frac{2}{\sqrt{3}}$.

$$\frac{\left|(1+\lambda)3 + (2-\lambda)(1) + (3+\lambda)(-1) - 2 - 3\lambda\right|}{\sqrt{(1+\lambda)^2 + (2-\lambda)^2 + (3+\lambda)^2}} = \frac{2}{\sqrt{3}}$$

$$\frac{\left|3 + 3\lambda + 2 - \lambda - 3 - \lambda - 2 - 3\lambda\right|}{\sqrt{1+\lambda^2+2\lambda+4+\lambda^2-4\lambda+9+\lambda^2+6\lambda}} = \frac{2}{\sqrt{3}}$$

$$\frac{|-2\lambda|}{\sqrt{3\lambda^2+4\lambda+14}} = \frac{2}{\sqrt{3}}$$

$$\frac{2\lambda}{\sqrt{3\lambda^2+4\lambda+14}} = \frac{2}{\sqrt{3}}$$

$$\sqrt{3}\lambda = \sqrt{3\lambda^2+4\lambda+14}$$

squaring

$$3\lambda^2 = 3\lambda^2 + 4\lambda + 14$$

$$4\lambda + 14 = 0$$

$$4\lambda = -14$$

$$\lambda = -\frac{14}{4} = -\frac{7}{2}$$

Sub $\lambda = -\frac{7}{2}$ in

$$(x+2y+3z-2) + \lambda(x-y+z-3) = 0$$

$$(x+2y+3z-2) - \frac{7}{2}(x-y+z-3) = 0$$

$$\frac{2x+4y+6z-4-7x+7y-7z+21}{2} = 0$$

$$\frac{-5x+11y-z+17}{2} = 0$$

$$-5x+11y-z+17 = 0$$

$$\boxed{5x-11y+z-17=0}$$

---

3) Find the angle betn the line $\bar{r} = (2\bar{i}-\bar{j}+\bar{k}) + t(\bar{i}+2\bar{j}-2\bar{k})$ and the plane $\bar{r}\cdot(6\bar{i}+3\bar{j}+2\bar{k}) = 8$

Given eqn of the line and the plane are $\bar{r} = (2\bar{i}-\bar{j}+\bar{k}) + t(\bar{i}+2\bar{j}-2\bar{k})$ and $\bar{r}\cdot(6\bar{i}+3\bar{j}+2\bar{k}) = 8$.

Here $\bar{b} = \bar{i}+2\bar{j}-2\bar{k}$

$\bar{n} = 6\bar{i}+3\bar{j}+2\bar{k}$

angle betn the line and the plane is

$$\alpha = \sin^{-1}\left[\frac{|\bar{b}\cdot\bar{n}|}{|\bar{b}||\bar{n}|}\right]$$

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$$\theta = \sin^{-1}\left[\frac{|(\hat{i}+2\hat{j}-2\hat{k})\cdot(6\hat{i}+3\hat{j}+2\hat{k})|}{|\hat{i}+2\hat{j}-2\hat{k}|\,|6\hat{i}+3\hat{j}+2\hat{k}|}\right]$$

$$= \sin^{-1}\left[\frac{6+6-4}{\sqrt{1+4+4}\ \sqrt{36+9+4}}\right]$$

$$= \sin^{-1}\left[\frac{|8|}{\sqrt{9}\ \sqrt{49}}\right]$$

$$= \sin^{-1}\left[\frac{8}{3\times 7}\right] = \sin^{-1}\left(\frac{8}{21}\right)$$

---

4) Find the angle betn the planes $\vec{r}\cdot(\hat{i}+\hat{j}-2\hat{k}) = 3$ and $2x-2y+z = 2$

Given planes are

$\vec{r}\cdot(\hat{i}+\hat{j}-2\hat{k}) = 3$ and $2x-2y+z = 2$

The normal vectors of the planes are $\vec{n_1} = \hat{i}+\hat{j}-2\hat{k}$ and $\vec{n_2} = 2\hat{i}-2\hat{j}+\hat{k}$

Required angle $\theta = \cos^{-1}\left[\frac{|\vec{n_1}\cdot\vec{n_2}|}{|\vec{n_1}|\,|\vec{n_2}|}\right]$

$$\theta = \cos^{-1}\left[\frac{|2-2-2|}{\sqrt{1+1+4}\ \sqrt{4+4+1}}\right]$$

$$\theta = \cos^{-1}\left(\frac{|-2|}{\sqrt{6}\ \sqrt{9}}\right)$$

$$= \cos^{-1}\left(\frac{2}{\sqrt{2}\sqrt{3}\ 3}\right) = \cos^{-1}\left(\frac{2}{3\sqrt{6}}\right)$$ //

---

5) Find the eqn of the plane which passes thrwsh the point $(3,4,-1)$ and is parellel the plane $2x-3y+5z+7=0$. Also, find the distance betn two planes.

The eqn of the given plane is

$2x-3y+5z+7=0$ ——— (1)

The eqn of the ll plane to the plane (1)

$2x-3y+5z+d=0$ ——— (2)

This plane passes thrwsh $(3,4,-1)$

$\therefore$ (2) $\Rightarrow$ $2(3)-3(4)+5(-1)+d=0$

$6-12-5+d=0$

$-11+d=0 \Rightarrow d=11$

$\therefore$ The parelld plane is

$2x-3y+5z+11=0$ ——— (3)

Distance betn the given plane (1) and parelld plane (2) is

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$$\delta = \frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$$

$$\therefore \delta = \frac{|7-11|}{\sqrt{4+9+25}} = \frac{|-4|}{\sqrt{38}}$$

$$= \boxed{\frac{4}{\sqrt{38}}}$$

---

6) Find the length of the perpendicular from the point $(1,-2,3)$ to the plane $x-y+z=5$.

The eqn of the given plane is

$x-y+z-5=0$.

Length of the ⊥r from the point (x_1, y_1, z_1) to the plane $ax+by+cz+d=0$ is $\delta = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

Length of the ⊥r from $(1,-2,3)$ to the plane $x-y+z-5=0$ is

$$\delta = \frac{|1+2+3-5|}{\sqrt{1+1+1}} = \frac{|1|}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$ units,

7) Find the point of intersection of the line $x-1 = \frac{y}{2} = z+1$ with the plane $2x-y+2z=2$. Also find the angle betn the line and the plane.

Given line is $x-1 = \frac{y}{2} = z+1 = t$ (say)

Any point on the line is $(t+1, 2t, t-1)$

This point lies in the plane.

$\therefore 2(t+1) - 2t + 2(t-1) = 2$

$2t+2-2t+2t-2-2=0$

$2t-2=0$

$2t=2$

$t=1$

∴ The required point of intersection is $(1+1, 2(1), 1-1)$ i.e. $(2,2,0)$

---

8) Find the coordinates of the foot of the perpendicular and length of the perpendicular from the point $(4,3,2)$ to the plane $x+2y+3z=2$

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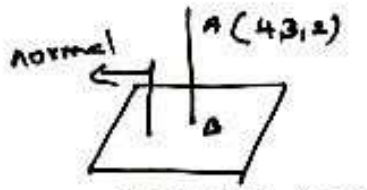

The eqn of the given plane is

$x+2y+3z-2=0$ ——— (1)

Given pt be $A(4,3,2)$

AB is dr to the plane.

B is foot of the dr from A to the plane.

Let the coordinate of B be (x_1, y_1, z_1)

Direction ratios of AB are

$x_1-4,\ y_1-3,\ z_1-2$

Direction ratio of normal to the plane $x+2y+3z-2=0$ is $(1,2,3)$

Now AB is parallel to normal to the plane.

$\therefore \frac{x_1-4}{1} = \frac{y_1-3}{2} = \frac{z_1-2}{3} = \lambda$ (say)

$x_1 = \lambda+4$

$y_1 = 2\lambda+3$

$z_1 = 3\lambda+2$

$\therefore (x_1, y_1, z_1) = (\lambda+4, 2\lambda+3, 3\lambda+2)$

The point is on the plane $x+2y+3z-2=0$

$\therefore \lambda+4+2(2\lambda+3)+3(3\lambda+2)-2=0$

$\lambda+4+4\lambda+6+9\lambda+6-2=0$

$14\lambda+14=0$

$14\lambda = -14$

$\boxed{\lambda = -1}$

$\therefore$ The coordinates of B $(-1+4, 2(-1)+3, 3(-1)+2)$

i.e. $B(3,1,-1)$

Distance betn the points $A(4,3,2)$ and $B(3,1,-1)$ is

$= \sqrt{(4-3)^2+(3-1)^2+(2+1)^2}$

$= \sqrt{1+4+9} = \sqrt{14}$

$\therefore$ length of the dr is $\sqrt{14}$

Ex 6.5 no 7th sum and 6.37 same.

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Chapter 7 – Applications of Differential Calculus

* Exercise are hyperlink with page number. To get into particular exercise, please click exercise number

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Application of Differential calculus – chapter – 7.

a) Average rate $= \frac{f(b)-f(a)}{b-a}$

$= b+a$.

b) Instantaneous rate: $f'(x)$ or $\frac{dy}{dx}$.

Example 7.1 For the function $f(x) = x^2$, $x \in [0,2]$. Compute the average rate of changes in the sub intervals $[0,0.5]$, $[0.5,1]$, $[1,1.5]$, $[1.5,2]$ and the instaneous rate of changes at the point $x = 0.5, 1.5, 2$.

Given $f(x) = x^2$

(1) $[0, 0.5]$, Average rate $= 0.5$

2) $[0.5, 1]$ Average rate $= 1.5$

3) $[1, 1.5]$ Average rate $= 2.5$

4) $[1.5, 2]$ Average rate $= 3.5$

II Instantaneous rate:-

$f(x) = x^2$

instantaneous rate $f'(x) = 2x$

(i) $x = 0.5$, $f'(0.5) = 2(0.5) = 1$

(ii) $x = 1$, $f'(1) = 2(1) = 2$

(iii) $x = 1.5$, $f'(1.5) = 2(1.5) = 3$

(iv) $x = 2$, $f'(2) = 2(2) = 4$

a	b	x	average rate	Instantaneous rate
0	0.5	0.5	0.5	1
0.5	1	1	1.5	2
1	1.5	1.5	2.5	3
1.5	2	2	3.5	4

Note: Velocity $V(t) = \frac{ds}{dt}$

Acceleration $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

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Ex 7.2 The temperature in celsius in a long rod of length 10m, insulated at both ends, is a function of length x given $T = x(10-x)$. prove that the rate of change of temperature at the midpt of the rod is zero.

We are given that,

$$T = 10x - x^2$$

Rate of change of temperature

$$\frac{dT}{dt} = 10 - 2x$$

midpt is $x = 5$

∴ rate of change of temperature at $x = 5$ is

$$\frac{dT}{dt} = 10 - 2(5) = 0$$

Eg 7.3 A person learnt 100 words for an English Test. The number of words the person remembers in t days after learning is given by $w(t) = 100(1-0.1t)^2$, $0 \le t \le 10$, what is the rate at which the person forgets the words 2 days after learning

Given $w(t) = 100(1-0.1t)^2$

$$\frac{dw}{dt} = 100 \times 2(1-0.1t)(-0.1)$$

$$= -20(1-0.1t)$$

$$\frac{dw}{dt}(t=2) = -20(1-0.1(2))$$

$$= -20[1-0.2]$$

$$= -20[0.8]$$

$$= -16$$

He forget at the rate of 16 words after 2 days. of learning.

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Eg 7.4 A particle moves so that the distence moved is according to the law $s(t) = \frac{t^3}{3} - t^2 + 3$. At what time the velocity and acceleration are zero respectively?

Distance $s(t) = \frac{t^3}{3} - t^2 + 3$

velocity $s'(t) = \frac{1}{3} 3t^2 - 2t$

$= t^2 - 2t$

acceleration $s''(t) = 2t - 2$

velocity is zero

$\Rightarrow t^2 - 2t = 0$

$t(t-2) = 0$

$\Rightarrow \boxed{t = 0 \text{ or } t = 2}$.

Acceleration is zero

$\Rightarrow 2t - 2 = 0$

$2t = 2$

$\boxed{t = 1}$

Eg 7.5 A particle is fired straight up from the ground to reach a height of s feet in t seconds where $s(t) = 128t - 16t^2$

a) compute the max height of the particle reached.

b) what is the velocity when the particle hits the ground?

$s(t) = 128t - 16t^2$.

a) The max height of the particle reached.

At the maximum height, the velocity is zero.

velocity $V = \frac{ds}{dt} = 128 - 32t$

$128 - 32t = 0 \Rightarrow 32t = 128$

$t = \frac{128}{32}$

$\boxed{t = 4 \text{ sec}}$

$s(t)_{\text{at } t=4} = 128(4) - 16(4)^2$

$= 512 - 16 \times 16 = 512 - 256$

Maximum height $= \boxed{256 \text{ ft}}$ $= 256 \text{ ft}$

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b) when the particle hits the ground, $s(t) = 0$

$128t - 16t^2 = 0$

$t(128 - 16t) = 0$

$t = 0$ (or) $128 - 16t = 0$

$16t = 128$

$t = \frac{128}{16} = 8$

The velocity at $t = 8$

$v = \frac{ds}{dt}(t=8) = 128 - 32(8)$

$= 128 - 256$

$= -128$ ft/sec.

Note:- Speed is absolute value of velocity

Speed $= |v(t)| = \left|\frac{ds}{dt}\right|$.

2) when the particle is at rest then $v(t) = 0$.

3) when the particle moving forward then $v(t) > 0$.

4) when the particle moving backward then $v(t) < 0$.

5) when the particle changes in direction, $v(t)$ then changes the sign.

6) If t_c is the time point betn the time point t_1 and t_2 $(t_1 < t_c < t_2)$ where the particle changes direction then the total distance travelled from time t_1 to time t_2 is calculated as

$|s(t_1) - s(t_c)| + |s(t_c) - s(t_2)|$

7) Slope is the rate of change in vertical length with respect to the horizontal length.

8) Velocity is the rate of displacement with respect to time.

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9) Acceleration is the rate of change in velocity with respect to time.

EXERCISE 7.1

A point moves along a st line in a such a way that after t seconds its distance from the origin is $s = 2t^2 + 3t$ metres.

a) Find the average velocity of the points between $t = 3$ and $t = 6$ seconds.

b) Find the instentaneous velocities at $t = 3$ and $t = 6$ sec.

$$s(t) = 2t^2 + 3t$$

$a = 3, \quad b = 6$

$$s(3) = 2(3)^2 + 3(3)$$
$$= 2(9) + 9$$
$$= 18 + 9$$
$$= 27$$

$$s(6) = 2(6^2) + 3(6)$$
$$= 2(36) + 18$$
$$= 72 + 18$$
$$= 90$$

$$\text{Average velocity} = \frac{s(b) - s(a)}{b - a}$$
$$= \frac{90 - 27}{6 - 3} = \frac{63}{3}$$
$$= 21 \text{ m/s}$$

(OR)

~~$s'(3) =$~~ $s(t) = 2t^2 + 3t$

$$s'(t) = 4t + 3$$

$$s'(3) = 4(3) + 3 = 12 + 3 = 15$$

$$s'(6) = 4(6) + 3 = 24 + 3 = 27$$

$$\text{Average velocity} = \frac{s'(3) + s'(6)}{2}$$
$$= \frac{15 + 27}{2} = \frac{42}{2} = 21 \text{ m/s}.$$

(11) $s(t) = 2t^2 + 3t$

$$s'(t) = 2t + 3$$

Instantaneous velocity at $t = 3$

$$s'(3) = 4(3) + 3 = 12 + 3 = 15 \text{ m/s}$$

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Instantaneous velocity at $t=6$

$s'(6) = 4(6)+3$

$= 24+3 = 27$ m/s.

---

2) A camera is accidentally knocked off an edge of a cliff 400 ft high. The camera falls a distance of $s = 16t^2$ in t sec.

a) How long does the camera fall before it hits the ground?

b) what is the average velocity with which the camera falls during the last 2 sec?

c) what is the instantaneous velocity of the camera when it hits the ground?

Given $s(t) = 16t^2$

a) when the camera hits the ground $s(t) = 16t^2 = 400$

$t^2 = \frac{400}{16} = 25$

$t^2 = 25$

$t = 5$ sec.

b) average velocity in last two seconds.

$t = 3, \quad t = 5$

average velocity $\frac{s(5) - s(3)}{5-3}$

$= \frac{16(5^2) - 16(3^2)}{5-3}$

$= \frac{16(25) - 16(9)}{2}$

$= \frac{400 - 144}{2} = \frac{256}{2}$

$= 128$ ft/s

c) Instantaneous velocity $= s'(t)$ at $(t=5)$

$= 16(2t)$

$= 32t$

$= 32(5)$

$= 160$ ft/sec.

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3) A particle moves along a line according to the law $s(t) = 2t^3 - 9t^2 + 12t - 4$, $t \geq 0$

a) At what time the particle changes direction!

b) Find the total distance travelled by the particle in the first 4 sec.

c) Find the particles acceleration each time the velocity is zero.

Given $s(t) = 2t^3 - 9t^2 + 12t - 4$

$s'(t) = 6t^2 - 18t + 12$

a) when changes of direction

$s'(t) = 0$

$\Rightarrow 6t^2 - 18t + 12 = 0$

$t^2 - 3t + 2 = 0$

$(t-2)(t-1) = 0$

$\boxed{t = 1 \text{ or } t = 2}$

b) Total distance travelled in first 4 secs.

$s(t) = 2t^3 - 9t^2 + 12t - 4$

$t = 0, \quad s(0) = -4$

$t = 1, \quad 2 - 9 + 12 - 4 = 14 - 13 = 1$

$t = 2, \quad 2(8) - 9(4) + 12(2) - 4$

$16 - 36 + 24 - 4 = 40 - 40 = 0$

$t = 3, \quad 2(27) - 9(9) + 12(3) - 4$

$= 54 - 81 + 36 - 4$

$= 90 - 85 = 5$

$t = 4, \quad 2(64) - 9(16) + 12(4) - 4$

$128 - 144 + 48 - 4$

$176 - 148 = 28$

$= |s(1) - s(0)| + |s(2) - s(1)| + |s(3) - s(2)| + |s(4) - s(3)|$

$= |1 - (-4)| + |0 - 1| + |5 - 0| + |28 - 5|$

$= 5 + 1 + 1 + 23 =$

$= |1 - (-4)| + |0 - 1| + |28 - 0|$

$= 5 + 1 + 28 = 34 \text{ m}.$

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(iii) $s'(t) = 6t^2 - 18t + 12$

$s''(t) = 12t - 18$

velocity $= 0 \Rightarrow t = 1, 2$

acc at $t = 1$, $s''(1) = 12 - 18 = -6$ m/s^2

acc at $t = 2$, $s''(2) = 24 - 18$

$= 6$ m/s^2

---

4) If the volume of a cube of side length x is $V = x^3$, find the rate of change of the volume with respect to x when $x = 5$ units.

$V = x^3$

rate of change of $V = \frac{dV}{dt} = 3x^2$

$\frac{dV}{dt}(t=5) = 3(5)^2$

$= 3(25)$

$= 75$ units.

5) If the mass $m(x)$ (in kgs) of a thin rod of length x (in metres) is given by $m(x) = \sqrt{3}x$ then what is the rate of change of mass w.r to the length when it is $x = 3$ and $x = 27$ mts.

$m(x) = \sqrt{3x} = \sqrt{3}\, x^{\frac{1}{2}}$

rate of change $= \frac{dm}{dt}$

$= \sqrt{3}\, \frac{1}{2\sqrt{x}}$

when $x = 3$, $\frac{dm}{dt} = \sqrt{3}\, \frac{1}{2\sqrt{3}} = \frac{1}{2}$ kg/m

when $x = 27$, $\frac{dm}{dt} = \sqrt{3}\, \frac{1}{2\sqrt{27}}$

$= \sqrt{3}\, \frac{1}{2 \times 3\sqrt{3}} = \frac{1}{6}$ kg/m

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6) A stone is dropped into pond causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a content at 2 cm per second. when the radius is 5 cm find the rate of changing of the total area of the disturbed water?

Given $\frac{dr}{dt} = 2$ cm/sec, $r = 5$

Area of the circle $A = \pi r^2$

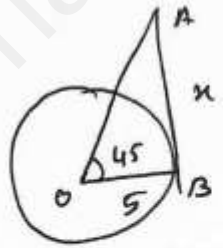
Diff w.r.t t

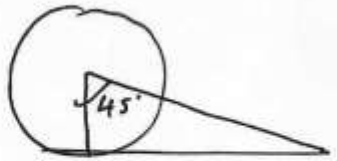
$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

$$= 2\pi(5) \cdot 2$$

$$= 20\pi \text{ sq cm/sec}.$$

7) A beacon makes one revolution every 10 secs. It is located on a ship which anchored 5 kms from a st shore line. How fast is the beam moving along the shore line when it makes an angle of 45° with the shore.





Given the beacon makes one revolution in 10 secs. i.e. 10 sec = 2π

$\therefore$ θ in 1 sec = $\frac{2\pi}{10} = \frac{\pi}{5}$

i.e. $\frac{d\theta}{dt} = \frac{\pi}{5}$.

In ΔOAB, $\tan\theta = \frac{x}{OB}$

$\tan\theta = \frac{x}{5} \Rightarrow x = 5\tan\theta$

$$\frac{dx}{d\theta} = 5\sec^2\theta \cdot \frac{d\theta}{dt}$$

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$\theta = 45°, \quad x = 5$

$\therefore \frac{dx}{dt} = \pi \sec^2(45) \cdot \frac{\pi}{8}$

$= (\sqrt{2})^2 \cdot \pi$

$= 2\pi$ km/sec.

---

8) A conical water tank with vertex down of 12 m height has a radius of 5 m at the top. If water flows into the tank at a rate 10 cubic m/min how fast is the depth of the water increases when the water is 8 m deep?

Given $h = 12m, \quad r = 5m.$

$\frac{r}{h} = \frac{5}{12} \Rightarrow r = \frac{5h}{12}$

Volume of the cone

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \left(\frac{25h^2}{144}\right) h$

$= \frac{25}{432}\pi h^3$

XII ADC 7.1 / P10

$\frac{dV}{dt} = \frac{25}{144}\pi(3h^2) = \frac{25}{144}\pi h^2 \frac{dh}{dt}$

Given that $\frac{dV}{dt} = 10$ cubic m/min

and $h = 8$

$\therefore \quad 10 = \frac{25}{144}\pi 64 \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{\cancel{10} \times 144}{\cancel{25} \times \cancel{64} \times \pi} \cancel{18}^{9}$

$= \frac{9\pi}{10}$ m/min

$\therefore$ Depth of water increases $\frac{9}{10}$ m/min

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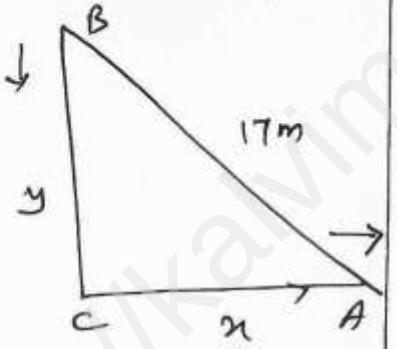
9) A ladder 17m long is leaning against the wall. The base of the ladder is pulled away from the wall at a rate of 5m/s. when the base of the ladder is 8m from the wall,

(i) how fast is the top of the ladder moving down the wall?

(ii) at what rate, the area of the triangle formed by the ladder, wall, and the floor, is changing?



Let AB be the ladder of length 17 m.

Let x and y be the horizondal and vertical moments of the ladder.

Given $\frac{dx}{dt} = 5$, when $x = 8$

In ΔABC, $AB^2 = AC^2 + BC^2$

$17^2 = x^2 + y^2$

$x^2 + y^2 = 289$

$8^2 + y^2 = 289$

$64 + y^2 = 289$

$y^2 = 289 - 64 = 225$

$y = \sqrt{225} = 15m$.

$x^2 + y^2 = 289$

Diff w.r.to t

$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0.$

$x \frac{dx}{dt} + y \frac{dy}{dt} = 0$

$8(5) + 15 \frac{dy}{dt} = 0 \Rightarrow 15 \frac{dy}{dt} = -40$

$\frac{dy}{dt} = -\frac{40}{15} = -\frac{8}{3}$

∴ The top of the ladder is moving down the wall $= \frac{8}{3}$ m/s

(ii) Area of the triangle $\Delta = \frac{1}{2} xy$

$\frac{d\Delta}{dt} = \frac{1}{2}\left[x \frac{dy}{dt} + y \frac{dx}{dt}\right]$

$= \frac{1}{2}\left[8\left(-\frac{8}{3}\right) + 15(5)\right]$

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$= \frac{1}{2}\left[-\frac{64}{3} + 75\right]$

$= \frac{1}{2}\left[\frac{225-64}{3}\right]$

$= \frac{1}{2}\left[\frac{161}{3}\right] = \frac{161}{6} = 26.83$

∴ rate of change of area of triangle $= 26.83\ m^2/sec$.

---

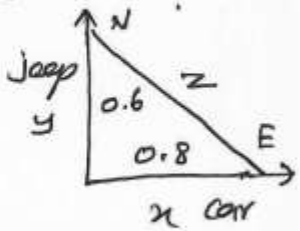
10) A police jeep, approaching an orthogonal intersection from the northern direction, is chasing a speeding car that has turned and moving straight east, when the jeep is 0.6 km north of the intersection and the car is increasing at 20 km/hr. If the jeep is moving at 60 km/hr at the instant of measurement what is the speed of the car?

Let x and y be the distance moving by car and jeep.

Given speed of jeep $\frac{dy}{dt} = -60$

(take − for approaching).

Find $\frac{dx}{dt}$, when $x = 0.8, y = 0.6$.



From the Δ,

$x^2 + y^2 = z^2$ ——— (1)

∴ $(0.8)^2 + (0.6)^2 = z^2$

$6.4 + 3.6 = z^2$

$\Rightarrow z^2 = 1 \Rightarrow \boxed{z = 1}$

Given $\frac{dz}{dt} = 20$

Diff (1) w.r to t

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$

$(0.8)\frac{dx}{dt} + 0.6(-60) = 1(20)$

$0.8\frac{dx}{dt} - 36 = 20$

$0.8\frac{dx}{dt} = 56 \Rightarrow \frac{dx}{dt} = \frac{56}{0.8}$

$= \frac{560}{8}$

$= 70\ km/s$

∴ Speed of the car = 70 km/s

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Ex 7.6 A particle moves along a horizontal line such that its position at any time $t \geq 0$ is given by $s(t) = t^3 - 6t^2 + 9t + 1$, where s is measured in metres and t in seconds.

1) At what time the particle is at rest?

2) At what time the particle changes direction?

3) Find the total distance travelled by the particle in the first 2 secs.

Given $s(t) = t^3 - 6t^2 + 9t + 1$

(1) At what time the particle is rest.

$$s(t) = t^3 - 6t^2 + 9t + 1$$

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$$

The particle is rest $\Rightarrow v(t) = 0$

i.e. $3t^2 - 12t + 9 = 0$

$\div 3$ $t^2 - 4t + 3 = 0$

$(t-1)(t-3) = 0$

$\Rightarrow t = 1$ or $t = 3$.

∴ The particle is at rest when $t = 1$ and $t = 3$ secs.

2) At what time the particle changes direction.

$$v(t) = 3t^2 - 12t + 9$$
$$= 3(t^2 - 4t + 3)$$
$$= 3(t-1)(t-3)$$

If $0 \leq t < 1$ then $t - 1 < 0$ and $t - 3 < 0$

Hence $v(t) > 0$

If $1 < t < 3$ then $t - 1 > 0$, $t - 3 < 0$

Hence $v(t) < 0$.

If $t > 3$, then $t - 1 > 0$, $t - 3 > 0$

Hence $v(t) > 0$.

∴ The particle changes direction when $t = 1$ and $t = 3$.

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Find the total distance travelled by the particle in the first two seconds.

The total distance travelled by the particle from time $t=0$ to $t=2$ is given by

$$d = |s(0) - s(1)| + |s(1) - s(2)| \quad \text{——— (1)}$$

$$s(t) = t^3 - 6t^2 + 9t + 1$$

$$s(0) = 0 - 0 + 0 + 1 = 1$$

$$s(1) = 1 - 6 + 9 + 1 = 11 - 6 = 5$$

$$s(2) = 8 - 24 + 18 + 1$$

$$= 27 - 24 = 3$$

$\therefore$ From (1) $d = |1-5| + |5-3|$

$= |-4| + |2|$

$= 4 + 2 = 6$

$\therefore$ Total distance travelled by the particle in first 2 seconds is 6m.

Ex 7.7 If we blow air into the balloon of spherical shape at a rate of 1000 cm^3 per second. At what rate the radius of the balloon changes when the radius is 7cm. Also compute the rate at which the surface area changes.

Volume of the baloon of radius r is $V = \frac{4}{3}\pi r^3$

Given $\frac{dV}{dt} = 1000\ cm^3$.

$$V = \frac{4}{3}\pi r^3$$

$\therefore \frac{dV}{dt} = \frac{4}{3}\pi\, 3r^2 \frac{dr}{dt}$

find $\frac{dr}{dt}$ when $r = 7cm$.

$\therefore 1000 = 4\pi r^2 \frac{dr}{dt}$

$$\frac{1000}{4\pi r^2} = \frac{dr}{dt}$$

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$$\frac{dV}{dt}(r=7) = \frac{1000}{4\pi(7)^2}$$

$$= \frac{1000}{4\times 49\pi}$$

$$= \frac{250}{49\pi}$$

The surface area S of the baloon is $S = 4\pi r^2$

$\therefore \frac{dS}{dt} = 4\pi(2r)\frac{dr}{dt}$

sub $\frac{dr}{dt} = \frac{250}{49\pi}$, and $r = 7$

$\therefore \frac{dS}{dt} = 8\pi(7)\frac{250}{49\pi}$

$$= \frac{2000}{7}$$

$\therefore$ rate of change of radius is $\frac{250}{49\pi}$.

rate of change of surface area is $\frac{2000}{7}$ cm^2/sec.

Ex 7.8 The price of a product is related to the number of units available (supply) by the equation $px + 3p - 16x = 234$, where p is the price of the product per unit in Rupees and x is the number of units. Find the rate at which the price is changing with respect to time when 90 units are available and the supply is increasing at the rate of 15 units / week.

Given $px + 3p - 16x = 234$ —①

where p is price per unit and x is no of units.

from ① $p(x+3) = 16x + 234$

$$p = \frac{16x+234}{x+3}$$

Diff w.r.t t

$$\frac{dp}{dt} = \frac{(x+3)16 - (16x+234)(1)}{(x+3)^2}\cdot\frac{dx}{dt}$$

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$$= \frac{16x + 48 - 16x - 234}{(x+3)^2} \cdot \frac{dx}{dt}$$

$$= \frac{-186}{(x+3)^2} \cdot \frac{dx}{dt}$$

Given $x = 90$ and $\frac{dx}{dt} = 15$

$$\therefore \frac{dp}{dt} = \frac{-186}{(90+3)^2} \cdot 15$$

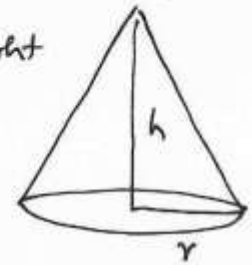
$$= -\frac{186}{93 \times 93} \times 15 = -\frac{10}{31}$$

$$= -0.32$$

∴ The price is decreasing at the rate of ₹0.32 per unit

Eg: 7.9 Salt is poured from a conveyer belt at the rate of 30 cubic metre per minute forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10m high?

Let h and r be the height and base radius of the cone.



Given that $h = 2r$

Let V be the volume of the salt cone.

$$V = \frac{1}{3}\pi r^2 h \quad \Big| \quad h = 2r,\ r = \frac{h}{2}$$

$$\therefore V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 \cdot h$$

$$= \frac{1}{3}\pi \frac{h^3}{4} = \frac{1}{12}\pi h^3$$

Diff w.r.to t.

$$\frac{dV}{dt} = \frac{1}{12}\pi (3h^2) \frac{dh}{dt}$$

$$= \frac{\pi h^2}{4} \frac{dh}{dt}$$

Given $\frac{dV}{dt} = 30\ m^3/min$, $h = 10m$

$$\therefore 30 = \frac{\pi \times 100}{4} \times \frac{dh}{dt}$$

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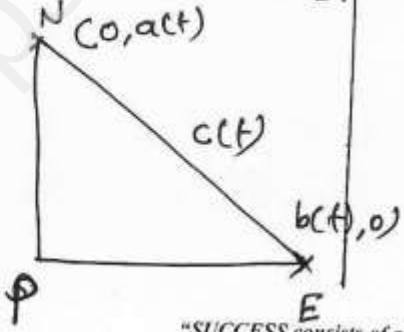
$\therefore \frac{dh}{dt} = \frac{30 \times 4}{\pi \times 100} = \frac{30}{25\pi} = \frac{6}{5\pi}$

$\therefore$ Rate of increasing of height is $\frac{6}{5\pi}$ m/min

---

Eg 7.10 A road running north to South crosses a road going east to west at the point P. Car A is driving north along the first road and car B is driving east along the second road. At a particular time car A 10 km to the north of P and travelling at 80 km/hr, while car B is 15 km to the east of P and travelling at 100 km/hr. How fast is the distance betn the two cars changing?



pN is north direction to the point P.
pE is east direction to the point P.

Let a(t) be the distance of a car A to the north of P.

Let b(t) be the distance of car B to the east of P.

Let c(t) be the distance from car A to car B.

By pythagoras Th.

$$(c(t))^2 = [a(t)]^2 + [b(t)]^2$$

Diff w.r to t

$$2c(t)\frac{dc}{dt} = 2a(t)\cdot\frac{da}{dt} + 2b(t)\frac{db}{dt}$$

$$c(t)\frac{dc}{dt} = a(t)\frac{da}{dt} + b(t)\frac{db}{dt}.$$

$$\frac{dc}{dt} = \frac{a(t)\,a' + b(t)\,b'}{c(t)}$$

Given $a = 10$ km, $\frac{da}{dt} = a' = 80$ km/hr

$b = 15$ km, $\frac{db}{dt} = b' = 100$ km/hr.

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$$\frac{dc}{dt} = \frac{10 \times 80 + 15 \times 100}{\sqrt{a^2+b^2}}$$

$$= \frac{800 + 1500}{\sqrt{100+225}} = \frac{2300}{\sqrt{325}}$$

$$= \frac{2300}{\sqrt{25 \times 13}} = \frac{2300}{5\sqrt{13}}$$

$$= \frac{460}{\sqrt{13}}$$

$$= 127.6 \text{ km/hr}$$

∴ The rate of change of the distance betn the cars A and B is 127.6 km/hr.

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CHAPTER-7 EXERCISE 7.2

Equation of Tangent:-

The equation of a tangent to a curve $y = f(x)$ at a given point $P(x_1, y_1)$ is given by

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

Equation of normal:-

The equation of normal to a curve $y = f(x)$ at a given point $P(x_1, y_1)$ is given by

$$y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} (x - x_1)$$

$$\frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} = -\frac{1}{\text{slope of tgt at } P(x_1, y_1)}$$

Important result:

1) If $\frac{dy}{dx} > 0$, the tgt makes an acute angle with x-axis.

2) If $\frac{dy}{dx} < 0$, the tgt makes an obtuse angle with x-axis.

3) If $\frac{dy}{dx} = 0$, the tgt is $\parallel^l$ to x-axis.

4) If the tgt is $\perp_r$ to x-axis then $\frac{dy}{dx} = \infty$ ie $\frac{dx}{dy} = 0$.

5) If the tgt is equally inclined to the axes, then $\frac{dy}{dx} = \tan 45^\circ$ or $\tan 135^\circ = \pm 1$

Eg 7.11 Find the eqns of tgt and normal to the curve $y = x^2 + 3x - 2$ at the point $(1, 2)$

The eqn of curve is $y = x^2 + 3x - 2$

$$\frac{dy}{dx} = 2x + 3$$

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$\left(\frac{dy}{dx}\right)_{(1,2)} = 2(1)+3 = \boxed{5}$

∴ The eqn of tgt is

$y - y_1 = \frac{dy}{dx}(x - x_1)$

i $y - 2 = 5(x-1)$

$y - 2 = 5x - 5 \Rightarrow 5x - y - 5 + 2 = 0$

$\boxed{5x - y - 3 = 0}$.

∴ slope of normal is $-\frac{1}{5}$

∴ eqn of normal is

$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)}(x - x_1)$

$y - 2 = -\frac{1}{5}(x-1)$

$5(y-2) = -(x-1)$

$5y - 10 = -x + 1$

$x + 5y - 10 - 1 = 0$

i $\boxed{x + 5y - 11 = 0}$

7.12 For what value of x the tgt of the curve $y = x^3 - 3x^2 + x - 2$ is parallel to the line $y = x$.

Eqn of the curve is

$y = x^3 - 3x^2 + x - 2$

Slope $\frac{dy}{dx} = 3x^2 - 6x + 1$ —— ①.

Eqn of the line is $y = x$

i $y - x = 0$

Slope of the line $\frac{dy}{dx} = 1$ —— ②

Given that they are parallel.

i slopes are equal.

i $3x^2 - 6x + 1 = 1$

$\Rightarrow 3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0$

$\Rightarrow \boxed{x = 0, \text{ or } x = 2}$

when $x = 2$, $y = 8 - 12 + 2 - 2$

$y = -4$

when $x = 0$, $y = -2$

∴ At $(0,-2)$ and $(2,-4)$ the tgt is ∥ to the line.

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Eg 7.13 Find the eqn of the tangent and normal to the Lissajous curve given by $x = 2\cos 3t$, $y = 3\sin 2t$, $t \in R$

The eqn of the given curve is

$x = 2\cos 3t$, $y = 3\sin 2t$.

$\frac{dx}{dt} = 2(-\sin 3t)\cdot 3 = -6\sin 3t$

$\frac{dy}{dt} = 3\cos 2t(2) = 6\cos 2t$

$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6\cos 2t}{-6\sin 3t} = -\frac{\cos 2t}{\sin 3t}$

∴ The eqn of tgt at $(2\cos 3t, 3\sin 2t)$ having slope $\frac{dy}{dx} = -\frac{\cos 2t}{\sin 3t}$ is

$y - y_1 = \frac{dy}{dx}(x - x_1)$

i.e. $y - 3\sin 2t = -\frac{\cos 2t}{\sin 3t}(x - 2\cos 3t)$

$\sin 3t(y - 3\sin 2t) = -\cos 2t(x - 2\cos 3t)$

$y\sin 3t - 3\sin 3t\sin 2t = -x\cos 2t + 2\cos 2t\cdot\cos 3t.$

$x\cos 2t + y\sin 3t = 2\cos 2t\cdot\cos 3t + 3\sin 3t\sin 2t.$

This is the required eqn of tgt.

Eqn of normal is $y - y_1 = -\frac{1}{\frac{dy}{dx}}(x - x_1)$

i.e. $(y - 3\sin 2t) = -\frac{1}{-\frac{\cos 2t}{\sin 3t}}(x - 2\cos 3t)$

$(y - 3\sin 2t) = \frac{\sin 3t}{\cos 2t}(x - 2\cos 3t)$

$y\cos 2t - 3\sin 2t\cos 2t = x\sin 3t - 2\sin 3t\cos 3t.$

$x\sin 3t - y\cos 2t = 2\sin 3t\cos 3t - 3\sin 2t\cos 2t.$

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$x \sin 3t - y \cos 2t$

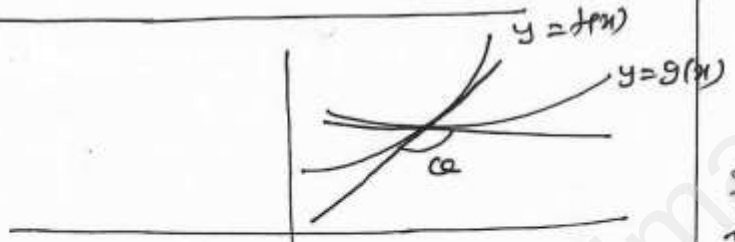
$= \sin 2(3t) - \frac{3}{2} \times 2 \sin 2t \cos 2t$

$= \sin 6t - \frac{3}{2} \sin 2(2t)$

$= \sin 6t - \frac{3}{2} \sin 4t$

This is required eqn of normal.

Angle betn two curves



The angle betn two curves is nothing but the angle between two tgts at the point of intersection of given two curves

If α is angle betn two curves, Then

$$\tan\alpha = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

Result:-

1) If $m_1 m_2 = -1$ then the given two curves cut at right angles or cut orthogonally.

2) If $m_1 = m_2$, then the given two curves touch each other.

Given curves are

$y = x^2$ ——— ①

$y = (x-3)^2$ ——— ②

Solving ① and ② we get the pt of interseection.

From ① and ②

$x^2 = (x-3)^2$

$x^2 = x^2 + 9 - 6x$.

$9 - 6x = 0 \Rightarrow 6x = 9$

$x = \frac{9}{6}$.

$x = 3/2$

put $x = \frac{3}{2}$ in ①

$y = (3/2)^2 = 9/4$

∴ The point of interseection is $(3/2, 9/4)$.

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Diff ① w.r.to x

$$\frac{dy}{dx} = 2x$$

$\therefore$ Slope $m_1 = \left(\frac{dy}{dx}\right)_{(3/2, 9/4)} = 2\left(\frac{3}{2}\right) = \boxed{3}$

Diff ② w.r.to x

$$\frac{dy}{dx} = 2(x-3)(1)$$

$\therefore$ Slope $m_2 = \left(\frac{dy}{dx}\right)_{(3/2, 9/4)}$

$$= 2\left(\frac{3}{2} - 3\right)$$

$$= 2\left(\frac{3-6}{2}\right) = \boxed{-3}$$

If θ is angle betn the curves then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{3-(-3)}{1+3(-3)}\right|$$

$$= \left|\frac{3+3}{1-9}\right| = \left|\frac{6}{-8}\right| = \left|\frac{3}{-4}\right| = \boxed{\frac{3}{4}}$$

Hence $\theta = \tan^{-1}\left(\frac{3}{4}\right)$

Eg 7.15 Find the acute angle betn the curves $y = x^2$ and $x = y^2$ at their point of intersection $(0,0)$, $(1,1)$

Given curves are

$$y = x^2 \quad \text{——} \quad (1)$$

$$x = y^2 \quad \text{——} \quad (1)$$

Diff ① w.r.to x

$$m_1 = \frac{dy}{dx} = 2x$$

Diff ② w.r.to x.

$$1 = 2y\frac{dy}{dx}$$

$$\therefore m_2 = \frac{dy}{dx} = \frac{1}{2y}$$

Let θ_1 and θ_2 be the acute angle at $(0,0)$ and $(1,1)$

$$\therefore \tan\theta_1 = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

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$$\tan\theta_1 = \left| \frac{2x - \frac{1}{2y}}{1 + 2x\left(\frac{1}{2y}\right)} \right|$$

$$\tan\theta_1 = \lim_{(x,y)\to(0,0)} \left| \frac{\frac{4xy-1}{2y}}{\frac{2y+2x}{2y}} \right|$$

$$= \lim_{(x,y)\to(0,0)} \left| \frac{4xy-1}{2x+2y} \right| = \left| \frac{-1}{0} \right| \infty$$

$\therefore \tan\theta_1 = \infty \Rightarrow \theta_1 = \tan^{-1}(\infty)$

$= \boxed{\pi/2}$

$$\tan\theta_2 = \lim_{(x,y)\to(1,1)} \left| \frac{4xy-1}{2(x+y)} \right|$$

$$= \left| \frac{4-1}{2(2)} \right| = \frac{3}{4}$$

$\tan\theta_2 = 3/4$

$\therefore \boxed{\theta_2 = \tan^{-1}(3/4)}$

Eg 7.16 Find the angle of intersection of the curve $y = \sin x$ with the positive x-axis.

The eqn of the given curve is $y = \sin x$ ——— (1)

The curve intersect the positive x-axis.

To get the point of interseation put $y = 0$ in (1)

$\sin x = 0 \Rightarrow x = n\pi$, $n = 1, 2, \ldots$

$y = \sin x$

$\frac{dy}{dx} = \cos x$

$\therefore$ Slope $m = \tan\theta = \frac{dy}{dx}(x = n\pi)$

$= \cos n\pi$

$= (-1)^n$

When $n = 1$, $\tan\theta = -1 \Rightarrow \theta = \tan^{-1}(-1)$

$= \pi - \pi/4$

$= 3\pi/4$

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When $n=2$, tance $=1$

$\Rightarrow \alpha = \tan^{-1}(1) = \pi/4$

When $n=3$, tance $=-1$

$\Rightarrow \alpha = \tan^{-1}(-1)$

$= \pi - \pi/4 = \frac{3\pi}{4}$

When $n=4$, tance $=1$

$\Rightarrow \alpha = \tan^{-1}(1)$

$= \pi/4$

$\therefore \alpha = \begin{cases} \pi/4, & \text{when } n \text{ is even} \\ \frac{3\pi}{4}, & \text{when } n \text{ is odd.} \end{cases}$

Eg 7.17 If the curves $ax^2+by^2=1$ and $cx^2+dy^2=1$ intersect each other orthogonally, Then

$$\frac{1}{a}-\frac{1}{b}=\frac{1}{c}-\frac{1}{d}.$$

Given curves are $ax^2+by^2=1$ — (1)

$cx^2+dy^2=1$ — (2)

Let (x_1, y_1) be the point of intersection of two curves.

$\therefore$ we have $ax_1^2+by_1^2=1$ — (3)

$cx_1^2+dy_1^2=1$ — (4).

Let $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

$\Delta x_1^2 = \begin{vmatrix} 1 & b \\ 1 & d \end{vmatrix} = d-b$

$\Delta y_1^2 = \begin{vmatrix} a & 1 \\ c & 1 \end{vmatrix} = a-c$

By cramer's rule,

$$x_1^2 = \frac{\Delta x_1^2}{\Delta} = \frac{d-b}{ad-bc}$$

$$y_1^2 = \frac{\Delta y_1^2}{\Delta} = \frac{a-c}{ad-bc}$$

Diff (1) w.r.to x

$$2ax + 2by\frac{dy}{dx} = 0$$

$$2by\frac{dy}{dx} = -2ax$$

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$$\frac{dy}{dx} = \frac{-2ax}{2by} = \frac{-ax}{by}$$

$\therefore$ Slope $m_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}$

$$= -\frac{ax_1}{by_1}$$

Diff eqn (2) w.r.to x

$$2cx + 2dy\frac{dy}{dx} = 0$$

$$2dy\frac{dy}{dx} = -2cx$$

$$\frac{dy}{dx} = -\frac{2cx}{2dy} = -\frac{cx}{dy}$$

$\therefore$ Slope $m_2 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}$

$$= -\frac{cx_1}{dy_1}$$

Given that the curves (1) and (2) intersect orthogonally.

$\therefore m_1 m_2 = -1$

i.e. $-\frac{ax_1}{by_1} \times -\frac{cx_1}{dy_1} = -1$

i.e. $\frac{acx_1^2}{bdy_1^2} = -1$

$$\frac{ac\left(\frac{d-b}{ad-bc}\right)}{bd\left(\frac{a-c}{ad-bc}\right)} = -1$$

$$\frac{ac(d-b)}{bd(a-c)} = -1$$

$\Rightarrow ac(d-b) = -(bd)(a-c)$

$$\frac{d-b}{bd} = \frac{-(a-c)}{ac}$$

$$\frac{d}{bd} - \frac{b}{bd} = \frac{c}{ac} - \frac{a}{ac}$$

$$\frac{1}{b} - \frac{1}{d} = \frac{1}{a} - \frac{1}{c}$$

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$$

Hence the proof.

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Eg 7.18 prove that the ellipse $x^2+4y^2=8$, and the hyperbola $x^2-2y^2=4$ intersect orthogonally

Given curves are

$x^2+4y^2=8$ ——— (1)

$x^2-2y^2=4$ ——— (2).

Let (x_1, y_1) be the point of intersection of two given curves.

∴ we have $x_1^2+4y_1^2=8$ — (3)

$x_1^2-2y_1^2=4$ — (4).

By Cramer's rule,

Let $\Delta = \begin{vmatrix} 1 & 4 \\ 1 & -2 \end{vmatrix} = -2-4 = -6$

$\Delta x_1^2 = \begin{vmatrix} 8 & 4 \\ 4 & -2 \end{vmatrix} = -16-16 = -32$

$\Delta y_1^2 = \begin{vmatrix} 1 & 8 \\ 1 & 4 \end{vmatrix} = 4-8 = -4$

$\therefore x_1^2 = \frac{\Delta x_1^2}{\Delta} = \frac{-32}{-6} = \boxed{\frac{16}{3}}$

$y_1^2 = \frac{\Delta y_1^2}{\Delta} = \frac{-4}{-6} = \boxed{+\frac{2}{3}}$.

Diff (1) w.r to x

$2x + 8y\frac{dy}{dx} = 0$

$8y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-2x}{8y}$

∴ slope $m_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\frac{x_1}{4y_1}$

Diff (2) w.r to x.

$2x - 4y\frac{dy}{dx} = 0$

$-4y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{-4y}$

$\therefore \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{x_1}{2y_1}$

∴ slope $m_2 = \frac{x_1}{2y_1}$

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Now $m_1 m_2 = \left(-\frac{x_1}{4y_1}\right)\left(\frac{x_1}{2y_1}\right)$

$= -\frac{x_1^2}{8y_1^2}$

$= -\frac{16}{\frac{3}{8\left(+\frac{2}{3}\right)}} = \frac{-\frac{16}{3}}{\frac{16}{3}}$

$= \boxed{-1}$

$m_1 \times m_2 = -1$

∴ The given curves ① and ② intersect orthogonally.

EXERCISE 7.2

1) Find the slope of the tgt to the curves at the respective given points.

(i) $y = x^4 + 2x^2 - x$ at $x = 1$

Given curve is $y = x^4 + 2x^2 - x$

Diff w.r.to x

$\frac{dy}{dx} = 4x^3 + 4x - 1$

$\frac{dy}{dx}$ at $x=1$ $= 4 + 4 - 1 = 8 - 1 = 7$

∴ slope of the tgt to the given curve is 7.

(ii) $x = a\cos^3 t$, $y = b\sin^3 t$ at $t = \pi/2$

Given that

$x = a\cos^3 t$

Diff x w.r.to t

$\frac{dx}{dt} = 3a\cos^2 t(-\sin t)$

$= -3a\cos^2 t \cdot \sin t$

$y = b\sin^3 t$

$\frac{dy}{dt} = 3b\sin^2 t$ (cost)

$= 3b\sin^2 t \cos t$.

$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$= \frac{3b\sin^2 t \cdot \cos t}{-3a\cos^2 t \cdot \sin t}$

$= -\frac{b\sin t}{a\cos t} = -\frac{b}{a}\tan t$.

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$\frac{dy}{dx}$ at $t=\pi/4$ $= -\frac{b}{a}\tan\pi/2$

$= -\frac{b}{a}\infty$

$= \infty$

∴ Slope of tgt to the given curve at $x=\pi/2$ is ∞

---

2) Find the point on the curve $y = x^2-5x+4$ at which the tgt is parallel to the line $3x+y=7$.

Given curve is $y = x^2-5x+4$ —— (1)

Given line is $3x+y=7$ —— (2).

Slope of the curve m_1 $\frac{dy}{dx} = 2x-5$

Slope of the line $3x+y-7=0$ is

$m_2 = 3+\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -3$

Given that the tgt to the curve is ∥l to the line.

∴ Slope of curve = slope of the line.

$2x-5 = -3$

$2x = -3+5$

$2x = 2$

$x = 1$

Sub $x=1$ in (1).

$y = 1-5+4 = 0$.

∴ The required point is $(1,0)$

---

3) Find the points on the curve $y = x^3-6x^2+x+3$ where the normal is parallel to the line $x+y=1729$

Given curve is $y = x^3-6x^2+x+3$ —— (1)

∴ $\frac{dy}{dx} = 3x^2-12x+1$

slope of tgt to the curve (1) is $3x^2-12x+1$

∴ slope of normal to the curve (1) is

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$$m_1 = -\frac{1}{3x^2-12x+1}$$

Given line is $x+y=1729$ —— (2)

slope of the line is $m_2 = -1$

Given that normal to the curve is parallel to the line.

$$\Rightarrow \quad m_1 = m_2$$

$$-\frac{1}{3x^2-12x+1} = -1$$

$$\Rightarrow \quad 3x^2-12x+1 = 1$$

$$\Rightarrow \quad 3x^2-12x = 0$$

$$3x(x-4) = 0 \Rightarrow x = 0 \text{ or } x = 4$$

when $x=0$, we have

$\boxed{y=3}$

when $x=4$, we have

$$y = 4^3 - 6(4)^2 + 4 + 3$$

$$= 64 - 6(16) + 4 + 3$$

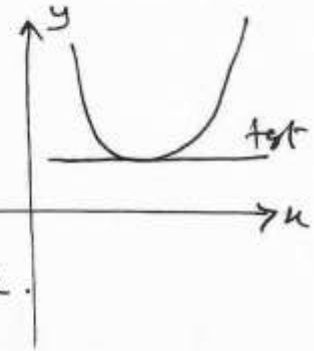
$$= 64 - 96 + 7$$

$$= 71 - 96 = -25$$

Hence the required points are $(0,3)$ and $(4,-25)$.

---

4) Find the points on the curve $y^2 - 4xy = x^2 + 5$ for which the tgt is horizontal.



Given curve is

$$y^2 - 4xy = x^2 + 5 \quad \text{—— (1)}$$

Diff w.r to x

$$2y\frac{dy}{dx} - 4\left[x\frac{dy}{dx} + y\right] = 2x.$$

$$2y\frac{dy}{dx} - 4x\frac{dy}{dx} - 4y = 2x.$$

$$\frac{dy}{dx}\left[2y - 4x\right] = 2x + 4y$$

$$\therefore \frac{dy}{dx} = \frac{2(x+2y)}{2(y-2x)} = \frac{x+2y}{y-2x}.$$

$\therefore$ Slope of tgt to the curve

$$m = \frac{x+2y}{y-2x}$$

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Given that the tgt is horizontal

ie the tgt is $\parallel^l$ to x-axis

$\therefore$ eqn of tgt is $y = c$.

$\therefore \frac{dy}{dx} = 0$.

ie $\frac{x+2y}{y-2x} = 0 \Rightarrow x+2y = 0$

$\Rightarrow x = -2y$ —— (2)

Sub (2) in (1).

$y^2 - 4(-2y)y = (-2y)^2 + 5$

$y^2 + 8y^2 = 4y^2 + 5$

$9y^2 - 4y^2 = 5 \Rightarrow 5y^2 = 5$

$y^2 = 1$

$y = \pm 1$

when $y = 1$ in (2), we have $x = -2$

when $y = -1$, we have $x = 2$.

$\therefore$ The required points are $(-2, 1)$ and $(2, -1)$

—— x —— x ——

5) Find the tangent and normal to the following curves at the given points on the curve.

(i) $y = x^2 - x^4$ at $(1, 0)$

Given $y = x^2 - x^4$

$\frac{dy}{dx} = 2x - 4x^3$

Slope of tgt $m = \frac{dy}{dx}_{(1,0)}$

$= 2(1) - 4(1)$

$= 2 - 4 = \boxed{-2}$

slope of normal $= -\frac{1}{m} = \frac{-1}{-2}$

$= \boxed{\frac{1}{2}}$

The eqn of tgt at $(1, 0)$ having slope $m = -2$ is

$y - y_1 = m(x - x_1)$

$y - 0 = -2(x - 1)$

$y = -2x + 2$

$\boxed{2x + y - 2 = 0}$.

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The eqn of normal is

$y - y_1 = -\frac{1}{m}(x - x_1)$

i.e. $y - 0 = \frac{1}{2}(x - 1)$

$2y = x - 1 \Rightarrow \boxed{x - 2y - 1 = 0}$.

(ii) $y = x^4 + 2e^x$ at $(0, 2)$

$y = x^4 + 2e^x$.

$\frac{dy}{dx} = 4x^3 + 2e^x$

slope of tgt $m = \frac{dy}{dx}(0,2)$

$= 4(0) + 2e^0$

$= 0 + 2(1) = \boxed{2}$

slope of normal $= -\frac{1}{m} = \boxed{-\frac{1}{2}}$

Eqn of tgt at $(0,2)$ having slope $m = 2$ is

$y - y_1 = m(x - x_1)$

i.e. $y - 2 = 2(x - 0)$

$y - 2 = 2x \Rightarrow \boxed{2x - y + 2 = 0}$

Eqn of normal is $y - y_1 = -\frac{1}{m}(x - x_1)$

i.e. $y - 2 = -\frac{1}{2}(x - 0)$

$2y - 4 = -x$

$\boxed{x + 2y - 4 = 0}$

(iii) $y = x \sin x$ at $(\pi/2, \pi/2)$

$y = x \sin x$

$\frac{dy}{dx} = x(\cos x) + \sin x(1)$

$= x \cos x + \sin x$

slope of tgt $m = \frac{dy}{dx}(\pi/2, \pi/2)$

$= \frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}$

$= \frac{\pi}{2}(0) + 1 = 1$

∴ slope of normal $= -\frac{1}{m} = \frac{-1}{1}$

$= \boxed{-1}$

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∴ Eqn of tgt at $(\pi/2, \pi/2)$ having slope $m = 1$ is

$y - y_1 = m(x - x_1)$

$y - \pi/2 = 1(x - \pi/2)$

$x - \pi/2 - y + \pi/2 = 0$

$\boxed{x - y = 0}$

Eqn of normal is $y - y_1 = -\frac{1}{m}(x - x_1)$

$y - \pi/2 = -1(x - \pi/2)$

$y - \pi/2 = -x + \pi/2$

$x + y - \pi/2 - \pi/2 = 0$

$\boxed{x + y - \pi = 0}$

(iv) $x = \cos t$, $y = 2\sin^2 t$ at $t = \pi/3$

$x = \cos t$ | $y = 2\sin^2 t$

$\frac{dx}{dt} = -\sin t$ | $\frac{dy}{dt} = 2(2\sin t)(\cos t)$

$= 2\sin 2t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\sin 2t}{-\sin t}$

$= \frac{2(2\sin t \cdot \cos t)}{-\sin t}$

$= -4\cos t$

Slope of tgt $m = \frac{dy}{dx}(t = \pi/3)$

$= -4\cos(\pi/3)$

$= -4(\frac{1}{2}) = \boxed{-2}$

Slope of normal $= -\frac{1}{m} = \boxed{\frac{1}{2}}$

when $t = \pi/3$ $x = \cos \pi/3 = \frac{1}{2}$

$y = 2\sin^2(\pi/3)$

$= 2(\frac{\sqrt{3}}{2})^2$

$= 2 \times \frac{3}{4} = 3/2$

∴ Eqn of tgt at $(1/2, 3/2)$ having slope $m = -2$ is

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$y - y_1 = m(x - x_1)$

i) $y - \frac{3}{2} = -2\left(x - \frac{1}{2}\right)$

$\frac{2y-3}{2} = -2\left(\frac{2x-1}{2}\right)$

$2y - 3 = -2(2x-1)$

$2y - 3 = -4x + 2$

$\Rightarrow$ $\boxed{4x + 2y - 5 = 0}$

Eqn of normal is $y - y_1 = -\frac{1}{m}(x - x_1)$

i) $y - \frac{3}{2} = \frac{1}{2}\left(x - \frac{1}{2}\right)$

$\frac{2y-3}{2} = \frac{1}{2}\left(\frac{2x-1}{2}\right)$

$2y - 3 = \frac{1}{2}(2x-1)$

$4y - 6 = 2x - 1$

i) $\boxed{2x - 4y + 5 = 0}$

---

6) Find the eqn of tgts to the curve $y = 1 + x^3$ for which the tgt is orthogonal with the line $x + 12y = 12$

X11 | CH7 | 7.2 | P16

Given curve is $y = 1 + x^3$ — (1)

∴ Slope of the tgt to the curve is $\frac{dy}{dx} = 3x^2$ (say m_1)

Given line is $x + 12y = 12$ — (2)

Slope of the line is

$1 + 12\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{1}{12}$ (say m_2)

Given that tgt is orthogonal to the line.

$\Rightarrow$ product of their slope $= -1$

ie $3x^2 \times -\frac{1}{12} = -1$

$\frac{x^2}{4} = 1 \Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

put $x = 2$ in (1) then $y = 9$

when $x = -2$ then $y = -7$

∴ The points are $(2, 9)$ and $(-2, -7)$

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To find eqns of tgts.

Given that tgts are orthogonal to $x + 12y = 12$

∴ The eqns of tgts is of the form $12x - y = K$

The tgt passes through $(2, 9)$

∴ $12(2) - 9 = K \Rightarrow K = 24 - 9$

$\boxed{K = 15}$

∴ The eqn of tgt is $\boxed{12x - y - 15 = 0}$

The tgt passes through $(-2, -7)$

∴ $12(-2) - (-7) = K$

$-24 + 7 = K$

∴ $K = -17$.

∴ The eqn of tgt is

$\boxed{12x - y + 17 = 0}$

7) Find the eqns of the tgts to the curve $y = \frac{x+1}{x-1}$ which are parallel to the line $x + 2y = 6$.

Given curve is $y = \frac{x+1}{x-1}$

∴ Slope $m_1 = \frac{dy}{dx} = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}$

$= \frac{x - 1 - x - 1}{(x-1)^2} = \frac{-2}{(x-1)^2}$

Given line is $x + 2y = 6$

∴ Slope $m_2 = 1 + 2\frac{dy}{dx} = 0$

$m_2 = \frac{dy}{dx} = -\frac{1}{2}$

Given that tgts are ⊥ to the line

$\Rightarrow$ slope m_1 = slope m_2

i.e. $\frac{-2}{(x-1)^2} = -\frac{1}{2}$

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$\Rightarrow (x-1)^2 = 4$

$\Rightarrow x-1 = \pm 2$

$\therefore x = 1 \pm 2$

i.e. $x = 3$ or $x = -1$

when $x = 3$, $y = \frac{3+1}{3-1} = \frac{4}{2} = 2$

when $x = -1$, $y = \frac{-1+1}{-1-1} = \frac{0}{-2} = 0$

$\therefore$ The points are $(3,2)$ and $(-1,0)$

Given that the tgts are ⊥ to the line $x+2y = 6$.

$\therefore$ The eqn of tgts are of the form $x+2y = k$.

It passes through $(3,2)$

$\therefore 3+2(2) = k \Rightarrow k = 7$

$\therefore$ The eqn of tgt is $\boxed{x+2y-7=0}$

Also it passes through $(-1,0)$

$\therefore -1+0 = k \Rightarrow k = -1$

$\therefore$ The eqn of tgt is $\boxed{x+2y+1=0}$

---

8) Find the eqn of tgt and normal to the curve given by $x = 7\cos t$ and $y = 2\sin t$, $t \in \mathbb{R}$ at any point on the curve.

$x = 7\cos t$ | $y = 2\sin t$

$\frac{dx}{dt} = 7(-\sin t)$ | $\frac{dy}{dt} = 2\cos t$

$\therefore \frac{dy}{dx}(\text{at } t) = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t}{-7\sin t}$ (say m)

$(x_1, y_1) = (7\cos t, 2\sin t)$

$\therefore$ The eqn of tgt is

$y - y_1 = m(x - x_1)$

i.e. $y - 2\sin t = -\frac{2\cos t}{7\sin t}(x - 7\cos t)$

$7y\sin t - 14\sin^2 t = -2x\cos t + 14\cos^2 t$

$2x\cos t + 7y\sin t - 14\sin^2 t - 14\cos^2 t = 0$

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$2x\cos t + 7y\sin t - 14 = 0$.

Slope of normal is $-\frac{1}{m}$

i.e. $-\frac{1}{\frac{2\cos t}{-7\sin t}} = \frac{7\sin t}{2\cos t}$

$(x_1, y_1) = (7\cos t, 2\sin t)$

∴ Eqn of normal is

$y - y_1 = -\frac{1}{m}(x - x_1)$

i.e. $y - 2\sin t = \frac{7\sin t}{2\cos t}(x - 7\cos t)$

$2y\cos t - 4\sin t\cos t = 7x\sin t - 49\sin t\cos t$.

$7x\sin t - 2y\cos t - 45\sin t\cos t = 0$.

---

9) Find the angle betn the rectangular hyperbola $xy = 2$ and the parabola $x^2 + 4y = 0$

Given $xy = 2$ — ①

$x^2 + 4y = 0$ — ②

To get point of intersection, solve ① and ②

From ① $xy = 2 \Rightarrow y = \frac{2}{x}$

From ② $x^2 + 4\left(\frac{2}{x}\right) = 0$

$x^3 + 8 = 0 \Rightarrow x^3 = -8$

$\boxed{x = -2}$

when $x = -2$, then $y = \frac{2}{-2} = -1$

∴ The pt of intersection is $(-2, -1)$

Diff ① w.r to x

$x\cdot\frac{dy}{dx} + y(1) = 0$

$\therefore \frac{dy}{dx} = \frac{-y}{x}$

Slope $m_1 = \left(\frac{dy}{dx}\right)_{(-2,-1)} = -\frac{(-1)}{-2} = \boxed{-\frac{1}{2}}$

Diff ② w.r to x

$2x + 4\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x}{4}$

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Slope $m_2 = \left(\frac{dy}{dx}\right)_{(-2,-1)}$

$= \frac{-2(-2)}{4} = \boxed{1}$

If θ is angle betn the curves then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

i.e. $\tan\theta = \left|\frac{-\frac{1}{2} - 1}{1 + (-\frac{1}{2})(1)}\right|$

$= \left|\frac{-3/2}{1 - \frac{1}{2}}\right| = \left|\frac{-3/2}{1/2}\right| = |-3|$

$\tan\theta = |-3| = 3$

$\therefore \theta = \tan^{-1}(3)$

---

10) Show that the two curves $x^2 - y^2 = r^2$ and $xy = c^2$ cut orthogonally.

Given $x^2 - y^2 = r^2$

Diff w.r.t x

$2x - 2y\frac{dy}{dx} = 0$

$2x = 2y\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{x}{y}$

$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{x_1}{y_1}$ (say m_1)

$xy = c^2$

Diff w.r.t x

$x\frac{dy}{dx} + y = 0$

$\therefore \frac{dy}{dx} = -\frac{y}{x}$

$\therefore m_2 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\frac{y_1}{x_1}$

Now $m_1 m_2 = \frac{x_1}{y_1}\left(-\frac{y_1}{x_1}\right)$

$= -1$

$m_1 m_2 = -1 \Rightarrow$ the two curves cut orthogonally.

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Mean Value Theorem:-

Rolle's Theorem:-

Let $f(x)$ be a real valued function

* $f(x)$ is defined and continuous on $[a,b]$
* $f(x)$ is differentiable on (a,b)
* $f(a) = f(b)$

then there exists atleast one point $c \in (a,b)$ such that $f'(c) = 0$.

Eg 7.19 Compute the value of 'c' satisfied by the Rolle's theorem for the function $f(x) = x^2(1-x)^2$ $x \in [0,1]$.

$f(x) = x^2(1-x^2)$

$f(0) = 0(1-0) = 0$

$f(1) = 1(1-1) = 0$

$f(0) = f(1)$

$\therefore$ $f(x)$ is continuous in the interval $[0,1]$

$f(x)$ is differentiable on $(0,1)$

Now, $f'(x) = x^2[2(1-x)](-1) + (1-x)^2(2x)$

$= 2x(1-x)[1-x-x]$

$= 2x(1-x)(1-2x)$

$\therefore f'(c) = 0$

$\Rightarrow 2c(1-c)(1-2c) = 0$

$\Rightarrow c = 0, c = 1, c = \frac{1}{2}$

But $0, 1 \notin (0,1)$ and $\frac{1}{2} \in (0,1)$

$\therefore$ The required value of c is $\frac{1}{2}$

7.20 Find the values in the interval $(\frac{1}{2}, 2)$ satisfied by the Rolle's Theorem for the function $f(x) = x + \frac{1}{x}$, $x \in [\frac{1}{2}, 2]$

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$f(x) = x + \frac{1}{x}$, $x \in [\frac{1}{2}, 2]$

$f(\frac{1}{2}) = \frac{1}{2} + \frac{1}{1/2} = \frac{1}{2} + 2 = \frac{5}{2}$

$f(2) = 2 + \frac{1}{2} = 5/2$

$\therefore f(\frac{1}{2}) = f(2)$

$f(x)$ is continuous on $[\frac{1}{2}, 2]$ and differentiable on $(\frac{1}{2}, 2)$.

$\therefore \exists\ c \in (\frac{1}{2}, 2)$ s.t. $f'(c) = 0$

$f'(x) = 1 - \frac{1}{x^2}$

$f'(c) = 0 \Rightarrow 1 - \frac{1}{c^2} = 0$

$\Rightarrow \frac{c^2 - 1}{c^2} = 0 \Rightarrow c^2 - 1 = 0$

$c^2 = 1$

$c = \pm 1$

But $c = -1 \notin (\frac{1}{2}, 2)$ and $c = 1 \in (\frac{1}{2}, 2)$

$\therefore$ The required value of c is 1

———x———

7.21 Compute the value of 'c' satisfied by Rolle's Th for the function $f(x) = \log\left[\frac{x^2+6}{5x}\right]$ in $[2, 3]$

$f(x) = \log\left[\frac{x^2+6}{5x}\right]$

$f(2) = \log\left[\frac{10}{10}\right] = \log 1 = 0$

$f(3) = \log\left[\frac{9+6}{15}\right] = \log\frac{15}{15} = \log 1 = 0$

$\therefore f(2) = f(3)$

Hence $f(x)$ is continuous on $[2,3]$ and differentiable in $(2,3)$.

$$f'(x) = \frac{1}{\frac{x^2+6}{5x}} \times \frac{5x(2x) - (x^2+6)5}{(5x)^2}$$

$$= \frac{5x}{x^2+6} \times \frac{10x^2 - 5x^2 - 30}{25x^2}$$

$$= \frac{5x^2 - 30}{5x(x^2+6)}$$

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$$= \frac{\cancel{3}(x^2-6)}{\cancel{3}x(x^2+6)} = \frac{x^2-6}{x(x^2+6)}$$

$f'(c) = 0 \Rightarrow \dfrac{x^2-6}{x(x^2+6)} = 0$

$\Rightarrow x^2-6 = 0 \Rightarrow x^2 = 6$

$x = \pm\sqrt{6}$

$= \pm 2.236$

$c = -\sqrt{6} \notin (2,3)$ and $c = \sqrt{6} \in (2,3)$.

∴ The required value of c is $\sqrt{6}$

7.22 Without actually solving show that the eqn $x^4+2x^3-2=0$ has only one real root in $(0,1)$

$f(x) = x^4+2x^3-2$

$f(0) = -2$

$f(1) = 1+2-2 = 1$

∴ f is continuous on $[0,1]$ and differentiable on $(0,1)$.

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$f'(x) = 4x^3+6x^2$

$= 2x^2(2x+3)$

If $f'(x) = 0 \Rightarrow 2x^2(2x+3) = 0$

$\Rightarrow x = 0$ or $x = -3/2$

$x = -3/2 \notin (0,1)$

Hence $f'(x) > 0 \ \forall\ x \in (0,1)$.

7.23 prove using the Rolle's Theorem that between any two distinct real zeros of the polynomial

$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

there is a zero of the polynomial

$n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

Let $\alpha < \beta$ be two real zeros of $p(x)$

∴ $p(\alpha) = p(\beta) = 0$.

$\Rightarrow$ $p(x)$ is continuous on $[\alpha,\beta]$ and differentiable in (α,β).

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∴ By Rolle's Th $\exists\ r \in (\alpha, \beta)$ such that $p'(r) = 0$.

$p'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1$

Hence, between any two real zeros of the polynomial $p(x)$, there is a zero r of the polynomial

$p'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1$

7.24 Prove that there is a zero of the polynomial $2x^3 - 9x^2 - 11x + 12$ in the interval $(2, 7)$ given that 2 and 7 are zeros of the polynomial $x^4 - 6x^3 - 11x^2 + 24x + 28$.

Let $p(x) = x^4 - 6x^3 - 11x^2 + 24x + 28$

$p(x)$ is continuous on $[2, 7]$ and differentiable on $(2, 7)$

By Rolle's Th, $\exists\ r \in (2, 7)$ s.t. $p'(r) = 0$.

$p'(x) = 4x^3 - 18x^2 - 22x + 24$

$\frac{p'(x)}{2} = 2x^3 - 9x^2 - 11x + 12$

∴ There is zero of the polynomial $\frac{p'(x)}{2}$ in $(2, 7)$

Verification:

$\frac{p'(2)}{2} = 16 - 36 - 22 + 12$

$= 28 - 58 < 0$.

$\frac{p'(7)}{2} = 2(343) - 9(49) - 11(7) + 12$

$= 686 - 441 - 77 + 12$

$= 698 - 518 = 90 > 0$.

$\frac{p'(x)}{2}$ has sign changes in $(2, 7)$

∴ $\frac{p'(x)}{2}$ has a zero in $(2, 7)$

∴ The polynomial $p(x)$ has zero in the interval $(2, 7)$.

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Lagrange's Mean Value Theorem.

Let $f(x)$ be a real valued function

(i) $f(x)$ is defined and continuous on $[a,b]$

(ii) $f(x)$ is differentiable on (a,b)

Then there exists atleast one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

7.25 Find the values in the interval $(1,2)$ of the mean value Theorem satisfied by the function $f(x) = x - x^2$ for $1 \le x \le 2$.

$$f(x) = x - x^2$$

$$f(1) = 1 - 1 = 0$$

$$f(2) = 2 - 4 = -2$$

Clearly $f(x)$ is defined and differentiable in $1 \le x \le 2$.

$$f'(x) = 1 - 2x.$$

$$f'(c) = 1 - 2c.$$

$$f(a) = f(1) = 0.$$

$$f(b) = f(2) = -2$$

By lagrange's Mean Value Th.

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

$$1 - 2c = \frac{-2-0}{2-1} = -2$$

$$1 - 2c = -2$$

$$1 + 2 = 2c \Rightarrow c = 3/2 \in (1,2)$$

7.26 A truck travels on a toll road with a speed limit of 80 km/hr. The truck completes a 164 km journey in 2 hrs. At the end of toll road the trucker is issued with a speed violation ticket. Justify this using the Mean Value Theorem.

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Let $f(t)$ be the distance travelled by the trucker in time 't' hrs.

$f(t)$ is continuous in $[0,2]$ and $f(t)$ is differentiable in $(0,2)$.

when $t=0$, $f(0) = 0$

$t=2$, $f(2) = 164$

By Mean value theorem, $\exists$ a time c such that

$$f'(c) = \frac{f(2)-f(0)}{2-0} = \frac{164}{2} = 82 > 80.$$

$\therefore$ at some point of time, during the travel in 2 hrs, the trucker must have travelled with a speed more than 80 km.

$\therefore$ The trucker is issued with a speed violation ticket.

Eg 7.27 Suppose $f(x)$ is differentiable function for all x with $f'(x) \leq 29$ and $f(2) = 17$. what is the maximum value of $f(7)$?

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Given $f'(x) \leq 29$ and $f(2) = 17$

By Mean Value Th, There exists a $c \in (2,7)$ s.t

$$\frac{f(7)-f(2)}{7-2} = f'(c) \leq 29$$

$$\frac{f(7)-17}{5} \leq 29 \Rightarrow f(7)-17 \leq 145$$

$$f(7) \leq 145+17$$

$$f(7) \leq 162$$

$\therefore$ The maximum value of $f(7)$ is 162

Eg 7.28 prove using mean value theorem, that $|\sin\alpha - \sin\beta| \leq \alpha - \beta$, $\alpha, \beta \in R$

Let $f(x) = \sin\alpha$

$f(x)$ is differentiable in any open interval.

consider the interval (α, β)

By MVT, $\exists\ c \in (\alpha,\beta)$ s.t $\frac{f(\beta)-f(\alpha)}{\beta-\alpha} = f'(c)$

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$$\left|\frac{\sin\beta - \sin\alpha}{\beta - \alpha}\right| = |\cos c| \le 1$$

$$\frac{|\sin\beta - \sin\alpha|}{|\beta - \alpha|} \le 1 \Rightarrow |\sin\beta - \sin\alpha| \le |\beta - \alpha|$$

7.29 A thermometer was taken from a freezer and placed in a boiling water. It took 22 seconds for the thermometer to raise from $-10^\circ C$ to $100^\circ C$. Show that the rate of change of temperature at some time 't' is $5^\circ C$ per second.

Let $f(t)$ be the temperature at time 't'.

Given the thermometer raises from $-10^\circ C$ to $100^\circ C$ in 22 seconds.

By MVT,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{100^\circ - (-10^\circ)}{22}$$

$$= \frac{110}{22} = 5^\circ C$$

The rate of change of temperature at some time is $5^\circ C$ per second.

EXERCISE 7.3

1) Explain why Rolle's Th is not applicable to the following fun in respective interval.

(i) $f(x) = \left|\frac{1}{x}\right|, \quad x \in [-1, 1]$

$f(x) = \left|\frac{1}{x}\right|, \quad x \in [-1, 1]$

$f(-1) = \left|\frac{1}{-1}\right| = |-1| = 1$

$f(1) = \left|\frac{1}{1}\right| = |1| = 1$

$f(-1) = f(1) = 1$

$f(x)$ is not continuous at $x = 0$

∴ Rolle's Th is not applicable to $f(x)$.

(ii) $f(x) = \tan x, \quad x \in [0, \pi]$

$f(x) = \tan x$

$f(0) = \tan 0 = 0$

$f(\pi) = \tan\pi = 0.$

$f(x) = \tan x$ is not continuous at $x = \pi/2$. ∴ $f(x)$ is not continuous

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in $[0,\pi]$

$\therefore$ Rolle's Th is not applicable to $f(x)$.

(iii) $f(x) = x - 2\log x$, $x \in [2,7]$

$f(x) = x - 2\log x$

$f(2) = 2 - 2\log 2$

$f(7) = 7 - 2\log 7$

$f(2) \neq f(7)$

Hence Rolle's Th is not applicable to $f(x)$.

---

2) Using the Rolle's Th, determine the value of x, at which the tgt is parallel to the x-axis for the following function.

(i) $f(x) = x^2 - x$, $x \in [0,1]$

$f(x) = x^2 - x$

$f(0) = 0 - 0 = 0$

$f(1) = 1 - 1 = 0$

$f(0) = f(1) = 0$

$\therefore$ $f(x)$ is continuous in $[0,1]$ and differentiable on $(0,1)$.

By Rolle's Th, $\exists\ c \in (0,1)$ s.t. the tgt is parallel to x-axis.

$\therefore$ slope of tgt at $x = c$ is zero.

$f'(c) = 0$

$f'(x) = 2x - 1$

$f'(c) = 2c - 1$

$f'(c) = 0 \Rightarrow 2c - 1 = 0$

$\Rightarrow c = \frac{1}{2} \in (0,1)$

$\therefore$ The required value of c is $\frac{1}{2}$

(ii) $f(x) = \dfrac{x^2 - 2x}{x+2}$, $x \in [-1,6]$

$f(-1) = \dfrac{1 - 2(-1)}{-1+2} = \dfrac{1+2}{1} = \boxed{3}$

$f(6) = \dfrac{36 - 2(6)}{6+2} = \dfrac{36-12}{8} = \dfrac{24}{8} = \boxed{3}$

$f(-1) = f(6)$

f is continuous on $[-1,6]$ and f is differentiable on $(-1,6)$.

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$\therefore$ By Rolle's Th. $\exists\ c \in (-1,6)$

s.t $f'(c) = 0$.

$$f(x) = \frac{x^2-2x}{x+2}$$

$$f'(x) = \frac{(x+2)(2x-2)-(x^2-2x)(1)}{(x+2)^2}$$

$$= \frac{2x^2-2x+4x-4-x^2+2x}{(x+2)^2}$$

$$= \frac{x^2+4x-4}{(x+2)^2}$$

$$f'(c) = 0 \Rightarrow \frac{c^2+4c-4}{(c+2)^2} = 0$$

$$\Rightarrow c^2+4c-4 = 0.$$

$$c = \frac{-4 \pm \sqrt{16-4(1)(-4)}}{2}$$

$$= \frac{-4 \pm \sqrt{16+16}}{2}$$

$$= \frac{-4 \pm \sqrt{2\times16}}{2} = \frac{-4 \pm 4\sqrt{2}}{2}$$

$$\therefore \frac{2(-2 \pm 2\sqrt{2})}{2}$$

$$\therefore -2 \pm 2\sqrt{2}$$

$-2-2\sqrt{2} \notin (-1,6)$

and $-2+2\sqrt{2} \in (-1,6)$

$\therefore$ The required value of x is $-2+2\sqrt{2}$

(iii) $f(x) = \sqrt{x} - \frac{x}{3}$, $x \in [0,9]$

$$f(x) = \sqrt{x} - \frac{x}{3}$$

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$$

$$f(0) = 0-0 = 0$$

$$f(9) = \sqrt{9} - \frac{9}{3}$$

$$= 3-3$$

$$= 0$$

$f(0) = f(9)$

f is continuous on $[0,9]$ and differentiable on $(0,9)$

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$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$

$f'(c) = \frac{1}{2\sqrt{c}} - \frac{1}{3}$

By Rolle's Th, $f'(c) = 0$

$\Rightarrow \frac{1}{2\sqrt{c}} - \frac{1}{3} = 0 \Rightarrow \frac{1}{2\sqrt{c}} = \frac{1}{3}$

$\sqrt{c} = \frac{3}{2}$

$c = 9/4 = 2.25$

$\in [0, 9)$

∴ The required value of c is $\frac{9}{4}$

---

3) Explain with lagrange's mean Value Theorem is not applicable to the following function in the respective intervals

(i) $f(x) = \frac{x+1}{x}$, $x \in [-1, 2]$

$f(x) = \frac{x+1}{x}$

$f(-1) = \frac{-1+1}{1} = \frac{0}{1} = 0$

$f(2) = \frac{3}{2}$ ∴ $f(-1) \neq f(2)$

$f(x)$ is not continuous at $x = 0$.

∴ LMT is not applicable to $f(x)$.

(ii) $f(x) = |3x+1|$, $x \in [-3, 3]$

$f(x) = |3x+1|$

$f(-1) = |3(-1)+1| = |-3+1| = |-2| = 2$

$f(3) = |3(3)+1|$

$= |9+1| = 10.$

$f(-1) \neq f(3)$

$f(x)$ is continuous on $[-1, 3]$

But not differentiable at $x = -\frac{1}{3}$

Hence LMT is not applicable to $f(x)$.

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4) Using the Lagrange's MVT determine the value of x at which the tgt is parallel to the secant line at the end point of the given interval.

(i) $f(x) = x^3 - 3x + 2$, $x \in [-2, 2]$

$f(-2) = -8 + 6 + 2 = 0$

$f(2) = 8 - 6 + 2 = 4$

$f(x)$ is continuous and differentiable on $[-2, 2]$.

$f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$

$f'(c) = 3c^2 - 3$

By LMVT, $f'(c) = \frac{f(b) - f(a)}{b - a}$

$\Rightarrow 3c^2 - 3 = \frac{4 - 0}{2 + 2} = \frac{4}{4} = 1$

$3c^2 - 3 = 1$

$\Rightarrow 3c^2 = 4 \Rightarrow c^2 = 4/3$

$c = \pm 2/\sqrt{3} \in (-2, 2)$

∴ The required value of c is $x = \pm 2/\sqrt{3}$

(ii) $f(x) = (x - 2)(x - 7)$, $x \in [3, 11]$

$f(x) = x^2 - 9x + 14$.

$f(3) = 9 - 27 + 14 = 23 - 27 = -4$

$f(11) = 121 - 99 + 14$

$= 135 - 99 = 36$

∴ $f(x)$ is continuous and differentiable on $(3, 11)$

$f(x) = x^2 - 9x + 14$

$f'(x) = 2x - 9$

$f'(c) = 2c - 9$

By LMVT, $f'(c) = \frac{f(b) - f(a)}{b - a}$

$2c - 9 = \frac{36 + 4}{11 - 3} = \frac{40}{8} = 5$

$2c - 9 = 5 \Rightarrow 2c = 14$

$c = \frac{14}{2} = 7 \in (3, 11)$

∴ The required value of x is $x = 7$

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5) Show that the value in the conclusion of the MVT for

(i) $f(x) = \frac{1}{x}$ on a closed interval of positive numbers $[a,b]$ is $\sqrt{ab}$.

$f(x) = \frac{1}{x}$

$f(x)$ is continuous and defined on $[a,b]$

$f(x)$ is differentiable on (a,b)

$f(x) = \frac{1}{x}$

$\therefore f'(x) = -\frac{1}{x^2}$

$f(a) = \frac{1}{a}$

$f(b) = \frac{1}{b}$

$f'(c) = \frac{f(b) - f(a)}{b-a}$

$-\frac{1}{c^2} = \frac{\frac{1}{b} - \frac{1}{a}}{b-a}$

$-\frac{1}{c^2} = \frac{\frac{a-b}{ab}}{b-a}$

$-\frac{1}{c^2} = \frac{a-b}{ab(b-a)}$

$-\frac{1}{c^2} = \frac{-(b-a)}{ab(b-a)}$

$\frac{1}{c^2} = \frac{1}{ab} \Rightarrow c^2 = ab$

$\therefore c = \pm\sqrt{ab}$.

But a and b are positive integer

$\therefore c = \sqrt{ab}$.

(ii) $f(x) = Ax^2 + Bx + C$ on any interval $[a,b]$ is $\frac{a+b}{2}$

$f(x) = Ax^2 + Bx + c$

$f(x)$ is defined and continuous on $[a,b]$

$f(x)$ is differentiable on (a,b)

$f(x) = Ax^2 + Bx + C$

$f'(x) = 2Ax + B$

$f(a) = Aa^2 + Ba + c$

$f(b) = Ab^2 + Bb + c$

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$f'(c) = 2Ac + B$

By LMVT,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

ii. $$2Ac + B = \frac{Ab^2 + Bb + c - Aa^2 - Ba - c}{b-a}$$

$$2Ac + B = \frac{A(b^2 - a^2) + B(b-a)}{b-a}$$

$$2Ac + B = \frac{(b-a)[A(b+a) + B]}{b-a}$$

$$2Ac + B = A(b+a) + B$$

$$2Ac = A(b+a)$$

$$c = \frac{A(a+b)}{2A} = \frac{a+b}{2}$$

$$\therefore c = \frac{a+b}{2}$$

---

6) A race car driver is racing at 20th km. If his speed never exceeds 150 km/hr. what is the maximum distance he can cover in the next two hours?.

Let $f(t)$ represents the distance covered at 't' hrs.

Here the interval is $[0, 2]$ and $f(0) = 20$, $f'(t) \leq 150$.

By LMVT,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$f'(c) = \frac{f(2) - f(0)}{2-0} \leq 150.$$

$$f(2) - 20 \leq 300$$

$$f(2) \leq 320.$$

Hence, in the next two hrs he can cover 320 km.

---

7) Suppose that for a function $f(x)$, $f'(x) \leq 1$ for all $1 \leq x \leq 4$ show that $f(4) - f(1) \leq 3$

$f'(x) \leq 1 \quad \forall x \in [1, 4]$

$\therefore f(x)$ is defined and continuous on $[1, 4]$

$f(x)$ is differentiable on $(1, 4)$

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By LMVT, $f'(c) = \frac{f(b)-f(a)}{b-a}$.

i.e. $f(b) = f(4)$

$f(a) = f(1)$

$f'(c) \le 1$

$\therefore f'(c) = \frac{f(4)-f(1)}{4-1} \le 1$

$\therefore \frac{f(4)-f(1)}{3} \le 1 \Rightarrow f(4)-f(1) \le 3$

i.e. $\frac{4+1}{2} \le 2$

$\frac{5}{2} \le 2$

i.e. $2.5 \le 2$ But $2.5 \notin [0,2]$

∴ Does not exist differentiable function $f(x)$

---

8) Does there exists a differentiable fun $f(x)$ s.t $f(0) = -1$, $f(2) = 4$ and $f'(x) \le 2$ $\forall x$. Justify your answer.

Given $f(0) = -1$, $f(2) = 4$

∴ $f(x)$ is continuous in $[0,2]$.

By LMVT, $f'(x) = \frac{f(b)-f(a)}{b-a}$

$= \frac{f(2)-f(0)}{2-0} \le 2$

---

a) Show that there exists a point on the curve $f(x) = x(x+3)e^{\pi/2}$, $-3 \le x \le 0$ where tgt is ⊥l to x-axis.

$f(x) = x(x+3)e^{\pi/2}$

$= (x^2+3x)e^{\pi/2}$

The tgt is parallel to x-axis

i.e. $\frac{dy}{dx} = 0$ i.e. $f'(x) = 0$

i.e. $(2x+3)e^{\pi/2} = 0$

$\Rightarrow 2x+3 = 0 \Rightarrow x = -3/2 \in [-3,0]$

⇒ There lies a point in $[-3,0]$, where the tgt is parallel to x-axis.

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10) Using mean value theorem prove that for $a>0$, $b>0$,

$|e^{-a}-e^{-b}| < |a-b|$

Let $f(x) = e^{-x}$, $x \in [a,b]$

$f(x)$ is continuous in $[a,b]$ and differentiable in (a,b).

$f(a) = e^{-a}$; $f(b) = e^{-b}$.

By LMVT,

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{—— (1)}.$$

$f(x) = e^{-x}$

$\therefore f'(x) = -e^{-x}$

$f'(c) = -e^{-c}$

Now, (1) $\Rightarrow$ $-e^{-c} = \dfrac{e^{-b}-e^{-a}}{b-a}$

$$|-e^{-c}| = \left|\frac{e^{-b}-e^{-a}}{b-a}\right|$$

$$\left|\frac{e^{-a}-e^{-b}}{a-b}\right| = |-e^{-c}| < 1$$

$$\frac{|e^{-a}-e^{-b}|}{|a-b|} < 1$$

$\Rightarrow |e^{-a}-e^{-b}| < |a-b|$

Hence, proved.

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Series Expansion:-

Taylor's Series:-

Let $f(x)$ be a function infinitely differentiable at $x=a$. Then $f(x)$ can be expanded as a series, in an interval $(x-a, x+a)$, of the form.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \cdots\cdots + \frac{f^n(a)}{n!}(x-a)$$

Maclaurin's series:-

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots\cdots$$

$1! = 1$

$2! = 1 \times 2 = 2$

$3! = 1 \times 2 \times 3 = 6$

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Ex 7.30 Application of Differential calculus.

Expand $\log(1+x)$ as a Maclaurin's series upto 4 non-zero terms for $-1 < x \leq 1$.

Let $f(x) = \log(1+x)$

Function and its derivatives	$\log(1+x)$ & its derivatives	Value at $x=0$
$f(x)$	$\log(1+x)$	$\log 1 = 0$
$f'(x)$	$\frac{1}{1+x}$	1
$f''(x)$	$-1(1+x)^{-2} = -\frac{1}{(1+x)^2}$	-1
$f'''(x)$	$\frac{2}{(1+x)^3}$	2
$f^{IV}(x)$	$-\frac{6}{(1+x)^4}$	-6

∴ Maclaurin's series

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^n(0)}{n!}x^n + \cdots$$

i.e. $f(x) = 0 + \frac{1}{1!}x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{(-6)}{4!}x^4 + \cdots$

$$= x - \frac{x^2}{2} + \frac{2}{1\times2\times3}x^3 - \frac{6}{1\times2\times3\times4}x^4 + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

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7.31 Expand $\tan x$ in ascending power of x upto 5th power for $-\pi/2 < x < \pi/2$

$f(x) = \tan x$.

$f'(x) = \sec^2 x$

$f''(x) = 2\sec x \cdot \sec x \cdot \tan x$

$\therefore 2\sec^2 x \tan x$.

$f'''(x) = 2\left[\sec^2 x \sec^2 x + \tan x\, 2\sec x \cdot \sec x \cdot \tan x\right]$

$= 2\left[\sec^4 x + 2\sec^2 x \tan^2 x\right]$

$f^{IV}(x) = 2\left[4\sec^3 x \cdot \sec x \tan x + 2\left(\sec^2 x\, 2\tan x \cdot \sec^2 x + \tan^2 x\, 2\sec x \cdot \sec x \tan x\right)\right]$

$= 2\left[4\sec^4 x \tan x + 2\left(2\sec^4 x \tan x + 2\sec^2 x \tan^3 x\right)\right]$

$= 2\left[4\sec^4 x \tan x + 4\sec^4 x \tan x + 4\sec^2 x \tan^3 x\right]$

$= 2\left[8\sec^4 x \tan x + 4\sec^2 x \tan^3 x\right]$

$= 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x$.

$f^{V}(x) = 16\left[\sec^4 x \sec^2 x + \tan x\, 4\sec^3 x \cdot \sec x \tan x\right] + 8\left[\sec^2 x\, 3\tan^2 x \sec^2 x + \tan^3 x \cdot 2 \cdot \sec x \cdot \sec x \tan x\right]$

$= 16\sec^6 x + 64\sec^4 x \tan^2 x + 24\sec^4 x \tan^2 x + 16\tan^4 x \sec^2 x$

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$16\sec^6 x + 88\sec^4 x \tan^2 x + 16\sec^2 x \tan^4 x$.

$f(x) = \tan x$.

$f(0) = \tan 0 = 0$.

$f^1(x) = \sec^2 x$

$f^1(0) = \sec^2(0) = 1$

$f^{11}(x) = 2\sec^2 x \tan x$

$f^{11}(0) = 2\sec^2(0)\tan(0) = 0$

$f^{111}(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$.

$f^{111}(0) = 2\sec^4(0) + 4\sec^2(0)\tan^2(0)$

$= 2(1) + 4(0) = 2$

$f^{IV}(x) = 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x$

$f^{IV}(0) = 16\sec^4(0)\tan(0) + 8\sec^2(0)\tan(0)$

$= 0$.

$f^{V}(x) = 16\sec^6 x + 88\sec^4 x \tan^2 x + 16\sec^2 x \tan^4 x$

$f^{V}(0) = 16\sec^6(0) + 88\sec^4(0)\tan^2(0) + 16\sec^2(0)\tan^4(0)$

$= 16(1) + 0 + 0 = 16$.

Maclaurin's series

$$f(x) = f(0) + \frac{f^1(0)}{1!}x + \frac{f^{11}(0)}{2!}x^2 + \cdots$$

$$\therefore \tan x = 0 + \frac{1}{1!}x + 0 + \frac{2}{3!}x^3 + 0 + \frac{16}{5!}x^5 + \cdots$$

$$= x + \frac{1}{3}x^3 + \frac{16}{1\times2\times3\times4\times5}x^5 + \cdots$$

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$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$

Ex 7.32 Write the Taylor series expansion of $\frac{1}{x}$ about $x = 2$ finding the first three non-zero terms.

$f(x) = \frac{1}{x}$

$f'(x) = -\frac{1}{x^2} = -x^{-2}$

$f''(x) = -(-2)x^{-3} = 2x^{-3}$

$f'''(x) = -6x^{-4}$

$f(2) = 1/2$

$f'(2) = -1/4$

$f''(2) = \frac{2}{8} = \frac{1}{4}$

$f'''(2) \quad -\frac{6}{2^4} = -\frac{6}{16} = -\frac{3}{8}$

Taylor's series XII | ADC 7.4 | P5

$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$

$\therefore \frac{1}{x} = \frac{1}{2} + \frac{-1/4}{1}(x-2) + \frac{\frac{1}{4}}{2!}(x-2)^2 + \frac{-3/8}{3!}(x-2)^3 + \cdots$

$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{3}{48}(x-2)^3 + \cdots$

$\therefore \frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + \cdots$

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EXERCISE 7.4

1) Write the maclaurin series expansion

a) e^x

$f(x) = e^x$	$f(0) = e^0 = 1$
$f'(x) = e^x$	$f'(0) = e^0 = 1$
$f''(x) = e^x$	$f''(0) = e^0 = 1$
$f'''(x) = e^x$	$f'''(0) = e^0 = 1$
$f^{IV}(x) = e^x$.	$f^{IV}(0) = e^0 = 1$

The Maclaurin series of $f(x)$ is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

b) $f(x) = \sin x$.

$f(x) = \sin x$	$f(0) = \sin 0 = 0$
$f'(x) = \cos x$.	$f'(0) = \cos 0 = 1$
$f''(x) = -\sin x$.	$f''(0) = -\sin 0 = 0$
$f'''(x) = -\cos x$.	$f'''(0) = -\cos 0 = -1$
$f^{IV}(x) = \sin x$.	$f^{IV}(0) = \sin 0 = 0$.

∴ The Maclaurin series of $f(x)$ is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

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c) $f(x) = \cos x$ | $f(0) = \cos 0 = 1$

$f'(x) = -\sin x$ | $f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x$ | $f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x$ | $f'''(0) = \sin 0 = 0$

$f^{IV}(x) = \cos x$ | $f^{IV}(0) = \cos(0) = 1$

$f^{V}(x) = -\sin x$ | $f^{V}(0) = -\sin 0 = 0$

∴ The Maclaurin series of $f(x)$ is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

i $\cos x = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + \cdots$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24}x^4 + \cdots$$

d) $\log(1-x)$

Let $f(x) = \log(1-x)$

$f'(x) = \frac{1}{1-x}(-1) = -\frac{1}{1-x} = -(1-x)^{-1}$

$f''(x) = 1(1-x)^{-2}(-1) = -(1-x)^{-2}$

$f'''(x) = +2(1-x)^{-3}(-1) = -2(1-x)^{-3}$

$f^{IV}(x) = 6(1-x)^{-4}(-1) = -6(1-x)^{-4}$

$f^{V}(x) = +24(1-x)^{-5}(-1) = -24(1-x)^{-5}$

$f(0) = \log(1-0) = \log 1 = 0$

$f'(0) = -1$ $\quad f^{V}(0) = -24$

$f''(0) = -1$ $\quad \cdots\cdots$

$f'''(0) = -2$

$f^{IV}(0) = -6$

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∴ The Maclaurin series of $f(x)$ is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots .$$

(i) $$\log(1-x) = -\frac{x}{1!} - \frac{1}{2}x^2 - \frac{2}{1\times2\times3}x^3 - \frac{6}{1\times2\times3\times4}x^4 - \cdots .$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots .$$

$$= -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}\right). \text{ //}$$

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$\tan^{-1}(x)$

Let $f(x) = \tan^{-1}(x)$

$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$

$= 1 - x^2 + x^4 - x^6 + \cdots$

$f''(x) = -2x + 4x^3 - 6x^5 + \cdots$

$f'''(x) = -2 + 12x^2 - 30x^4 + \cdots$

$f^{IV}(x) = -0 + 24x - 120x^3 + \cdots$

$f^{V}(x) = 24 - 360x^2 + \cdots$

Maclaurin Series of $f(x)$ is

$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$ —— (1)

$f(0) = \tan^{-1}(0) = 0$

$f'(0) = 1$ $\quad f'''(0) = -2$

$f''(0) = 0$ $\quad f^{IV}(x) = 0$

XII / ADC q.4 / P9

$f^{V}(0) = 24$.

$\therefore$ (1) $\Rightarrow$

$\tan^{-1}(x) = 0 + \frac{1}{1!}x + 0 + \frac{-2}{3!}x^3 + 0 + \frac{24}{5!}x^5 + \cdots$

$= x - \frac{x^3}{3} + \frac{24}{1\times2\times3\times4\times5}x^5 + \cdots$

$= x - \frac{x^3}{3} + \frac{1}{5}x^5 - \cdots$

(VI) $\cos^2 x$.

Let $f(x) = \cos^2 x$.

$f'(x) = 2\cos x \cdot (-\sin x)$

$= -2\sin x\cos x$

$= -\sin 2x$.

$f''(x) = -\cos 2x \cdot 2$

$= -2\cos 2x$.

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$f'''(x) = -2(-\sin 2x).2$

$= +4\sin 2x$.

$f^{IV}(x) = 4(\cos 2x).2 = 8\cos 2x$.

$f^{V}(x) = 8(-\sin 2x).2 = -16\sin 2x$.

$f(0) = \cos^2(0) = 1$

$f'(0) = -\sin 2(0) = 0$

$f''(0) = -2\cos 0 = -2$

$f'''(0) = 4\sin 0 = 0$

$f^{IV}(0) = 8\cos 0 = 8$

$f^{V}(0) = -16\sin 0 = 0$

Maclaurin Series of $f(x)$ is

$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$

$\therefore \cos^2 x = 1 + 0 - \frac{2}{2!}x^2 + 0 + \frac{8}{4!}x^4 + \cdots$

+2 | ADC 3.4 | P10

$= 1 - x^2 + \frac{8}{24}x^4 + \cdots$

$= 1 - x^2 + \frac{1}{3}x^4 + \cdots$

2) Write down the Taylor series expansion of the function $\log x$ about $x = 1$ upto three non zero terms

$f(x) = \log x$

$f'(x) = \frac{1}{x} = x^{-1}$

$f''(x) = -1x^{-2} = -\frac{1}{x^2}$

$f'''(x) = 2x^{-3} = \frac{2}{x^3}$

$f^{IV}(x) = -6x^{-4} = -\frac{6}{x^4}$

$f(1) = \log 1 = 0$ | $f''(1) = -1$

$f'(1) = 1$ | $f'''(1) = 2$

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Taylor's series of f(x) is

$$f(x) = f(a) + \frac{f'(a)}{\lfloor 1}(x-a) + \frac{f''(a)}{\lfloor 2}(x-a)^2 + \cdots$$

i) $\log x = 0 + \frac{1}{\lfloor 1}(x-1) + \frac{(-1)}{\lfloor 2}(x-1)^2 + \frac{2}{\lfloor 3}(x-1)^3 + \cdots$

$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$

3) Expansion of $\sin x$ in ascending powers $x - \pi/4$ upto three non-zero terms

$f(x) = \sin x$

$f'(x) = \cos x$.

$f''(x) = -\sin x$.

$f'''(x) = -\cos x$

XII/ADCE 2.3/ P11

Taylor's series of f(x) is

$$f(x) = f(a) + \frac{f'(a)}{\lfloor 1}(x-a) + \frac{f''(a)}{\lfloor 2}(x-a)^2 + \cdots$$

$f(\pi/4) = \sin \pi/4 = 1/\sqrt{2}$

$f'(\pi/4) = \cos \pi/4 = +1/\sqrt{2}$

$f''(\pi/4) = -\sin \pi/4 = -1/\sqrt{2}$

$f'''(\pi/4) = -\cos \pi/4 = -1/\sqrt{2}$

$$\therefore f(x) = \frac{1}{\sqrt{2}} + \frac{1/\sqrt{2}}{\lfloor 1}(x-\pi/4) + \frac{-1/\sqrt{2}}{\lfloor 2}(x-\pi/4)^2 - \frac{1/\sqrt{2}}{\lfloor 3}(x-\pi/4)^3 + \cdots$$

$$= \frac{1}{\sqrt{2}}\left[1 + \frac{x-\pi/4}{1} - \frac{1}{2}(x-\pi/4)^2 - \frac{1}{6}(x-\pi/4)^3 + \cdots\right]$$

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$$\frac{\sqrt{2}}{2}\left[1+\frac{(x-\pi/4)}{\lfloor 1}-\frac{(x-\pi/4)^2}{\lfloor 2}-\frac{(x-\pi/4)^3}{\lfloor 3}-\cdots\right]$$

4) Expand the polynomial $f(x) = x^2-3x+2$ in power of $x-1$

$f(x) = x^2-3x+2$ | $f(1) = 0$

$f'(x) = 2x-3$ | $f'(1) = -1$

$f''(x) = 2$ | $f''(1) = 2$

$\therefore$ Taylor's series of $f(x)$ is

$$f(x) = f(a) + \frac{f'(a)}{\lfloor 1}(x-a) + \frac{f''(a)}{\lfloor 2}(x-a)^2 + \cdots$$

$$f(x) = 0 + \frac{x-1}{\lfloor 1}(-1) + \frac{(x-1)^2}{\lfloor 2}(2) \cdots$$

$$= -(x-1) + (x-1)^2$$

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Indeterminate forms-

$\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 1^{\infty}, 0^{0}, \infty^{0}$

Evaluating inderterminate form

we know that $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

if $\lim_{x\to a} g(x) \neq 0$

But if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

then we get the form $\frac{0}{0}$ which is indeterminate.

L' Hospital's rule: Let f and g be two function such that

a) If $\lim_{x\to a} f(x) = 0$ and

$\lim_{x\to a} g(x) = 0$

b) $f'(a)$ and $g'(a)$ exists and $g'(a) \neq 0$

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

7.33 Evaluate $\lim_{x\to 1} \left(\frac{x^2-3x+2}{x^2-4x+3}\right)$

$\lim_{x\to 1} \frac{x^2-3x+2}{x^2-4x+3} = \frac{1-3+2}{1-4+3} = \frac{0}{0}$

which is an indeterminate form.

Applying L Hopital's rule.

$$\lim_{x\to 1} \frac{x^2-3x+2}{x^2-4x+3} = \lim_{x\to 1} \frac{2x-3}{2x-4}$$

$$= \frac{2(1)-3}{2(1)-4} = \frac{2-3}{2-4}$$

$$= \frac{-1}{-2}$$

$$= \boxed{1/2}$$

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7.34 Compute the limit

$$\lim_{x \to a} \frac{x^n - a^n}{x - a}$$

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \frac{a^n - a^n}{a - a} = \frac{0}{0}$$ which is an indeterminate form

Applying L'Hopital's rule,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} \frac{nx^{n-1}}{1} = na^{n-1}$$

7.35 Evaluate $\lim_{x \to 0} \frac{\sin mx}{x}$.

$$\lim_{x \to 0} \frac{\sin mx}{x} = \frac{\sin m(0)}{0} = \frac{0}{0}$$

which is an indeterminate form.

∴ Applying L'Hopital's rule we have

$$\therefore \lim_{x \to 0} \frac{\sin mx}{x} = \lim_{x \to 0} \frac{\cos mx \cdot m}{1}$$

$$= \cos m(0) \cdot m$$

$$= \cos(0) \cdot m$$

$$= m \quad \because \cos 0 = 1$$

7.36 Evaluate the limit

$$\lim_{x \to 0} \left(\frac{\sin x}{x^2} \right)$$

$$\lim_{x \to 0} \frac{\sin x}{x^2} = \frac{\sin(0)}{0} = \frac{0}{0}$$ which is an indeterminate form.

$$\lim_{x \to 0} \frac{\sin x}{x^2} = \lim_{x \to 0} \frac{\cos x}{2x} = \frac{\cos 0}{2(0)}$$

$$= \frac{1}{0} = \infty$$

limit does not exist.

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7.37 If $\lim_{\theta \to 0} \left(\frac{1-\cos m\theta}{1-\cos n\theta} \right) = 1$

then prove that $m = \pm n$.

$$\lim_{\theta \to 0} \left(\frac{1-\cos m\theta}{1-\cos n\theta} \right) = \frac{1-\cos 0}{1-\cos 0} = \frac{1-1}{1-1} = \frac{0}{0}$$

which is indeterminate form.

$$\lim_{\theta \to 0} \frac{1-\cos m\theta}{1-\cos n\theta} = \lim_{\theta \to 0} \frac{\sin m\theta \cdot m}{\sin n\theta \cdot n}$$

$$= \lim_{\theta \to 0} \frac{m}{n} \frac{\sin m\theta}{\sin n\theta} = \frac{m}{n} \frac{\sin 0}{\sin 0} = \frac{0}{0}$$

which is indeterminate form.

$$\therefore \lim_{\theta \to 0} \frac{m}{n} \frac{\cos m\theta \cdot m}{\cos n\theta \cdot n} = \lim_{\theta \to 0} \frac{m^2}{n^2} \frac{\cos m\theta}{\cos n\theta}$$

$$= \frac{m^2}{n^2} \frac{\cos 0}{\cos 0}$$

$$= \frac{m^2}{n^2}$$

Given that

$$\lim_{\theta \to 0} \frac{1-\cos m\theta}{1-\cos n\theta} = 1$$

$$\Rightarrow \frac{m^2}{n^2} = 1 \Rightarrow m^2 = n^2 \Rightarrow m = \pm n.$$

7.38 Evaluate $\lim_{x \to 1} \frac{\log(1-x)}{\cot(\pi x)}$

$$\lim_{x \to 1} \frac{\log(1-x)}{\cot(\pi x)} = \frac{\log(1-1)}{\cot \pi} = \frac{\infty}{\infty}$$

which is indeterminate.

$$\lim_{x \to 1} \frac{\log(1-x)}{\cot(\pi x)} = \lim_{x \to 1} \frac{\frac{1}{1-x}(-1)}{-\operatorname{cosec}^2(\pi x) \cdot \pi}$$

$$= \lim_{x \to 1} \frac{\sin^2(\pi x)}{\pi(1-x)} = \frac{0}{0} \text{ which}$$

is indeterminate.

Applying L Hopital's rule,

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$$= \lim_{x \to 1} \frac{2 \sin \pi x \cdot \cos \pi x \cdot \pi}{-\pi}$$

$$= \lim_{x \to 1} - \sin 2\pi x = 0 \quad (\because \sin n\pi = 0)$$

×

7.39 Evaluate $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right) = \left(\frac{1}{0} - \frac{1}{e^0 - 1} \right)$$

$$= \frac{1}{0} - \frac{1}{1-1}$$

$$= \frac{1}{0} - \frac{1}{0} = \infty - \infty$$

which is indeterminate form.

Applying L Hopital's rule

$$\lim_{x \to 0} \left(\frac{e^x - 1 - x}{x(e^x - 1)} \right) = \frac{0}{0} \text{ form}$$

7.5 ADC/XII/P 4

$$\lim_{x \to 0} \left(\frac{e^x - 1}{xe^x + e^x - 1} \right) = \lim_{x \to 0} \frac{e^x - 1}{e^x(x+1) - 1}$$

$$= \frac{1-1}{1-1} = \frac{0}{0} \text{ form}$$

$$\therefore \lim_{x \to 0} \left(\frac{e^x}{xe^x + e^x \cdot 1 + e^x} \right)$$

$$= \lim_{x \to 0} \frac{e^x}{xe^x + 2e^x} = \frac{e^0}{0 + 2e^0} = \frac{1}{2} //.$$

×

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Eg 7.40 Evaluate $\lim_{x \to 0} x \log x$

$\lim_{x \to 0} x \log x = \lim_{x \to 0} \frac{\log x}{1/x}$

$= \frac{\log 0}{1/0} = \frac{\infty}{\infty}$ form

which is an indeterminate form.

∴ Applying L Hopital's rule.

$\lim_{x \to 0} \frac{\log x}{1/x} = \lim_{x \to 0} \frac{1/x}{-1/x^2}$

$= \lim_{x \to 0} \frac{1}{x}(-x^2) = \lim_{x \to 0} -x$

$= 0.$

Eg 7.41 Evaluate $\lim_{x \to \infty} \frac{x^2+17x+29}{x^4}$

$\lim_{x \to \infty} \frac{x^2+17x+29}{x^4} = \frac{\infty}{\infty}$ form

which is an indeterminate form.

Applying L Hopital's rule

$\lim_{x \to \infty} \frac{x^2+17x+29}{x^4} = \lim_{x \to \infty} \frac{2x+17}{4x^3} = \frac{\infty}{\infty}$

Again applying L Hopital's rule.

$\lim_{x \to \infty} \frac{2}{12x} = \frac{2}{12(\infty)} = 2(0) = 0.$

Eg 7.42 Evaluate $\lim_{x \to \infty} \left(\frac{e^x}{x^m}\right)$

$\lim_{x \to \infty} \frac{e^x}{x^m} = \frac{e^\infty}{\infty^m} = \frac{\infty}{\infty}$ form

This is indeterminate form.

Applying L Hopital's rule,

$\lim_{x \to \infty} \frac{e^x}{x^m} = \lim_{x \to \infty} \frac{e^x}{mx^{m-1}} = \frac{\infty}{\infty}$ form

Applying L Hopital's rule m times.

$\lim_{x \to \infty} \frac{e^x}{x^m} = \lim_{x \to \infty} \frac{e^x}{m!} = e^\infty = \infty.$

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Eg 7.43 prove that $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$

$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = 1^{\frac{1}{0}} = 1^{\infty}$.

Let $y = (1+x)^{\frac{1}{x}}$

Taking log, $\log y = \frac{1}{x}\log(1+x)$

Applying limit,

$\lim_{x\to 0}\log y = \lim_{x\to 0}\frac{1}{x}\log(1+x)$.

$= \frac{\log(1+0)}{0} = \frac{\log 1}{0} = \frac{0}{0}$

Applying L'Hopital's rule,

$\lim_{x\to 0}\log y = \lim_{x\to 0}\frac{\frac{1}{1+x}}{1}$

$= \lim_{x\to 0}\frac{1}{1+x} = \frac{1}{1} = 1$

$\lim_{x\to 0}\log y = 1$

$\therefore \log \lim_{x\to 0} y = 1$

Taking Antilog,

$\lim_{x\to 0} y = e^1 = e$

$\therefore \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$.

7.44 Evaluate $\lim_{x\to\infty}(1+2x)^{\frac{1}{2\log x}}$

$\lim_{x\to\infty}(1+2x)^{\frac{1}{2\log x}} = (1+2\infty)^{\frac{1}{2\log\infty}}$

$= (1+2\infty)^{\frac{1}{2\infty}} = \infty^0$

Let $y = (1+2x)^{\frac{1}{2\log x}}$.

Taking log, $\log y = \frac{1}{2\log x}\log(1+2x)$.

Taking limit.

$\lim_{x\to\infty}\log y = \lim_{x\to\infty}\frac{\log(1+2x)}{2\log x}$

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$$\lim_{x\to\infty} \log y = \frac{\log(1+\infty)}{2\log\infty} = \frac{\infty}{2\infty}$$

Applying L Hopital's rule

$$\lim_{x\to\infty} \log y = \lim_{x\to\infty} \frac{\frac{1}{1+2x}\cdot 2}{2\times\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{x}{1+2x} \times \frac{x}{x}$$

$$= \lim_{x\to\infty} \frac{x}{1+2x} = \frac{\infty}{\infty}.$$

Again applying L Hopital's rule,

$$\lim_{x\to\infty} \log y = \lim_{x\to\infty} \frac{1}{2}$$

$$= \frac{1}{2}$$

$$\Rightarrow \log\left(\lim_{x\to\infty} y\right) = \frac{1}{2}$$

XII/ADC
7.5/P7

Taking A.log,

$$\lim_{x\to\infty} y = e^{1/2} = \sqrt{e}$$

$$\Rightarrow \lim_{x\to\infty} (1+2x)^{\frac{1}{2\log x}} = \sqrt{e}.$$

7.45 Evaluate $\lim_{x\to 1} x^{\frac{1}{1-x}}$

$$\lim_{x\to 1} x^{\frac{1}{1-x}} = 1^{\frac{1}{1-1}} = 1^{\frac{1}{0}} = 1^{\infty} = 1$$

This is an indeterminate form.

Let $y = x^{\frac{1}{1-x}}$

$$\log y = \frac{1}{1-x}\log x.$$

$$\lim_{x\to 1} \log y = \lim_{x\to 1} \frac{1}{1-x}\log x.$$

$$= \frac{\log 1}{1-1} = \frac{0}{0}$$

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XII / AOC
7.5 / P 8

$$\lim_{x \to 1} \log y = \lim_{x \to 1} \frac{\frac{1}{x}}{0-1}$$

$$\therefore \lim_{x \to 1} -\frac{1}{x}$$

$$= -\frac{1}{1} = -1$$

$$\lim_{x \to 1} \log y = -1$$

$$\log \left(\lim_{x \to 1} y \right) = -1$$

Taking A.log

$$\lim_{x \to 1} y = e^{-1}$$

$$\lim_{x \to 1} x^{\frac{1}{1-x}} = \frac{1}{e} //$$

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EXERCISE 7.5

Evaluate the following limits, if necessary use L' Hospital's rule

1) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1 - \cos 0}{0} = \frac{1-1}{0} = \frac{0}{0}$

This is indeterminate form $(\frac{0}{0})$

Use L' Hospital's ~~rule~~ Rule.

$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{0 + \sin x}{2x}$

$= \frac{\sin 0}{2(0)} = \frac{0}{0}$

This is indeterminate form.

Again use L' Hospital's rule,

$\lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{\cos 0}{2}$

$= \frac{1}{2}$ //

2) $\lim_{x \to \infty} \frac{2x^2 - 3}{x^2 - 5x + 3}$

$= \lim_{x \to \infty} \frac{x^2(2 - 3/x^2)}{x^2(1 - \frac{5x}{x^2} + \frac{3}{x^2})}$

$= \lim_{x \to \infty} \frac{2 - \frac{3}{x^2}}{1 - \frac{5}{x} + \frac{3}{x}} = \frac{2 - \frac{3}{\infty}}{1 - \frac{5}{\infty} + \frac{3}{\infty}}$

$\therefore \frac{2 - \frac{3}{1/0}}{1 - \frac{5}{1/0} + \frac{3}{1/0}}$

$= \frac{2}{1} = 2$ //.

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II $\lim_{x\to\infty} \dfrac{2x^3-3}{x^2-5x+3}$

$= \lim_{x\to\infty} \dfrac{2(\infty)-3}{\infty^2-5(\infty)+3} = \dfrac{\infty}{\infty}$ form

This is Indeterminate form.

∴ Applying L'Hopital's rule

$\lim_{x\to\infty} \dfrac{2x^3-3}{x^2-5x+3} = \lim_{x\to\infty} \dfrac{6x-0}{2x-5} = \dfrac{\infty}{\infty}$ form

$\lim_{x\to\infty} \dfrac{6x}{2x-5} = \lim_{x\to\infty} \dfrac{6}{2} = 2$

3) $\lim_{x\to\infty} \dfrac{x}{\log x}$

$\lim_{x\to\infty} \dfrac{x}{\log x} = \dfrac{\infty}{\log\infty} = \dfrac{\infty}{\infty}$

which is indeterminate.

$\lim_{x\to\infty} \dfrac{x}{\log x} = \lim_{x\to\infty} \dfrac{1}{1/x}$

$= \lim_{x\to\infty} x = \infty$

4) $\lim_{x\to\pi/2} \dfrac{\sec x}{\tan x}$

$\lim_{x\to\pi/2} \dfrac{\sec x}{\tan x} = \dfrac{\sec\pi/2}{\tan\pi/2} = \dfrac{1/0}{\infty} = \dfrac{\infty}{\infty}$

which is indeterminate

$\lim_{x\to\pi/2} \dfrac{\sec x}{\tan x} = \lim_{x\to\pi/2} \dfrac{\sec x\tan x}{\sec^2 x}$

$= \lim_{x\to\pi/2} \dfrac{\tan x}{\sec x} = \lim_{x\to\pi/2} \dfrac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}}$

$= \lim_{x\to\pi/2} \sin x$

$= \sin\pi/2 = 1$ //

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MII $\lim_{x \to \pi/2} \frac{\sec x}{\tan x}$

$= \lim_{x \to \pi/2} \frac{1}{\cos x} \times \frac{\cos x}{\sin x} = \lim_{x \to \pi/2} \frac{1}{\sin x}$

$= \frac{1}{\sin \pi/2} = \frac{1}{1} = \boxed{1}$

5) $\lim_{x \to \infty} e^{-x} \sqrt{x}$

$\lim_{x \to \infty} e^{-x} \sqrt{x} = \lim_{x \to \infty} \frac{\sqrt{x}}{e^x} = \frac{\infty}{e^\infty} = \frac{\infty}{\infty}$

which is indeterminate form.

Applying L'Hopital's rule.

$\lim_{x \to \infty} \frac{\sqrt{x}}{e^x} = \lim_{x \to \infty} \frac{\frac{1}{2\sqrt{x}}}{e^x}$

$= \lim_{x \to \infty} \frac{1}{2\sqrt{x}\, e^x} = \frac{1}{\infty e^\infty} = \frac{1}{\infty}$

$= \boxed{0}$

6) $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$

$= \lim_{x \to 0} \frac{x - \sin x}{x \sin x} : \frac{0-0}{0} = \frac{0}{0}$

which is indeterminate form.

Applying L Hopital's rule.

$\lim_{x \to 0} \frac{x - \sin x}{x \sin x} = \lim_{x \to 0} \frac{1 - \cos x}{x \cos x + \sin x}$

$= \frac{1 - \cos 0}{0 \cos 0 + \sin 0} = \frac{1-1}{0} = \frac{0}{0}$

Applying L'Hopital's rule.

$\lim_{x \to 0} \frac{1 - \cos x}{x \cos x + \sin x} = \lim_{x \to 0} \frac{0 + \sin x}{x(-\sin x) + \cos x (1) + \cos x}$

$= \lim_{x \to 0} \frac{\sin x}{-x \sin x + 2 \cos x}$

$= \frac{0}{0 + 2(\cos 0)} = \frac{0}{2} = \boxed{0}$

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7) $\lim_{x \to 1} \left(\frac{2}{x^2-1} - \frac{x}{x-1} \right)$

$\lim_{x \to 1} \left(\frac{2}{(x+1)(x-1)} - \frac{x}{x-1} \right)$

$\lim_{x \to 1} \left[\frac{2-(x+1)x}{(x+1)(x-1)} \right]$

$\lim_{x \to 1} \frac{2-x^2-x}{(x+1)(x-1)} = \frac{2-1-1}{(1+1)(1-1)} = \frac{0}{0}$

$\lim_{x \to 1} \frac{2-x^2-x}{(x+1)(x-1)} = \lim_{x \to 1} \frac{2-x^2-x}{x^2-1}$

$= \lim_{x \to 1} \frac{-2x-1}{2x}$

$= \frac{-2-1}{2} = -3/2$

XII / ADC / 7.5 / P 82

8) $\lim_{x \to 0^+} x^x$

$\lim_{x \to 0^+} x^x = 0^0$ This is indeterminate form.

Let $y = x^x$

Taking log $\log y = x \log x$

Now $\lim_{x \to 0} \log y = \lim_{x \to 0} x \log x$

$= \lim_{x \to 0} \frac{\log x}{1/x} = \frac{\infty}{\infty}$

This is indeterminate form

Applying L Hopital rule,

$\lim_{x \to 0} \frac{\log x}{1/x} = \lim_{x \to 0} \frac{1/x}{-1/x^2} = \lim_{x \to 0}$

$= \lim_{x \to 0} \frac{1}{x}(-x^2)$

$= \lim_{x \to 0} -x = 0$

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XII/ADC/ 7.5/P93

i) $\lim_{x\to 0} \log y = 0$

$\log\left(\lim_{x\to 0} y\right) = 0$

taking Alog on both sides

$\lim_{x\to 0} y = e^0$

$\lim_{x\to 0} x^x = 1$ //

---

9) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \left(1+\frac{1}{\infty}\right)^{\infty}$

$= 1^{\infty}$

This is indeterminate form.

Let $y = \left(1+\frac{1}{x}\right)^x$

$\log y = x \log\left(1+\frac{1}{x}\right)$

Taking limit

$\lim \log y = \lim_{x\to\infty} \left[x \cdot \log\left(1+\frac{1}{x}\right)\right]$

$= \lim_{x\to\infty} \left[\frac{\log\left(1+\frac{1}{x}\right)}{1/x}\right]$

$= \frac{\log 1}{1/\infty} = \frac{0}{0}$

This is ~~indeterminate~~ indeterminate form

$\therefore \lim_{x\to\infty} \frac{\log\left(1+\frac{1}{x}\right)}{1/x}$

$= \lim_{x\to\infty} \frac{\left(\frac{1}{1+\frac{1}{x}}\right) \times \left(0-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$

$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = \frac{1}{1+0} = 1$

i) $\lim_{x\to\infty} \log y = 1$

$\log \lim_{x\to\infty} y = 1$ Taking A.log

$\lim_{x\to\infty} y = e^1 = e$ //.

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10) $\lim_{x \to \pi/2} (\sin x)^{\tan x}$

$\lim_{x \to \pi/2} (\sin x)^{\tan x} = (\sin \pi/2)^{\tan \pi/2}$

$= 1^{\infty}$

which is indeterminate form.

Let $y = \sin x^{\tan x}$

$\log y = \log \sin x^{\tan x}$

$= \tan x \log \sin x$

$\log y = \dfrac{\log \sin x}{\cot x}$

Taking limit

$\lim_{x \to \pi/2} \log y = \lim_{x \to \pi/2} \left(\dfrac{\log \sin x}{\cot x} \right)$

$= \dfrac{\log \sin \pi/2}{\cot \pi/2}$

$= \dfrac{\log 1}{0} = \dfrac{0}{0}$ form

which is indeterminate form.

Applying L'Hopital's rule.

$\lim_{x \to \pi/2} \dfrac{\log \sin x}{\cot x} = \lim_{x \to \pi/2} \dfrac{\frac{1}{\sin x}(\cos x)}{-\operatorname{cosec}^2 x}$

$= \lim_{x \to \pi/2} -\dfrac{\cos x}{\operatorname{cosec} x}$

$= \lim_{x \to \pi/2} (-\cos x \cdot \sin x)$

$= -\cos \pi/2 \sin \pi/2 = 0$

$\lim_{x \to \pi/2} \log y = 0 \Rightarrow \log \left(\lim_{x \to \pi/2} y \right) = 0$

Taking A.log, $\lim_{x \to \pi/2} y = e^0 = 1$

$\therefore \lim_{x \to \pi/2} (\sin x)^{\tan x} = 1$ //.

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11) $\lim_{x \to 0} (\cos x)^{\frac{1}{x^2}}$

$\lim_{x \to 0} (\cos x)^{\frac{1}{x^2}} = (\cos 0)^{\frac{1}{0}}$

$= 1^{\infty}$

which is indeterminate form.

Let $y = (\cos x)^{1/x^2}$

$\log y = \frac{1}{x^2} \log(\cos x)$

Taking limit

$\lim_{x \to 0} \log y = \lim_{x \to 0} \frac{\log(\cos x)}{x^2}$

$= \frac{\log \cos 0}{0} = \frac{\log 1}{0} = \frac{0}{0}$

which is indeterminate form.

$\lim_{x \to 0} \log y = \lim_{x \to 0} \frac{\frac{1}{\cos x}(-\sin x)}{2x}$

$= \lim_{x \to 0} \frac{-1}{2\cos x} \cdot \frac{\sin x}{x}$

$= \lim_{x \to 0} \frac{-1}{2\cos x} \cdot \lim_{x \to 0} \frac{\sin x}{x}$

$= \frac{-1}{2\cos 0} \cdot \lim_{x \to 0} \frac{\sin x}{x}$

$= -\frac{1}{2} \cdot (1) = -\frac{1}{2}$

$\therefore \lim_{x \to 0} \log y = -\frac{1}{2} \Rightarrow \log \lim_{x \to 0} y = -\frac{1}{2}$

Taking A.log $\lim_{x \to 0} y = e^{-\frac{1}{2}}$

$\therefore \lim_{x \to 0} (\cos x)^{\frac{1}{x^2}} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$ //

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12) If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is $A = A_0\left(1+\frac{r}{n}\right)^{nt}$. If the interest is compounded continuously (ie $n \to \infty$) show that the amount after t years is $A = A_0 e^{rt}$

$$\lim_{n\to\infty} A = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt}$$

$$= A_0\left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}\right)^{t}$$

$$= A_0 (e^{r})^{t}$$

$$= A_0 e^{rt}$$

$\because \lim_{n\to\infty}\left(1+\frac{k}{n}\right)^{n} = e^{k}$.

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Increasing and Decreasing function

Let I be an open interval,
Let $f: I \to R$ be a differentiable function.

1) a) A function $f(x)$ is increasing function on I iff $f'(x) \geq 0$ $\forall x \in I$

b) A function $f(x)$ is said to be an increasing function in an interval I if $a < b \Rightarrow f(a) \leq f(b)$, $\forall a, b \in I$.

2 (a) A function $f(x)$ is decreasing function on I iff $f'(x) \leq 0$ $\forall x \in I$.

b) A function $f(x)$ is decreasing function in an interval I if $a < b \Rightarrow f(a) \geq f(b)$ $\forall a, b \in I$.

Strictly increasing and strictly decreasing function

a) $f(x)$ is strictly increasing function on I iff $f'(x) > 0$ $\forall x \in I$

b) $f(x)$ is strictly decreasing function on I iff $f'(x) < 0$, $\forall x \in I$.

Monotonic function:-

If a function $f(x)$ is either increasing or decreasing on the entire interval I, then it is called Monotonic function.

Condition for non-monotonicity:-

If $f'(x)$ has different signs at different points, Then $f(x)$ is not monotonic function.

Critical number:-

A number c is called Critical number of a given function $f(x)$ if $f'(c) = 0$ or $f'(c)$ does not exists.

Stationary point:-

$(c, f(c))$ is called Stationary point at which $f'(c) = 0$.

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Absolute maximum and Absolute minimum:

procedure

1) Find the values of f at the critical number.
2) Find the value of f(a) and f(b)
3) The largest values from steps 1 and 2 is the obsolute maximum values, the smallest of these values is the obsolute minimum value.

Eg 7.46 prove that the function $f(x) = x^2 + 2$ is strictly increasing in the interval $(2,7)$ and strictly decreasing in the interval $(-2,0)$

$f(x) = x^2 + 2$

$f'(x) = 2x > 0 \quad \forall x \in (2,7)$

$\therefore f(x)$ is strictly increasing function on $(2,7)$.

$f(x) = x^2 + 2$

$f'(x) = 2x < 0 \quad \forall x \in (-2,0)$

$\therefore f(x)$ is strictly decreasing function on $(-2,0)$

7.47 prove that the function $f(x) = x^2 - 2x - 3$ is strictly increasing in $(2,\infty)$

$f(x) = x^2 - 2x - 3$

$f'(x) = 2x - 2$

$f'(x) > 0 \quad \forall x \in (2,\infty)$

$\therefore f(x)$ is strictly increasing on $(2,\infty)$

7.48 find the obsolute maximum and absolute minimum values of the function $f(x) = 2x^3 + 3x^2 - 12x$ on $[-3,2]$.

$f(x) = 2x^3 + 3x^2 - 12x$

$f'(x) = 6x^2 + 6x - 12$

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For critical no $f'(x) = 0$

i.e. $6x^2 + 6x - 12 = 0$

$6(x^2 + x - 2) = 0$

$6(x+2)(x-1) = 0$

$\Rightarrow x = 1$ or $x = -2 \in [-3, 2]$.

∴ The critical nos are $x = -2, 1$.

Now $f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2)$

$= 2(-8) + 3(4) + 24$

$= -16 + 12 + 24$

$= 36 - 16 = \boxed{20}$.

$f(1) = 2(1)^3 + 3(1)^2 - 12(1)$

$= 2 + 3 - 12 = 5 - 12$

$= \boxed{-7}$.

$f(-3) = 2(-3)^3 + 3(-3)^2 - 12(-3)$

$= 2(-27) + 3(9) + 36$

$= -54 + 27 + 36$

$= 63 - 54 = \boxed{9}$

$f(2) = 2(2)^3 + 3(2)^2 + 24 - 12(2)$

$= 2(8) + 3(4) \oplus 24$

$= 16 + 12 \oplus 24$

$= 28 - 24 = \boxed{4}$.

∴ Absolute max is 20

Absolute minimum is -7.

XII/ADC
7.6/p3

7.49 Find the absolute extrema of the function $f(x) = 3\cos x$ on $[0, 2\pi]$

$f(x) = 3\cos x$

$f'(x) = 3(-\sin x) = -3\sin x$.

For critical no $f'(x) = 0$

$\Rightarrow -3\sin x = 0$

$\Rightarrow \sin x = 0$

$\Rightarrow x = \pi \in (0, 2\pi)$.

∴ The critical no is $x = \pi$.

$f(\pi) = 3\cos\pi = 3(-1) = -3$

$f(0) = 3\cos 0 = 3(1) = 3$

$f(2\pi) = 3\cos 2\pi = 3(1) = 3$

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∴ Absolute maximum is 3
Absolute minimum is −3

7.50 Find the intervals of monotonicity and hence find the local extrema for the function $f(x) = x^2 - 4x + 4$.

$f(x) = x^2 - 4x + 4$

$f'(x) = 2x - 4$.

Now, $f'(x) = 0 \Rightarrow 2x - 4 = 0$

$2x = 4$

$\boxed{x = 2}$

∴ The intervals of monotonicity are $(-\infty, 2)$ and $(2, \infty)$.

In $(-\infty, 2)$, say $x = -1$

$f'(-1) = 2(-1) - 4 = -2 - 4 = -6 < 0$

∴ $f(x)$ is strictly decreasing on $(-\infty, 2)$

In $(2, \infty)$, say $x = 3$,

$f'(3) = 2(3) - 4 = 6 - 4 = 2 > 0$

∴ $f(x)$ is strictly increasing function.

Note that $f'(x)$ changes sign from negative to positive at $x = 2$.

∴ $f(x)$ has local minimum at $x = 2$.

∴ The local minimum value is $f(2)$.

$f(2) = 4 - 4(2) + 4 = 8 - 8 = 0$.

∴ Local minimum value is 0.

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Eg 7.51 Find the intervals of monotonicity and hence find the local extrema for the function

$f(x) = x^{2/3}$

$f(x) = x^{\frac{2}{3}}$

$f'(x) = \frac{2}{3}x^{\frac{2}{3}-1}$

$= \frac{2}{3}x^{-\frac{1}{3}}$

$= \frac{2}{3x^{\frac{1}{3}}}$

$f'(x) \neq 0 \ \forall x \in R$

$\therefore f(x)$ has no stationary point.

When $x = 0$,

$f'(x) = \frac{2}{0} = \infty$

$\therefore f'(x)$ does not exist at $x = 0$

Hence $x = 0$ is the critical point.

$\therefore$ The intervals of monotonicity are $(-\infty, 0)$ and $(0, \infty)$

In $(-\infty, 0)$ say $x = -1$

$f'(-1) = \frac{2}{3(-1)^{\frac{1}{3}}} = \frac{2}{3(-1)} = -\frac{2}{3} < 0$

$\therefore f(x)$ is decreasing fun. on $(-\infty, 0)$

In $(0, \infty)$ say $x = 1$

$f'(1) = \frac{2}{3(1)^{1/3}} = \frac{2}{3} > 0$.

$\therefore f(x)$ is increasing function on $(0, \infty)$

Interval	$f'(x)$	function
$(-\infty, 0)$	$f'(x) < 0$	Decreasing
$(0, \infty)$	$f'(x) > 0$	Increasing.

Eg 7.52 prove that the function $f(x) = x - \sin x$ is increasing on the real line. Also discuss for the existence of local extrema.

$f(x) = x - \sin x$

$f'(x) = 1 - \cos x \geq 0$ $\because$ $\cos x$ lies between -1 and 1

$f'(x) = 0 \Rightarrow 1 - \cos x = 0$

$\cos x = 1$

$x = 2n\pi$; $n \in Z$.

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$f'(x) \geq 0$ we have the function $f(x)$ is increasing on the real line.

$f(x)$ has no sign change in R when $x = 2n\pi$.

∴ by the first derivative test, there is no local extrema.

Eg 7.53 Discuss the monotonicity and local extrema of the fun $f(x) = \log(1+x) - \frac{x}{1+x}$, $x > -1$ and hence find the domain where $\log(1+x) > \frac{x}{1+x}$.

$$f(x) = \log(1+x) - \frac{x}{1+x}.$$

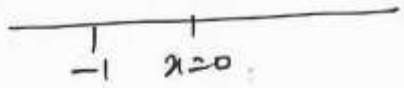
$$f'(x) = \frac{1}{1+x} - \left[\frac{(1+x)1 - x(1)}{(1+x)^2}\right]$$

$$= \frac{1}{1+x} - \left[\frac{1+x-x}{(1+x)^2}\right]$$

$$= \frac{1}{1+x} - \frac{1}{(1+x)^2}$$

$$= \frac{1+x-1}{(1+x)^2} = \frac{x}{(1+x)^2}$$

$f'(x) = 0 \Rightarrow \frac{x}{(1+x)^2} = 0 \Rightarrow x = 0$.



Hence The intervals are $(-1, 0)$, $x = 0$ and $(0, \infty)$

In $(-1, 0)$ say $x = -1/2$

$$f'(x) = \frac{-1/2}{(1-1/2)^2} = \frac{-1/2}{1/4} = -2 < 0$$

∴ $f(x)$ is strictly decreasing

At $x = 0$, $f'(x) = 0$.

In $(-1, \infty)$ say $x = 2$,

$$f'(2) = \frac{2}{9} > 0$$

∴ $f(x)$ is strictly increasing function.

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Intervals	$f(x)$	function
$(-1,0)$	$f'(x)$	Strictly decreasing function
$(0,\infty)$	$f'(x)$	Strictly Inc fun

a) $f'(x)$ changes sign from negative to positive at $x = 0$.

$\therefore$ $f(x)$ has local minimum at $x = 0$.

$\therefore$ The local minimum value is $f(0) = \log(1+0) - \frac{0}{1+0}$

$= \log 1$

$\boxed{f(0) = 0}$

for $x > 0$ we have $f(x) > f(0) = 0$

i.e. $\log(1+x) - \frac{x}{1+x} > 0$

$\Rightarrow \log(1+x) > \frac{x}{1+x}$

$\therefore$ The domain when $\log(1+x) > \frac{x}{1+x}$ is $(0,\infty)$.

Eg 7.54 Find the intervals of monotonicity and local extrema of the function $f(x) = x\log x + 3x$

$f(x) = x\log x + 3x$

$f'(x) = x\frac{1}{x} + \log x(1) + 3$

$= 1 + \log x + 3$

$= \log x + 4$

$f'(x) = 0 \Rightarrow \log x + 4 = 0$

$\log x = -4$

$x = e^{-4}$

Hence the stationary point is $x = e^{-4}$.

$\therefore$ The intervals are $(-\infty, e^{-4})$ and (e^{-4}, ∞)

In $(-\infty, e^{-4})$ say $x = e^{-5}$

$f'(e^{-5}) = \log e^{-5} + 4$

$= -5\log e + 4$

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$= -5+4 = -1 < 0$

$\therefore f(x)$ is strictly decreasing in $(0, e^{-4})$

In (e^{-4}, ∞) say $x = e^{-3}$

$f'(e^{-3}) = \log e^{-3} + 4$

$= -3 \log e + 4$

$= -3 + 4 = 1 > 0$

$\therefore f(x)$ is strictly increasing in (e^{-4}, ∞).

$f'(x)$ changes sign from negative to positive at $x = e^{-4}$

$\therefore f(x)$ has local minimum at $x = e^{-4}$

$\therefore$ The local minimum value is

$f(e^{-4}) = e^{-4} \log e^{-4} + 3e^{-4}$

$= -4e^{-4} \log e + 3e^{-4}$

$= -4e^{-4} + 3e^{-4}$

$= -e^{-4}$

$\therefore$ Local minimum value at $x = e^{-4}$ is $-e^{-4}$

Eg 7.55 Find the intervals of monotonicity and local extrema of the function

$f(x) = \frac{1}{1+x^2}$

$f(x) = \frac{1}{1+x^2}$

$f'(x) = \frac{(1+x^2) \cdot 0 - 1(2x)}{(1+x^2)^2}$

$= -\frac{2x}{(1+x^2)^2}$

$f'(x) = 0 \Rightarrow \frac{-2x}{(1+x^2)^2} = 0$

$\Rightarrow x = 0$.

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∴ The stationary point is $x = 0$.

Hence the intervals are
$(-\infty, 0)$ and $(0, \infty)$

In $(-\infty, 0)$ say $x = -1$

$$f'(-1) = \frac{-2(-1)}{(1+1)^2} = \frac{2}{4} = \frac{1}{2} > 0.$$

∴ $f(x)$ is strictly Increasing fun in $(-\infty, 0)$.

In $(0, \infty)$, say $x = 1$

$$f'(1) = \frac{-2(1)}{(1+1)^2} = -\frac{2}{4} = -\frac{1}{2} < 0$$

∴ $f(x)$ is strictly decreasing function in $(0, \infty)$.

$f'(x)$ changes sign from positive to negative at $x = 0$.

∴ $f(x)$ has local maximum at $x = 0$.

$\therefore f(0) = \frac{1}{1+0} = \frac{1}{1} = 1$

∴ local maximum value at $x = 0$ is 1

Eg 7.56 find the intervals of monotonicity and local extrema of the function $f(x) = \frac{x}{1+x^2}$

$$f(x) = \frac{x}{1+x^2}$$

$$f'(x) = \frac{(1+x^2)1 - x(2x)}{(1+x^2)^2}$$

$$= \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

$$f'(x) = 0 \Rightarrow \frac{1-x^2}{(1+x^2)^2} = 0$$

$$\Rightarrow 1 - x^2 = 0$$

$$\Rightarrow x^2 = 1$$

$$\Rightarrow x = \pm 1$$

∴ The intervals are $(-\infty, -1)$ $(-1, 1)$ and $(1, \infty)$

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In $(-\infty,-1)$ say $x=-2$

$f'(-2) = \frac{1-4}{(1+4)^2} = -\frac{3}{25} < 0$

$\therefore$ $f(x)$ is strictly decreasing in $(-\infty,-1)$

In $(-1,1)$ say $x=0$

$f'(0) = \frac{1-0}{(1+0)^2} = \frac{1}{1} = 1 > 0$.

$\therefore$ $f(x)$ is strictly increasing in $(-1,1)$

In $(1,\infty)$, say $x=2$

$f'(2) = \frac{1-4}{(1+4)^2} = -\frac{3}{25} < 0$

$\therefore$ $f(x)$ is strictly decreasing in $(1,\infty)$

Intervals	$f'(x)$	function
$(-\infty,-1)$	$f'(x)<0$	strictly dec function.
$(-1,1)$	$f'(x)>0$	strictly Inc function
$(1,\infty)$	$f'(x)<0$	strictly dec function.

a) $f'(x)$ changes sign from negative to positive at $x=-1$

$\therefore$ $f(x)$ has local minimum at $x=-1$

$\therefore$ local minimum value at $x=-1$ is $f(-1) = \frac{-1}{1+(-1)^2} = \frac{-1}{1+1} = \frac{-1}{2}$.

$\therefore$ local minimum value is $\boxed{\frac{-1}{2}}$.

b) $f'(x)$ changes sign from positive to negative.

$\therefore$ $f(x)$ has local maximum at $x=1$

$\therefore$ $f(1) = \frac{1}{1+1} = \boxed{\frac{1}{2}}$

$\therefore$ local maximum value at $x=1$ is $\frac{1}{2}$.

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First Derivative Test:-

1) If $f'(x)$ changes its sign +ve to -ve then $f(x)$ has local maximum $f(c)$.

2) If $f'(x)$ changes its sign from -ve to +ve then $f(x)$ has local minimum $f(c)$.

3) $f'(x)$ is +ve on both sides of c (or) -ve on both sides of c then $f(c)$ is neither local max. nor local minimum.

EXERCISE 7.6

Find the absolute extrema

(i) $f(x) = x^2 - 12x + 10$, $[1, 2]$

$f'(x) = 2x - 12$

$f'(x) = 0 \Rightarrow 2x - 12 = 0$

$2x = 12 \Rightarrow \boxed{x = 6}$

The critical no is 6.

$f(6) = 6^2 - 12(6) + 10$

$= 36 - 72 + 10$

$= 46 - 72$

$= -26$

$f(1) = 1 - 12 + 10 = 11 - 12 = -1$

$f(2) = 4 - 24 + 10 = 14 - 24 = -10$

∴ obsolute maximum $= -1$

obsolute minimum $= -26$

(ii) $f(x) = 3x^4 - 4x^3$, $[-1, 2]$

$f'(x) = 12x^3 - 12x^2$

$= 12x^2(x - 1)$

$f'(x) = 0 \Rightarrow 12x^2(x-1) = 0$

$\Rightarrow x = 0$ or $x = 1$

∴ critical nos are $x = 0, 1$

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$f(0) = 3(0) - 4(0) = \boxed{0}$

$f(1) = 3(1) - 4(1) = 3 - 4 = \boxed{-1}$

$f(-1) = 3(-1)^4 - 4(-1)^3$

$= 3(1) - 4(-1)$

$= 3 + 4 = \boxed{7}$

$f(2) = 3(2)^4 - 4(2)^3$

$= 3(16) - 4(8)$

$= 48 - 32$

$= 16$

∴ obSolute maximum $= 16$

obsolute minimum $= -1$

(iii) $f(x) = 6x^{4/3} - 3x^{1/3}$ $[-1, 1]$

$f'(x) = \frac{4}{3} x^{\frac{4}{3}-1} - 3\left(\frac{1}{3}\right) x^{\frac{1}{3}-1}$

$= 8x^{1/3} - x^{-2/3}$

$= x^{\frac{1}{3}}\left(8 - x^{-1}\right)$

$= x^{\frac{1}{3}}\left(8 - \frac{1}{x}\right)$

$= x^{1/3}\left(\frac{8x-1}{x}\right) = \frac{8x-1}{x^1 \cdot x^{-1/3}} = \frac{8x-1}{x^{-2/3}}$

XII/ADC/7.6
p.62

$f'(x) = 0 \Rightarrow 8x - 1 = 0$

$\Rightarrow 8x = 1$

$x = 1/8$

At $x = 0$, $f'(x)$ does not exist

∴ Critical numbers $x = 0, 1/8$

$f(0) = 0$

$f\left(\frac{1}{8}\right) = 6\left(\frac{1}{8}\right)^{\frac{4}{3}} - 3\left(\frac{1}{8}\right)^{\frac{1}{3}}$

$= 6\left(\frac{1}{2}\right)^4 - 3\left(\frac{1}{2}\right)$

$= \frac{6}{16} - \frac{3}{2} = \frac{6-24}{16} = \frac{-18}{16} = \boxed{\frac{-9}{8}}$

$f(-1) = 6(-1)^{4/3} - 3(-1)^{1/3} = 6 + 3 = \boxed{9}$

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$f(1) = 6 - 3 = 3$

$\therefore$ absolute maximum $= 9$

absolute minimum $= -\frac{9}{8}$

(iv) $f(x) = 2\cos x + \sin 2x \quad [0, 2\pi]$

$f'(x) = 2(-\sin x) + \cos 2x \cdot 2$

$= -2\sin x + 2\cos 2x$.

$f'(x) = 0 \Rightarrow 2\cos 2x - 2\sin x = 0$

$\Rightarrow 2(\cos 2x - \sin x) = 0$

$\Rightarrow \cos 2x - \sin x = 0$

$1 - 2\sin^2 x - \sin x = 0$

$2\sin^2 x + \sin x - 1 = 0$

$(\sin x + 1)(2\sin x - 1) = 0$

-2

$\frac{2}{2} \quad \frac{-1}{2}$

$1, -\frac{1}{2}$

$\sin x + 1 = 0 \quad | \quad 2\sin x - 1 = 0$

XII/ADC/7.6

PT 3

$\sin x = -1 \quad | \quad 2\sin x = 1$

$\sin x = 1/2$

~~$\sin x = \sin(-1)$~~

~~$x = n\pi + (-1)^n(-\pi/2)$~~

~~$\sin x = \sin \pi/6$~~

~~$x = n\pi + (-1)^n \pi/6$~~.

$\sin x = -1 \quad | \quad \sin x = 1/2$

~~$\sin x =$~~

$x = \sin^{-1}(-1) \quad | \quad x = \pi/6$.

$= 2\pi - \pi/2$

$= 3\pi/2$

$\therefore$ The critical nos are $3\pi/2, \pi/6$.

$F(3\pi/2) = 2\cos(3\pi/2) + \sin 2(3\pi/2)$

$= 2\cos\frac{3\pi}{2} + \sin 3\pi$

$= 2(0) + 0 = 0$

$=$ ~~[illegible]~~.

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$f(\pi/6) = 2\cos\pi/6 + \sin 2\pi/6$

$= 2\frac{\sqrt{3}}{2} + \sin\pi/3$

$= \frac{2\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

$f(0) = 2\cos(0) + \sin(0)$

$= 2(1) + 0 = 2$

$f(\pi/2) = 2\cos\pi/2 + \sin 2\pi/2$

$= 2(0) + \sin\pi = 2(0) + 0$

$= 0.$

∴ absolute maximum is $\frac{3\sqrt{3}}{2}$

absolute minimum is 0.

---

2) find the interval of monotonicity and hence find the local extremum.

(i) $f(x) = 2x^3 + 3x^2 - 12x$.

XII/ADC/ 7.6
P14

$f'(x) = 6x^2 + 6x - 12$.

$f'(x) = 0 \Rightarrow 6x^2 + 6x - 12 = 0$

$6(x^2 + x - 2) = 0$

$6(x+2)(x-1) = 0$

$\Rightarrow x = 1, -2$.

∴ The intervals are $(-\infty, -2)$ $(-2, 1)$, $(1, \infty)$.

In $(-\infty, -2)$ say $x = -3$

$f'(-3) = 6(-1)(-4) = 24 > 0$.

∴ $f(x)$ is strictly increasing fun.

In $(-2, 1)$, say $x = -1$

$f'(-1) = 6(1)(-2) = -12 < 0$

∴ $f(x)$ is strictly decreasing fun.

In $(1, \infty)$ say $x = 2$

$f'(2) = 6(4)(1) = 24 > 0$

$f(x)$ is strictly increasing fun.

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Interval	$f'(x)$	function
$(-\infty, -2)$	$f'(x) > 0$	Strictly Inc. fun
$(-2, 1)$	$f'(x) < 0$	Strictly dec function
$(1, \infty)$	$f'(x) > 0$	Strictly Inc. function.

a) $f'(x)$ changes sign from $+^{ve}$ to $-^{ve}$ through at $x = -2$.

$\therefore f(x)$ has local maximum at $x = -2$

$\therefore$ max value is $f(-2) = 2(-8) + 3(-2)^2 - 12(-2)$

$= -16 + 12 + 24$

$= 36 - 16 = 20.$

XII/ADC/7.6
P15

b) $f'(x)$ changes sign from $-^{ve}$ to $+^{ve}$ through $x = 1$

$\therefore f(x)$ has local minimum at $x = 1$

$\therefore$ minimum value is $f(1) = 2(1) + 3(1) - 12$

$= 2 + 3 - 12$

$= -7$

$\therefore$ Local maximum $= 20$

Local minimum $= -7$

(11) $f(x) = \frac{x}{x-5}$

$f'(x) = \frac{(x-5)1 - x(1)}{(x-5)^2} = \frac{x-5-x}{(x-5)^2}$

$= \frac{-5}{(x-5)^2}$

$f'(x) = 0 \Rightarrow \frac{-5}{(x-5)^2} \neq 0 \quad \forall x \in R$

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$f'(x)$ does not exist at $x=5$

There is no stationary point, but critical no at $x=5$.

∴ The intervals are $(-\infty,5), (5,\infty)$

In $(-\infty,5)$, say $x=-2$

$$f'(-2) = \frac{-5}{(-2-5)^2} = -\frac{5}{49} < 0.$$

∴ $f(x)$ is strictly decreasing fun

In $(5,\infty)$, say $x=2$,

$$f'(2) = \frac{-5}{(2-5)^2} = -\frac{5}{9} < 0.$$

∴ $f(x)$ is strictly decreasing function.

∴ $f(x)$ is strictly decreasing on both $(-\infty,5), (5,\infty)$

$f'(x)$ does not change its sign.

∴ There is no local extrema.

XII/ADC/ 7-6
pg6

(iii) $f(x) = \frac{e^x}{1-e^x}$

$$f'(x) = \frac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2}$$

$$= \frac{e^x - e^{2x} + e^{2x}}{(1-e^x)^2} = \frac{e^x}{(1-e^x)^2} > 0$$

$f'(x) > 0 \ \forall x \in R$

∴ $f(x)$ is strictly increasing in $(-\infty,\infty)$

$f'(x)$ does not change its sign.

∴ There is no local maxima extrema

× ×

(iv) $f(x) = \frac{x^3}{3} - \log x$

$$f'(x) = \frac{1}{3}3x^2 - \frac{1}{x} = x^2 - \frac{1}{x} = \frac{x^3-1}{x}$$

$f'(x)=0 \Rightarrow \frac{x^3-1}{x} = 0 \Rightarrow x^3-1=0$

$\Rightarrow x^3 = 1$

$x = 1$

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∴ The intervals are $(-\infty,1), (1,\infty)$

In $(-\infty,1)$, say $x=$

when $x=0$, $f'(x)$ does not exist

∴ critical numbers are $x=0,1$

∴ Intervals are $(0,1), (1,\infty)$

In $(0,1)$, say $x=\frac{1}{2}$

$$f'\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)^3-1}{\frac{1}{2}} = \frac{\frac{1}{8}-1}{\frac{1}{2}} = -\frac{7}{8}\times\frac{2}{1}$$

$$= -\frac{7}{4} < 0$$

∴ $f(x)$ is strictly Decreasing function on $(0,1)$.

In $(1,\infty)$, say $x=2$

$$f'(2) = \frac{2^3-1}{2} = \frac{7}{2} > 0.$$

∴ $f(x)$ is strictly increasing function on $(1,\infty)$.

$f'(x)$ changes sign from -ve to +ve at $x=1$.

∴ $f(x)$ has a local minimum at $x=1$

∴ Local minimum value $= f(1)$

$$= \frac{1}{3} - \log 1$$

$$= \frac{1}{3} //$$

(v) $f(x) = \sin x \cos x + 5$, $x \in (0, 2\pi)$

$f(x) = \sin x \cos x + 5$

$f'(x) = \sin x(-\sin x) + \cos x \cos x$.

$= \cos^2 x - \sin^2 x$

$= \cos 2x$.

$f'(x) = 0 \Rightarrow \cos 2x = 0$

$2x = \pi/2, 3\pi/2, 5\pi/2, 7\pi/2 \ldots$

∴ $x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4, \ldots$

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∴ critical nos are $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4, \ldots$

∴ The intervals are

$(0, \pi/4), (\pi/4, 3\pi/4), (3\pi/4, \frac{5\pi}{4}), (\frac{5\pi}{4}, \frac{7\pi}{4}), \ldots$

$(\frac{7\pi}{4}, 2\pi)$

In $(0, \pi/4)$, say $x = \pi/6$

$f'(x) = \cos 2x = \cos 2\pi/6 = \cos \pi/3 = \boxed{\frac{1}{2}} > 0$

∴ $f(x)$ is strictly increasing function.

In $(\pi/4, 3\pi/4)$ say $x = \pi/3$.

$f'(x) = \cos 2x = \cos 2(\pi/3) = \cos \frac{2\pi}{3}$

$= \cos(\pi - \pi/3)$

$= -\cos \pi/3 = \boxed{-\frac{1}{2}} < 0$

∴ $f(x)$ is strictly decreasing function on $(\pi/4, 3\pi/4)$

In $(\frac{3\pi}{4}, \frac{5\pi}{4})$ say $x = \pi$

$f'(x) = \cos 2x = \cos 2\pi = 1 > 0$

$f(x)$ is strictly Inc fun on $(\frac{3\pi}{4}, \frac{5\pi}{4})$

In $(\frac{5\pi}{4}, \frac{7\pi}{4})$ say $x = 3\pi/2$

$f'(x) = \cos 2x = \cos 2\frac{3\pi}{2}$

$= \cos 3\pi = -1 < 0$

∴ $f(x)$ is strictly dec function on $(\frac{5\pi}{4}, \frac{7\pi}{4})$.

In $(\frac{7\pi}{4}, 2\pi)$ say $x = 300°$

$f'(x) = \cos 2x = \cos 2(2\pi - \pi/3)$

$= \cos 2(\frac{5\pi}{3})$

$= \cos \frac{10\pi}{3} = \cos(3\pi + \frac{\pi}{3})$

$= \cos \pi/3 = 1/2 > 0$

$f(x)$ is strictly increasing function.

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$\therefore$ $f(x)$ is strictly increasing on $(0, \pi/4)$, $\left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$, $\left(\frac{7\pi}{4}, 2\pi\right)$ and strictly decreasing on $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$, $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

$f'(x)$ changes sign from positive to negative at $x = \pi/4, 5\pi/4$

$\therefore$ local max at $x = \frac{\pi}{4}, \frac{5\pi}{4}$

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} \cdot \cos\frac{\pi}{4} + 5$$

$$= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 5 = \frac{1}{2} + 5 = \boxed{\frac{11}{2}}$$

$$f\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{4} \cos\frac{5\pi}{4} + 5$$

$$= \sin\left(\pi + \frac{\pi}{4}\right) \cos\left(\pi + \frac{\pi}{4}\right) + 5$$

$$= -\sin\frac{\pi}{4}\left(-\cos\frac{\pi}{4}\right) + 5$$

$$= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 5$$

$$= \frac{1}{2} + 5 = \boxed{\frac{11}{2}}$$

Also $f'(x)$ changes sign from negative to positive at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$\therefore$ Local minimum at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$$\therefore f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} \cdot \cos\frac{3\pi}{4} + 5$$

$$= \sin\left(\pi - \frac{\pi}{4}\right) \cdot \cos\left(\pi - \frac{\pi}{4}\right) + 5$$

$$= \sin\frac{\pi}{4}\left(-\cos\frac{\pi}{4}\right) + 5$$

$$= \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) + 5$$

$$= 5 - \frac{1}{2} = \boxed{\frac{9}{2}}$$

$$f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} \cdot \cos\frac{7\pi}{4} + 5$$

$$= \sin\left(2\pi - \frac{\pi}{4}\right) \cos\left(2\pi - \frac{\pi}{4}\right) + 5$$

$$= -\sin\frac{\pi}{4}\left(\cos\frac{\pi}{4}\right) + 5$$

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$= -\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 5 = 5 - \frac{1}{2} = \frac{9}{2}$

$\therefore$ Local max is $\frac{11}{2}$

Local min is $\frac{9}{2}$

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Application of Second derivatives

Test of concavity:-

1) If $f''(x) > 0$ on an open interval I, then $f(x)$ is Concave up on I

2) If $f''(x) < 0$ on I, then $f(x)$ is concave down on I.

point of inflection:-

(i) If $f''(c)$ exists and $f''(c)$ changes sign +ve to −ve or −ve to +ve through c Then $(c, f(c))$ is called point of inflection.

(ii) If $f''(c)$ exists at point of inflection $f''(c) = 0$.

The Second derivative test

a) $f'(x)$ exist and give the critical point $x = c$

b) find $f''(x)$ then

$f''(c) < 0$ at $x = c$ then $f(x)$ has relative maximum.

$f''(x) > 0$ at $x = c$ then $f(x)$ has relative minimum.

$f''(x) = 0$ at $x = c$ The test is not informative.

Exercise 7.7

1) Find the interval of concavity and point of inflection

(i) $f(x) = x(x-4)^3$

$f'(x) = x\,3(x-4)^2 + (x-4)^3$

$= (x-4)^2\{3x + x - 4\}$

$= (x-4)^2(4x-4)$

$f''(x) = (x-4)^2(4) + (4x-4)\cdot 2(x-4)(1)$

$= 4(x-4)^2 + 4(x-1)\,2(x-4)$

$= 4(x-4)\{x-4+2(x-1)\}$

$= 4(x-4)(3x-6)$

$= 12(x-4)(x-2)$

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$12(x-4)(x-2)$

$f''(x) = 0 \Rightarrow x-4=0$ or $x-2=0$

$x = 4$ or $x = 2$

∴ The intervals are $(-\infty, 2), (2,4), (4,\infty)$

I	value	$f''(x)$	Sign of $f''(x)$	concavity
$(-\infty,2)$	0	$12(-4)(-2) = 96$	+	concave up
$(2,4)$	3	$12(-1)(1) = -12$	−	concave down.
$(4,\infty)$	5	$12(1)(3) = 36$	+	concave up

∴ The curve is concave up on $(-\infty,2)$ and $(4,\infty)$ and concave down on $(2,4)$

$f''(x)$ changes its sign when it passes through $x = 2$ and $x = 4$

$\therefore f(2) = 2(-2)^3 = -16$

$f(4) = 4(0) = 0$

∴ point of inflection $(2,-16)$ $(4,0)$

II $f(x) = \sin x + \cos x \quad 0 < x < 2\pi$

$f'(x) = \cos x - \sin x$.

$f''(x) = -\sin x - \cos x$

$f''(x) = 0 \Rightarrow -\sin x = \cos x$

$-\tan x = 1 \Rightarrow \tan x = -1$

$\tan(-45), \alpha = -\pi/4$

∴ $x = n\pi - \pi/4 \quad n \in Z$.

$x = 3\pi/4, 7\pi/4$

∴ intervals are $(0, \frac{3\pi}{4}), (\frac{3\pi}{4}, \frac{7\pi}{4})$ $(\frac{7\pi}{4}, 2\pi)$

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I Value	$f''(x)$	$f''(x)$ sign	concavity
	$= -(\sin x + \cos x)$		
$(0, \frac{3\pi}{4})$ $\frac{\pi}{2}$	$= -(\sin\frac{\pi}{2} + \cos\frac{\pi}{2})$		
	$= -(1+0)$	-ve	concave down.
	$= -1$		
$(\frac{3\pi}{4}, \frac{7\pi}{4})$ π	$-(\cos\pi + \sin\pi)$		
	$= -(-1) = 1$	+ve	Concave up
$(\frac{7\pi}{4}, 2\pi)$ $\frac{11\pi}{6}$ $= 330$	$-(\cos 330 + \sin 330)$		
	$-(\cos(2\pi - \frac{\pi}{6}) + \sin(2\pi - \frac{\pi}{6}))$		
	$= -(\cos\frac{\pi}{6} - \sin\frac{\pi}{6})$		
	$= -(\frac{\sqrt{3}}{2} - \frac{1}{2})$		
	$= \frac{1-\sqrt{3}}{2}$	-ve	Concave down

$\therefore$ $f(x)$ is concave up on $(\frac{3\pi}{4}, \frac{7\pi}{4})$, and concave down $(0, \frac{3\pi}{4})$, $(\frac{7\pi}{4}, 2\pi)$.

$f''(x)$ change its sign through $x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

$\therefore f(\frac{3\pi}{4}) = \sin 135 + \cos 135$

$= \sin(90+45) + \cos(90+45)$

$= \cos 45 - \sin 45$

$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$

$f(\frac{7\pi}{4}) = \sin 315 + \cos 315$

$= \sin(360-45) + \cos(360-45)$

$= -\sin 45 + \cos 45$

$= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0$

$\therefore$ pt of inflection $(\frac{3\pi}{4}, 0)$, $(\frac{7\pi}{4}, 0)$

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iii) $f(x) = \frac{1}{2}(e^x - e^{-x})$

$f'(x) = \frac{1}{2}(e^x + e^{-x})$

$f''(x) = \frac{1}{2}(e^x - e^{-x})$

$f''(x) = 0 \Rightarrow e^x - e^{-x} = 0$

$e^x = e^{-x} \Rightarrow x = 0$

Intervals are $(-\infty, 0)$, $(0, \infty)$

I	Value	$f''(x)$ $\frac{e^x - e^{-x}}{2}$	Sign of $f''(x)$	Concavity
$(-\infty, 0)$	-1	$\frac{e^{-1} - e^{+1}}{2}$	$-$	Concave down
$(0, \infty)$	1	$\frac{e^1 - e^{-1}}{2}$	$+$	Concave up

∴ $f(x)$ is concave up on $(0, \infty)$

Concave down on $(-\infty, 0)$

$f''(x)$ changes its sign through $x = 0$

∴ $f(0) = \frac{e^0 - e^0}{2} = \frac{1-1}{2} = 0$

∴ point of inflection $= (0, 0)$

---

2) Find the local extrema for the function using second derivative test

(i) $f(x) = -3x^5 + 5x^3$

$f'(x) = -15x^4 + 15x^2$

$= 15x^2(1 - x^2)$

$= 15x^2(1+x)(1-x)$

$f'(x) = 0 \Rightarrow 15x^2(1+x)(1-x) = 0$

$\Rightarrow x = 0, x = -1, x = 1$

∴ The critical numbers are $x = 0, -1, 1$.

$f''(x) = 15\{x^2(-2x) + (1-x^2)(2x)\}$

$= -30x^3 + 30x(1-x^2)$

$= -30x^3 + 30x - 30x^3$

$= -60x^3 + 30x$

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$x = -1,\ f''(-1) = 60 - 30$
$= 30 > 0$

$\therefore f(x)$ has local minimum at $x = -1$

$\therefore$ local minimum value is $f(-1)$

$f(-1) = 3 - 5 = -2$

$x = 1,\ f''(1) = -60 + 30 = -30 < 0$

$\therefore f(x)$ has local maximum at $x = 1$

$\therefore$ local max value $f(1) = -3+5$
$= 2$

$x = 0,\ f''(0)$

$\therefore$ Second derivative test is not informative of local extremum of $f(x)$.

$\therefore$ Local max $= 2$
local min $= -2$

2(11) $f(x) = x \log x$

$f'(x) = x\frac{1}{x} + \log x$

$= 1 + \log x$

$f'(x) = 0 \Rightarrow 1 + \log x = 0$

$\log x = -1$

$x = e^{-1}$

$x = \frac{1}{e}$

$\therefore$ critical no is $x = \frac{1}{e}$

$f''(x) = \frac{1}{x}$

$f''\left(\frac{1}{e}\right) = \frac{1}{1/e} = e > 0$

$\therefore f(x)$ has local min at $x = 1/e$.

$\therefore$ local minm value $= f\left(\frac{1}{e}\right)$

$= \frac{1}{e}\log\left(\frac{1}{e}\right)$

$= \frac{1}{e}(\log 1 - \log e)$

$= \frac{1}{e}(0 - 1)$

$= -\frac{1}{e}$

$\therefore$ local min is $-1/e$

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2(iii) $f(x) = x^2 e^{-2x}$

$f'(x) = x^2(-2e^{-2x}) + e^{-2x}(2x)$

$= -2x^2 e^{-2x} + 2xe^{-2x}$

$= 2xe^{-2x}(1-x)$

$= 2e^{-2x}(x-x^2)$

$f'(x) = 0 \Rightarrow 2e^{-2x}(x-x^2) = 0$

$e^{-2x} \neq 0 \quad \therefore x - x^2 = 0$

$x(1-x) = 0$

$x = 0 \mid x = 1$

$\therefore$ The critical nos are 0, 1

$f''(x) = 2\{e^{-2x}(1-2x) + (x-x^2)e^{-2x}(-2)\}$

$= 2e^{-2x}(1-2x) - 4e^{-2x}(x-x^2)$

$= 2e^{-2x}\{1 - 2x - 2x + 2x^2\}$

$= 2e^{-2x}\{2x^2 - 4x + 1\}$

(i) when $x = 0$, $f''(0) = 2e^0(0+0+1)$

$= 2e^0$

$= 2 > 0$

$\therefore f(x)$ has local minm at $x = 0$

$\therefore$ local minm value is $f(0)$

$= 0$.

(ii) when $x = 1$, $f''(1) = 2e^{-2}(2-4+1)$

$= -2e^{-2} < 0$

$\therefore f(x)$ is local max at $x = 1$

$\therefore$ local max value is $f(1)$

$= 1e^{-2} = \frac{1}{e^2}$

$\therefore$ local min $= 0$

local max $= \frac{1}{e^2}$

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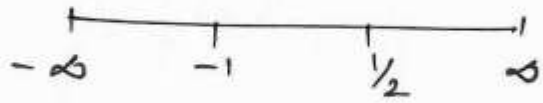
3) For the function $f(x) = 4x^3 + 3x^2 - 6x + 1$ find the intervals of monotonicity, local extrema, intervals of concavity and point of inflection.

$f'(x) = 12x^2 + 6x - 6$

$f'(x) = 6(2x^2 + x - 1)$

$= 6(x+1)(2x-1)$ — ①

$f'(x) = 0 \Rightarrow x = -1, \; x = \frac{1}{2}$



Intervals are $(-\infty, -1)$, $(-1, \frac{1}{2})$, $(\frac{1}{2}, \infty)$

I	Value x	$f'(x)$	Sign of $f'(x)$	monotonicity
$(-\infty, -1)$	-2	$6(-1)(-5) = 30$	+ve	Strictly Inc function
$(-1, \frac{1}{2})$	0	$6(1)(-1) = -6$	−ve	Strictly Dec. fun
$(\frac{1}{2}, \infty)$	1	$6(2)(1) = 12$	+ve	Strictly Inc fun

$\therefore f(x)$ is strictly inc on $(-\infty, -1)$, $(\frac{1}{2}, \infty)$

Strictly Dec. on $(-1, \frac{1}{2})$

$f'(x)$ changes from +ve to −ve at $x = -1$

$\therefore f(x)$ has local max at $x = -1$

$\therefore$ local max. Value is

$f(-1) = -4 + 3 + 6 + 1 = 6$

$f'(x)$ changes sign −ve to +ve through $x = \frac{1}{2}$

$\therefore f(x)$ has local minimum at $x = \frac{1}{2}$

$\therefore$ local min value $= f(\frac{1}{2})$

$= 4 \times \frac{1}{8} + 3 \times \frac{1}{4} - \frac{6}{2} + 1$

$= \frac{1}{2} + \frac{3}{4} - 3 + 1$

$= \frac{2 + 3 - 12 + 4}{4} = -\frac{3}{4}$

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$\therefore$ Local max $= 6$

Local min $= -3/4$

(11)

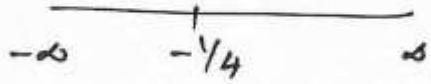
$f'(x) = 6(2x^2 + x - 1)$

$f''(x) = 6(4x + 1)$

$f''(x) = 0 \Rightarrow 4x + 1 = 0$

$4x = -1$

$x = -\frac{1}{4}$



Interval	value	$f''(x) = 6(4x+1)$	$f''(x)$ sign	concavity
$(-\infty, -\frac{1}{4})$	-1	$6(-4+1)$ -18	$-ve$	concave down.
$(-\frac{1}{4}, \infty)$	0	$6(1) = 6$	$+ve$	Convex up

$\therefore$ $f(x)$ is concave downward on $(-\infty, -\frac{1}{4})$

convex upward on $(-\frac{1}{4}, \infty)$

$f''(x)$ changes its sign through $x = 1/4$

$f(-\frac{1}{4}) = 4(-\frac{1}{64}) + 3(\frac{1}{16}) - 6(-\frac{1}{4}) + 1$

$= -\frac{1}{16} + \frac{3}{16} + \frac{6}{4} + 1$

$= \frac{-1 + 3 + 24 + 16}{16}$

$= \frac{42}{16} = \frac{21}{8}$

$\therefore$ point of inflection $(-\frac{1}{4}, \frac{21}{8})$

EX 7.57 Determine the intervals of concavity of $f(x) = (x-1)^3(x-5)$ and point of inflection.

$f(x) = (x-1)^3(x-5)$

$f'(x) = (x-1)^3(1) + (x-5)\,3(x-1)^2(1)$

$= (x-1)^2\{x - 1 + 3(x-5)\}$

$= (x-1)^2(x - 1 + 3x - 15)$

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$f'(x) = (x-1)^2(4x-16)$

$= 4(x-1)^2(x-4)$

$f''(x) = 4\{(x-1)^2(1) + (x-4)2(x-1)\}$

$= 4(x-1)\{x-1+2x-8\}$

$= 4(x-1)(3x-9)$

$= 12(x-1)(x-3)$

$f''(x) = 0 \Rightarrow x = 1, \; x = 3$

I	Value	$f''(x) = 12(x-1)(x-3)$	Sign of f''	Concavity
$(-\infty, 1)$	0	$12(-1)(-3) = +36$	+ve	concave up
$(1, 3)$	2	$12(1)(-1) = -12$	−ve	concave down
$(3, \infty)$	4	$12(3)(1) = 36$	+ve	concave up

XII | 7.7 | P9

The curve is concave up on $(-\infty, 1)$, $(3, \infty)$

The curve is concave down on $(1, 3)$

$f''(x)$ changes its sign through $x = 1$ and 3

$\therefore f(1) = (1-1)^2(4-16) = 0$

$f(3) = (3-1)^2(12-16)$

$= 4(-4) = -16$

$\therefore$ pt of inflections are $(1, 0)$ and $(3, -16)$

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7.58 Determine the intervals of concavity of $y = 3 + \sin x$.

$y = 3 + \sin x$

$y' = \cos x$

$y'' = -\sin x$

$y'' = 0 \Rightarrow -\sin x = 0$

$\Rightarrow n\pi = 0.$

$-\pi, \pi$

$\therefore$ Intervals are $(-\pi, 0)$ $(0, \pi)$

In $(-\pi, 0)$, say $-\pi/2$, $f''(-\pi/2) = -\sin(-\pi/2)$

$= -(-1)$

$= \sin \pi/2 = 1 > 0$

$\therefore f(x)$ is concave up on $(-\pi, 0)$

In $(0, \pi)$, say $x = \pi/2$, $f''(\pi/2) = -\sin \pi/2 = -1 < 0$

$\therefore f(x)$ is concave down on $(0, \pi)$

$f''(x)$ changes its sign through $x = 0$

$\therefore f(0) = 3 + \sin(0) = 3$

$\therefore$ point of inflection is $(0, 3)$

7.59 Find the local extremum of $f(x) = x^4 + 32x$

$f(x) = x^4 + 32x$

$f'(x) = 4x^3 + 32$

$f''(x) = 12x^2.$

$f'(x) = 0 \Rightarrow 4(x^3 + 8) = 0$

$x^3 + 8 = 0$

$x^3 = -8$

$x = -2$

$f''(-2) = 12(-2)^2 = 48 > 0$

$= -24 < 0.$

$\therefore f(x)$ is local max at $x = -2$

$\therefore$ local max value is $f(-2)$

$= 16 + 32(-2)$

$= 16 - 64 = -48$

$\therefore$ local max is -48

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7.60 Find the local extrema of $f(x) = 4x^6 - 6x^4$

$f(x) = 4x^6 - 6x^4$

$f'(x) = 24x^5 - 24x^3$

$= 24x^3(x^2-1)$

$= 24x^3(x+1)(x-1)$

$f'(x) = 0 \Rightarrow x = 0, x = -1, x = 1$

∴ The critical nos are $x = 0, -1, 1$

$f''(x) = 24\{x^3(2x) + (x^2-1)3x^2\}$

$= 24\{2x^4 + 3x^4 - 3x^2\}$

$= 24\{5x^4 - 3x^2\}$

$= 24x^2(5x^2-3)$

(i) when $x = 0$, $f''(0) = 0$

Second derivative does not give any information about local extrema at $x = 0$.

(ii) $f''(-1) = 24(1)(5-3)$

$= 24(2) = 48 > 0$

∴ $f(x)$ has local minum at at $x = -1$

∴ Local min value is $f(-1)$

$= 4 - 6 = -2$

(iii) $f''(1) = 24(1)(5-3) = 48 > 0$

∴ $f(x)$ has local min at $x = 1$

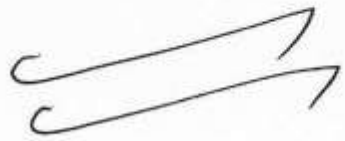
∴ Local min value is $f(1)$

$= 4 - 6 = -2$.

By the 1st derivative test $f(x)$ has local max at $x = 0$ and local max. value is 0.

∴ Local max = 0

Local min = −2



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EXERCISE 7.8

1. Find two positive numbers whose sum is 12 and their product is maximum.

Let the two positive numbers be x and y.

Given $x+y=12 \Rightarrow y = 12-x$ ——— (1)

Their product is $f = xy$

$= x(12-x)$

$= 12x - x^2$

$\therefore f' = 12 - 2x$

$f'' = -2$

For maximum or minimum $f' = 0$.

i.e. $12 - 2x = 0$

$2x = 12$

$\boxed{x = 6}$

$\therefore$ The critical number is 6.

when $x = 6$, $f'' = -2 < 0$

$\therefore$ The product is maximum when $x = 6$.

put $x = 6$ in (1)

$y = 12 - 6 = 6$

Hence the required nos are 6, 6. //

2) Find two positive numbers whose product is 20 and their sum is minimum.

Let the two positive numbers be x and y.

Given $xy = 20 \Rightarrow y = \frac{20}{x}$. ——— (1)

Their sum is $f = x + y$

$= x + \frac{20}{x}$

Now $f' = 1 + 20\left(-\frac{1}{x^2}\right)$

$f'' = -20\left(-\frac{2}{x^3}\right) = \frac{40}{x^3}$.

For maximum or minimum

$f' = 0 \Rightarrow 1 - \frac{20}{x^2} = 0$

$\Rightarrow \frac{20}{x^2} = 1 \Rightarrow x^2 = 20$

$x = \sqrt{20}$

$= \sqrt{4 \times 5}$

$= 2\sqrt{5}$

when $x = 2\sqrt{5}$, $f'' = \frac{40}{(2\sqrt{5})^3} > 0$.

Hence f is minimum when $x = 2\sqrt{5}$

put $x = 2\sqrt{5}$ in (1)

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$$y = \frac{20}{2\sqrt{5}} = \frac{4\times5}{2\sqrt{5}} = 2\sqrt{5}$$

Hence the required positive numbers are $2\sqrt{5}$, $2\sqrt{5}$.

---

3) Find the smallest possible value of x^2+y^2 given that $x+y=10$.

Given $x+y=10$

$y = 10-x$ ——— (1)

Let $f(x) = x^2+y^2 \Rightarrow x^2+(10-x)^2$

$f(x) = x^2+100+x^2-20x$

$= 2x^2-20x+100$

$f'(x) = 4x-20$

$f''(x) = 4$

For maxima or minima $f'(x)=0$

i.e. $4x-20=0$

$4x=20$

$x=5$

when $x=5$, $f''(x)=4>0$

$\therefore$ $f(x)$ is minimum when $x=5$

put $x=5$ in (1)

$y = 10-5 = 5$

$\therefore$ The smallest possible value of x^2+y^2 is $x^2+y^2 = 5^2+5^2 = 25+25 = 50$.

---

4) A garden is tobe laid out in a rectangular area and protected by wire fence. what is the largest possible area of the fenced garden with 40 metre of wire.?

Let l and b be the length and breadth of the rectangular garden.

Given $2(l+b) = 40$

$\therefore l+b = 20$.

Let $l = x$ m

$\therefore$ $b = 20-l \Rightarrow b = 20-x$ ——— (1)

Area $A = lb$

$= x(20-x)$

$= 20x-x^2$

$A'(x) = 20-2x$

$A''(x) = -2$

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For maxima or minima

$A' = 0 \Rightarrow 20 - 2x = 0$

$2x = 20$

$x = 10$.

$x = 10$, $A''(x) = -2 < 0$.

$\therefore$ Area is maximum when $x = 10$.

put $x = 10$, in ①

$b = 20 - 10 = 10$.

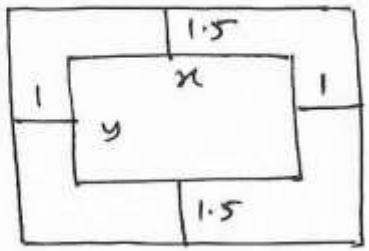
$\therefore$ Area $= lb = 10 \times 10 = 100\ m^2$.

$\therefore$ The largest possible area of the garden is $100\ m^2$.

---

5) A rectangular page is to contain $24\ cm^2$ of print. The margins at the top and bottom of the page are 1.5cm and the margin at other sides of the page is 1cm. what should be the dimension of the page so that the area of the paper used is minimum.

Let x and y be the length and breadth of the printed area.



Given that

Area $= 24$

i.e. $xy = 24$

$y = \frac{24}{x}$ ——— ①

Length of the page with margin $= x + 1 + 1$

$= x + 2$

Breadth of the page with margin $= y + 1.5 + 1.5$

$= y + 3$.

Area of the page $A = (x+2)(y+3)$

$A = xy + 3x + 2y + 6$

$= x\left(\frac{24}{x}\right) + 3x + 2\left(\frac{24}{x}\right) + 6$

$= 24 + 3x + \frac{48}{x} + 6$

$= 30 + 3x + \frac{48}{x}$

$A'(x) = 3 + 48\left(-\frac{1}{x^2}\right)$

$A''(x) = -48\left(-\frac{2}{x^3}\right) = \frac{96}{x^3}$

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For maxima or minima

$A' = 0 \Rightarrow 3 - \frac{48}{x^2} = 0$

$\frac{48}{x^2} = 3$

$3x^2 = 48 \Rightarrow x^2 = \frac{48}{3}$

$x^2 = 16$

$x = \pm 4$

$x = -4$ is not possible.

$\therefore \boxed{x = 4}$

when $x = 4$, $A'' = \frac{96}{4^3} > 0$.

$\therefore$ A is minimum when $x = 4$

put $x = 4$ in (1) $y = \frac{24}{4} = 6$

$\therefore$ length of the page $= x + 2 = 4 + 2 = \boxed{6\text{cm}}$

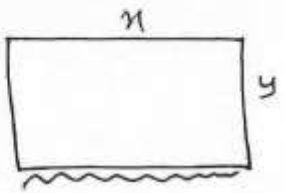
Breadth of the page $= y + 3$

$= 6 + 3 = \boxed{9\text{cm}}$.

---

6) A farmer plans to fence a rectangular pasture adjacent to a river. The posture must contain 1,80,000 sq.m. In order to provide enough grass for herds. No fencing is needed along the river. what is the length of the minimum needed fencing material?

Let x and y be the length and breadth of the posture.



Given Area $= 1,80,000$

i $xy = 1,80,000$

$y = \frac{1,80,000}{x}$ ——— (1)

length of the fencing $= x + 2y$

i $L(x) = x + 2\left(\frac{1,80,000}{x}\right)$

$L'(x) = 1 + 360000\left(-\frac{1}{x^2}\right)$

$L''(x) = -360000\left(-\frac{2}{x^3}\right)$

$= \frac{720000}{x^3}$

For maxima or minima,

$L'(x) = 0 \Rightarrow 1 - \frac{360000}{x^2} = 0$

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$$\frac{360000}{x^2} = 1 \Rightarrow x^2 = 360000$$

$$x = \pm 600$$

$x = -600$ is not possible.

$\therefore$ $\boxed{x = 600\ m}$.

when $x = 600$ m, $L''(x) = \frac{720000}{600^3} > 0$

$\therefore$ $L(x)$ is minimum when $x = 600$.

put $x = 600$ in (1)

$$y = \frac{1,80,000}{600} = \boxed{300}.$$

$\therefore$ minimum length of wire for fencing $= x + 2y$

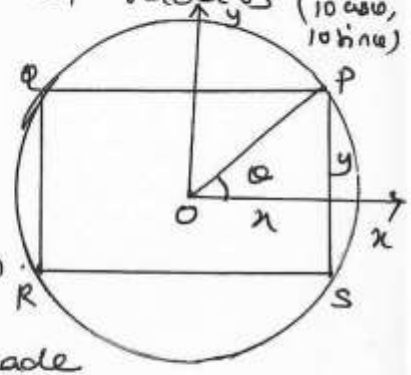
$= 600 + 2(300)$

$= 600 + 600$

$= 1200$m.

XII/CH7/7.8/P5

7) Find the dimensions of the rectangle with maximum area that can be incribed in a circle of radius 10cm.



Let P, Q, R, S be a rectangle inscribed in a circle with centre O and radius 10cm.

Let θ be the angle made by OP with x-axis.

Let P be any point on the circle and coordinates of P is

$P(10\cos\theta, 10\sin\theta)$

PQ and PS be the dimension of the rectangle.

Area of the rectangle $A = PQ \times PS$

$= 2x \times 2y$

$= 2(10\cos\theta) \times 2(10\sin\theta)$

$= 2 \times 10 \times 10\ (2\sin\theta\cos\theta)$

$= 200(\sin 2\theta)$

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$A' = 200(\cos 2\theta).2$

$= 400(\cos 2\theta)$

$A'' = 400(-\sin 2\theta).2$

$= -800 \sin 2\theta$

For maxima or minima,

$A' = 0 \Rightarrow 400 \sin 2\theta = 0$

$\Rightarrow \sin 2\theta = 0$

$\Rightarrow 2\theta = \pi/2$

$\therefore \theta = \pi/4$

when $\theta = \pi/4$, $A'' = -800 \sin \pi/2 < 0$

$\therefore$ Area is largest or maximum at $\theta = \pi/4$.

when $\theta = \pi/4$, $x = 10\cos\theta$

$= 10\cos \pi/4$

$= 10 \frac{1}{\sqrt{2}} = \frac{5 \times 2}{\sqrt{2}}$

$= \boxed{5\sqrt{2}}$

when $\theta = \pi/4$, $y = 10\sin\theta$

$= 10 \sin \pi/4$

$= 10 \times \frac{1}{\sqrt{2}} = \boxed{5\sqrt{2}}$

$\therefore$ length of the rectangle $2x$

$= 2 \times 5\sqrt{2}$

$= 10\sqrt{2}$ cm.

Breadth of the rectangle $2y$

$= 2 \times 5\sqrt{2}$

$= 10\sqrt{2}$ cm

8) Prove that among all the rectangles of the given perimeter, the square has the maximum area.

Let x and y be the dimensions of the rectangle with a given perimeter p.

$\therefore p = 2x + 2y$ ——— (1)

$\because 2y = p - 2x$

$y = \frac{p-2x}{2}$

Area $= xy$

$= x\left(\frac{p-2x}{2}\right)$

$= \frac{px}{2} - x^2$

$A' = \frac{p}{2} - 2x$

$A'' = -2$

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for maxima or minima,

$A' = 0$

i.e. $\frac{p}{2} - 2x = 0 \Rightarrow 2x = \frac{p}{2}$

$$\boxed{x = p/4}$$

when $x = p/4$, $A'' = -2 < 0$.

∴ Area is maximum at $x = p/4$

put $x = p/4$ in ①

$$y = \frac{p - 2(p/4)}{2}$$

$$= \frac{p - p/2}{2} = \frac{p/2}{2} = \boxed{p/4}$$

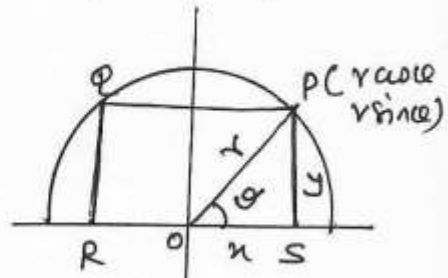
∴ $x = y = p/4$

Hence, for a given perimeter the area is maximum if the rectangle is a square.

---

9) Find the dimensions of the largest rectangle that can be inscribed in a semi circle of radius r cm.

Let PQRS be a rectangle inscribed in a semicircle with centre O and radius r.



Let θ be the angle made by OP with x-axis

Let P be any point on the semi circle and coordinates of P are

P(r cos θ, r sin θ)

PQ, PS be the dimensions of the rectangle.

Length of the rectangle PQ = 2x

= 2r cos θ

Breadth of the rectangle PS = y

= r sin θ.

Area of the rectangle A = PQ × PS

= 2r cos θ × r sin θ

= $2r^2 \sin\theta\cos\theta$

= $r^2 \sin 2\theta$

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$A' = r^2 \cos 2\theta \cdot 2$

$A'' = 2r^2(-\sin 2\theta) \cdot 2$

$= -4r^2 \sin 2\theta$

For maxima or minima $A' = 0$.

$\Rightarrow 2r^2 \cos 2\theta = 0$

$\cos 2\theta = 0$

$2\theta = \pi/2$

$\theta = \pi/4$

At $\theta = \pi/4$, $A''(\theta) = -4r^2 \sin 2\theta$

$= -4r^2 \sin \pi/2$

$= -4r^2 < 0$

∴ Area is largest or maximum

At $\theta = \pi/4$, length of the rectangle

$PQ = 2x$

$= 2r \cos\theta$

$= 2r \cos \pi/4$

$= 2r \frac{1}{\sqrt{2}} = \boxed{r\sqrt{2}}$

Breadth of the rectangle $PS = y$

$= r \sin\theta$

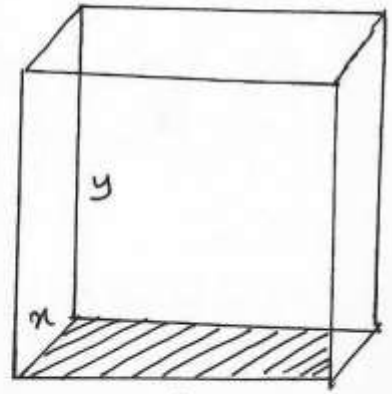
$= r \sin \pi/4$

$= \boxed{\frac{r}{\sqrt{2}}}$

∴ Length $= r\sqrt{2}$, Breadth $= \frac{r}{\sqrt{2}}$

10) A manufacturer wants to design an open box having a square base and a surface area of 108 sq cm. Determine the dimensions of the box for the maximum volume.

Let x, x, and y be the length, breadth and height of the box.



Area of the base $= x^2$

Area of 4 sides $= 4xy$

Given,

Surface area $= 108$

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i.e. $x^2 + 4xy = 108$

$$y = \frac{108 - x^2}{4x}$$

Volume of the box $V = x \times x \times y$

$$V = x^2 y = x^2\left[\frac{108 - x^2}{4x}\right]$$

$$V = \frac{x(108 - x^2)}{4}$$

$$V = \frac{108x - x^3}{4}$$

$$V' = \frac{1}{4}\left[108 - 3x^2\right]$$

$$V'' = \frac{1}{4}\left[-6x\right] = -\frac{3x}{2} < 0.$$

For maxima, $V' = 0$

$$\Rightarrow \frac{1}{4}\left[108 - 3x^2\right] = 0$$

$$108 - 3x^2 = 0$$

$$3x^2 = 108$$

$$x^2 = \frac{108}{3} = 36$$

$$\boxed{x = 6}$$

when $x = 6$, $V'' = -\frac{3x}{2} = -\frac{3}{2} \times 6$

$= -9 < 0$.

∴ Volume of the box is maximum when $x = 6$.

$$\therefore y = \frac{108 - x^2}{4x} = \frac{108 - 36}{4 \times 6}$$

$$= \frac{72}{24} = 3$$

∴ The dimensions are 6, 6, 3 cm.

---

11) The volume of a cylinder is given by the formula $V = \pi r^2 h$. Find the greatest and least values of V if $r + h = 6$.

Given that $r + h = 6$

$$h = 6 - r \quad \text{——} \ (1)$$

Volume of the cylinder $V = \pi r^2 h$

$$V = \pi r^2 (6 - r)$$

$$= \pi (6r^2 - r^3)$$

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$\therefore V' = \pi[12r - 3r^2]$

$V'' = \pi[12 - 6r]$

For maximum or minimum $V' = 0$

i.e. $\pi[12r - 3r^2] = 0$

$12r - 3r^2 = 0$

$3r(4 - r) = 0 \Rightarrow r = 0$ or $r = 4$

When $r = 0$, $V'' = \pi(12 - 0) = 12\pi > 0$.

$\therefore$ Volume is minimum when $r = 0$.

$\therefore$ least value of volume $V = \pi r^2 h$

$= \pi(0)h$

$= 0$.

When $r = 4$, $V'' = \pi[12 - 6(4)]$

$= \pi(12 - 24)$

$= -12\pi < 0$

$\therefore$ Volume is max when $r = 4$.

$\therefore$ greatest value of volume

$V = \pi r^2 h$

$= \pi(4)^2(6 - 4)$ $\quad \because r + h = 6$

$= \pi \times 16 \times 2$ $\quad h = 6 - r$

$= 32\pi$. $\quad = 6 - 4$

$= 2$

$\therefore$ least value $= 0$

greatest value $= 32\pi$

---

12) A hollow cone with base radius a cm and height b cm is placed on a table. Show that the volume of the largest cylinder that can be hidden underneath is $\frac{4}{9}$ times volume of cone.

Given: Radius of the cone = a cm

Height of the cone = b cm.

Let r be the radius of the cylinder. and y be the distance from the top of the cone and top of the cylinder..

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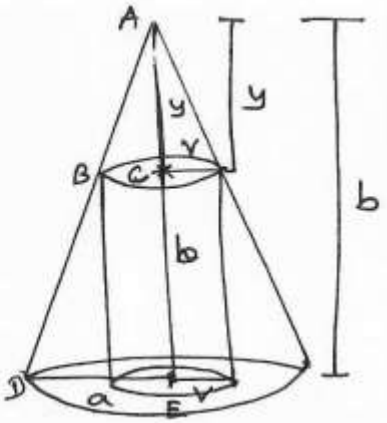

∴ Height of the cylinder = $b-y$

New Volume of the cylinder = $\pi r^2 h$

$= \pi r^2 (b-y)$

From the diagram

$\triangle ABC \sim \triangle ADE$

$\therefore \quad \frac{y}{r} = \frac{b}{a} \Rightarrow ar = by$

$\Rightarrow \quad y = \frac{br}{a}$

$\therefore$ Volume of cylinder $V = \pi r^2 \left(b - \frac{br}{a}\right)$

$= \pi b \left(r^2 - \frac{r^3}{a}\right)$

New $\quad V' = \pi b \left(2r - \frac{3r^2}{a}\right)$

$V'' = \pi b \left(2 - \frac{6r}{a}\right)$

For max / min, $V' = 0$

$\Rightarrow \quad \pi b \left(2r - \frac{3r^2}{a}\right) = 0$

$\Rightarrow \quad 2r - \frac{3r^2}{a} = 0$

$r\left(2 - \frac{3r}{a}\right) = 0$

$\Rightarrow \quad r = 0 \text{ or } \frac{3r}{a} = 2$

$r = \frac{2a}{3}$

$r = 0$ is not possible

$\therefore \quad r = \frac{2a}{3}$

When $r = \frac{2a}{3}$, $V'' = \pi b \left[2 - \frac{b}{a}\left(\frac{2a}{3}\right)\right]$

$= \pi b \left[2 - \frac{4a}{a}\right]$

$= \pi b \left[2 - 4\right]$

$= -2\pi b < 0$

when $r = \frac{2a}{3}$, $V'' = -2\pi b < 0$

$\therefore$ Volume of the cylinder is maximum when $r = \frac{2a}{3}$.

New Volume of the cylinder $= \pi r^2 h$

$= \pi \times \left(\frac{2a}{3}\right)^2 (b-y)$

$= \pi \times \frac{4a^2}{9}\left[b - \frac{b}{a}\left(\frac{2a}{3}\right)\right]$

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$$= \pi \times \frac{4}{9} a^2 \left[b - \frac{2b}{3} \right]$$

$$= \pi \times \frac{4}{9} a^2 \left[\frac{b}{3} \right]$$

$$= \frac{4}{9} \left[\frac{1}{3} \pi a^2 b \right]$$

$$= \frac{4}{9} \left[\text{V. of the cone} \right]$$

∴ Volume of the cylinder = $\frac{4}{9}$ times volume of cone.

I.62 We have a 12 square unit piece of thin material and want to make an open box by cutting small squares from the corners of our material and folding the side up. The question is which cut produces the box of maximum volume?

Let x be the length of the side which cut from the corner of the square box.

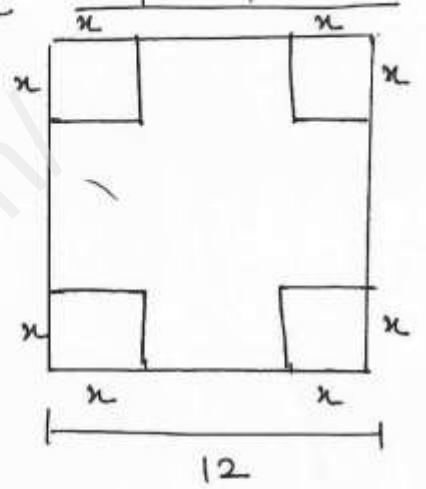
Length of the side of the given square box = 12

After folding,

Length of the box = $12 - 2x$

Breadth = $12 - 2x$

Height = x



∴ Volume of the folded box V = lbh

$$= (12-2x)(12-2x)x$$

$$= x(12-2x)^2$$

$$= x\left[144 + 4x^2 - 48x\right]$$

$$= 4\left[x^3 - 12x^2 + 36x\right]$$

Now $V' = 4\left[3x^2 - 24x + 36\right]$

$$= 12\left[x^2 - 8x + 12\right]$$

$$= 12\left[x-6\right]\left[x-2\right]$$

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$V'' = 12[2x-8]$

$= 24[x-4]$

For max / min, $V' = 0$.

i.e. $12(x-6)(x-2) = 0$

$\Rightarrow x = 6$ or $x = 2$.

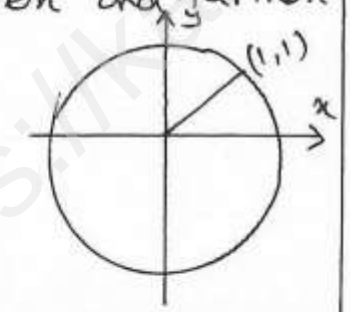
when $x = 2$, $V'' = 24(2-4)$

$= 24(-2) = -48 < 0$

$\therefore$ Volume of the box is maximum when $x = 2$.

$x = 6$ is not possible $\because 6 \notin (0,6)$

7.63 Find the points on the unit circle $x^2+y^2=1$ nearest and farthest from $(1,1)$.



Eqn of the circle is

$x^2+y^2 = 1$ ——— (1)

Diff w.r to x

$2x + 2y\frac{dy}{dx} = 0$

$x + y\frac{dy}{dx} = 0$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

The distance from the point $(1,1)$ to any point (x,y) is

$d = \sqrt{(x-1)^2+(y-1)^2}$

$d^2 = (x-1)^2+(y-1)^2$

$= x^2-2x+1+y^2-2y+1$

$= x^2+y^2-2x-2y+2$

$d^2 = D = 3-2x-2y \quad (\because x^2+y^2=1)$

Diff w.r to x

$D' = -2-2\frac{dy}{dx}$

$= -2-2\left(-\frac{x}{y}\right) \quad \left(\because \frac{dy}{dx} = -\frac{x}{y}\right)$

$= -2+\frac{2x}{y}$

$D'' = 2\left[\frac{y(1)-x\,dy/dx}{y^2}\right]$

$= 2\left[\frac{y-x\left(-\frac{x}{y}\right)}{y^2}\right]$

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$$= \frac{2y + \frac{2x^2}{y}}{y^2} = \frac{2y^2 + 2x^2}{y^3}$$

$$= \frac{2(x^2+y^2)}{y^3} \quad \text{——} \quad (2)$$

For max/min $D' = 0$

$$-2 + \frac{2x}{y} = 0$$

$$\frac{2x}{y} = 2 \Rightarrow \frac{x}{y} = 1$$

$$\Rightarrow x = y \quad \text{——} \quad (3)$$

Sub $x = y$ in (1)

$$x^2 + x^2 = 1 \text{ i.e. } 2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$x = \pm \frac{1}{\sqrt{2}}$$

when $x = \frac{1}{\sqrt{2}}$ in (3) $y = \frac{1}{\sqrt{2}}$

when $x = -\frac{1}{\sqrt{2}}$, Then $y = -\frac{1}{\sqrt{2}}$

when $x = \frac{1}{\sqrt{2}}$, $D'' = \frac{2(1)}{\left(\frac{1}{\sqrt{2}}\right)^3} > 0$

∴ The distance is minimum when $x = \frac{1}{\sqrt{2}}$

∴ The nearest pt of $(1,1)$ is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

when $x = -\frac{1}{\sqrt{2}}$, $D'' = \frac{2}{y^3}$

$$= \frac{2}{\left(-\frac{1}{\sqrt{2}}\right)^3} < 0$$

∴ The distance is max when $x = -\frac{1}{\sqrt{2}}$

∴ The farthest point of $(1,1)$ is $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Nearest distance of $(1,1)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is

$$= \sqrt{\left(1 - \frac{1}{\sqrt{2}}\right)^2 + \left(1 - \frac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{2\left(1 - \frac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{2}\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= \sqrt{2}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)$$

$$= \boxed{\sqrt{2}-1}$$

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Farthest distance betn $(1,1)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ is

$$d = \sqrt{\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1+\frac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{2\left(1+\frac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{2}\left(1+\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\frac{\sqrt{2}+1}{\sqrt{2}}\right)$$

$$= \boxed{\sqrt{2}+1}$$

7.64 A steel plant is capable of producing x tonnes per day of low grade steel and y tonnes per day of high grade steel, where $y = \frac{40-5x}{10-x}$. If the fixed market price of low grade steel is half that of high grade steel, then what should be optimal productions in low grade steel and high-grade steel in order to have maximum receipts.

Let p be the price of low grade steel per tonne.

∴ The price of high grade steel per tonnes $= 2p$.

Total receipt per day $(R) = px + 2py$

Given $y = \frac{40-5x}{10-x}$ —— (1)

$$\therefore R = px + 2p\left(\frac{40-5x}{10-x}\right)$$

$$\text{New } R' = p + 2p\left[\frac{(10-x)(-5) - (40-5x)(-1)}{(10-x)^2}\right]$$

$$= p + 2p\left[\frac{-50+5x+40-5x}{(10-x)^2}\right]$$

$$= p + 2p\left(\frac{-10}{(10-x)^2}\right)$$

$$R'' = \frac{2p(-10)(+2)}{(10-x)^3}$$

$$= \frac{-40p}{(10-x)^3}$$

$\frac{d}{dx}(10-x)^{-2}$

$= -2(10-x)^{-3}(-1)$

$= \frac{+2}{(10-x)^3}$

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For max / min

$R' = 0$ i.e. $p - \frac{20p}{(10-x)^2} = 0$

$\frac{20p}{(10-x)^2} = p \Rightarrow \frac{20}{(10-x)^2} = 1$

$(10-x)^2 = 20$

$10 - x = \pm 2\sqrt{5}$

$x = 10 \mp 2\sqrt{5}$

$\therefore x = 10 + 2\sqrt{5}$ and $x = 10 - 2\sqrt{5}$

when $x = 10 - 2\sqrt{5}$, $R'' = \frac{-40p}{(10 - 10 + 2\sqrt{5})^3}$

$= \frac{-40p}{(2\sqrt{5})^3} < 0$

when $x = 10 - 2\sqrt{5}$,

then ① $\Rightarrow y = \frac{40 - 5(10 - 2\sqrt{5})}{10 - (10 - 2\sqrt{5})}$

$= \frac{40 - 50 + 10\sqrt{5}}{10 - 10 + 2\sqrt{5}}$

$= \frac{-10 + 10\sqrt{5}}{2\sqrt{5}}$

$= \frac{-10}{2\sqrt{5}} + \frac{10\sqrt{5}}{2\sqrt{5}}$

$= -\sqrt{5} + 5$

$\boxed{y = 5 - \sqrt{5}}$

∴ The steel plant must produce low grade and high grade steels respectively in tonnes per day are $10 - 2\sqrt{5}$ and $5 - \sqrt{5}$

7.65 prove that among all the rectangles of the given area square has the least perimeter.

Let x and y be the dimensions of the rectangle with given area A.

∴ $A = xy$ ——— ①

we with to minimize the perimeter p

∴ $p = 2x + 2y$

$= 2x + 2\left(\frac{A}{x}\right)$

$p' = 2 + 2A\left(-\frac{1}{x^2}\right)$

$p'' = -2A\left(-\frac{2}{x^3}\right) = \frac{4A}{x^3}$

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For maximum

$$p' = 0 \Rightarrow \quad 2 - \frac{2A}{x^2} = 0$$

$$\frac{2A}{x^2} = 2$$

$$\frac{A}{x^2} = 1 \Rightarrow A = x^2$$

$$\therefore x = \pm\sqrt{A}$$

$x = -\sqrt{A}$ is not possible.

$$\therefore x = \sqrt{A}.$$

when $x = \sqrt{A}$, $p'' = \frac{4\sqrt{A}}{x^3} > 0$

$\therefore$ perimeter is minimum when $x = +\sqrt{A}$

when $x = \sqrt{A}$ ① $\Rightarrow$ $A = \sqrt{A}\,y$

$$y = \frac{A}{\sqrt{A}} = \sqrt{A}.$$

$\therefore$ $x = y = \sqrt{A}$.

$\therefore$ minimum perimeter rectangle of a given area is a square.

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Symmetry and Asymptotes:-

Asymptotes:-

An asymptotes for the curve $y = f(x)$ is a straight line which is tangent at ∞ to the curve.

Horizontal asymptotes:-

Which is parallel to x-axis. The line $y = L$ is said to be a horizontal asymptotes for the curve $y = f(x)$ if either $\lim_{x \to +\infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$.

Vertical asymptotes:-

Which is parallel to y-axis. The line $x = a$ is said to be vertical asymptotes for the curve $y = f(x)$ if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

Slant asymptotes:-

A slant (oblique) asymptotes occurs when the polynomial in the numerator is a higher degree than the polynomial in the denominator.

EXERCISE 7.9

1. Find the asymptotes of the following curves.

(i) $f(x) = \dfrac{x^2}{x^2 - 1}$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2}{x^2\left(1 - \frac{1}{x^2}\right)}$$

$$= \lim_{x \to \infty} \frac{1}{1 - \frac{1}{x^2}} = \frac{1}{1 - \frac{1}{\infty}}$$

$$= \frac{1}{1 - 0} = \boxed{1}$$

$\therefore y = 1$ is the horizontal asymptotes of the given curve.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} \frac{x^2}{x^2 - 1}$$

$$= \frac{1}{1 - 1} = \frac{1}{0} = \infty.$$

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$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2}{x^2-1} = \frac{1}{1-1} = \frac{1}{0} = \infty$

$\therefore x = 1$ and $x = -1$ are vertical asymptotes.

(ii) $f(x) = \frac{x^2}{x+1}$

a) Horizontal asymptote:

$\lim_{x\to\infty} \frac{x^2}{x+1} = \lim_{x\to\infty} \frac{x^2}{x(1+\frac{1}{x})}$

$= \lim_{x\to\infty} \frac{x}{1+\frac{1}{x}}$

$= \frac{\infty}{1+\frac{1}{\infty}} = \frac{\infty}{1+0} = \infty$

Hence the curve has no horizontal asymptotes.

b) Vertical asymptote:-

$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{x^2}{x+1}$

$= \frac{1}{-1+1} = \frac{1}{0} = \infty$

$\therefore x = -1$ is the vertical asymptote.

c) Slant asymptote:-

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2 \\ & x^2+x \\ \hline & -x \\ & -x-1 \\ \hline & +1 \end{array}$$

$\therefore \frac{x^2}{x+1} = x-1 + \frac{1}{x+1}$

Hence $y = x-1$ is the slant asymptote of the given curve.

(iii) $f(x) = \frac{3x}{\sqrt{x^2+2}}$

a) Horizontal asymptote:-

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x}{\sqrt{x^2+2}}$

$= \lim_{x\to\infty} \frac{3x}{\sqrt{x^2(1+\frac{2}{x^2})}}$

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$$= \lim_{x\to\infty} \frac{3x}{\pm x\sqrt{1+\frac{2}{x^2}}} = \lim_{x\to\infty} \frac{\pm 3}{\sqrt{1+\frac{2}{x^2}}}$$

$$= \frac{\pm 3}{\sqrt{1+\frac{2}{\infty}}} = \frac{\pm 3}{\sqrt{1+0}} = \pm 3$$

$\therefore$ $y=3$, $y=-3$ are horizontal asymptotes.

b) vertical asymptote:-

It is impossible to find an $a \in R$ other than ∞.

$$\therefore \lim_{x\to \pm a} \frac{3x}{\sqrt{x^2+2}} = \pm\infty$$

$\therefore$ The curve has no vertical asymptote.

(IV) $f(x) = \dfrac{x^2-6x-1}{x+3}$

$$\lim_{x\to\infty} \frac{x^2-6x-1}{x+3}$$

$$= \lim_{x\to\infty} \frac{x^2\left(1-\frac{6x}{x^2}-\frac{1}{x^2}\right)}{x\left(1+3/x\right)}$$

$$= \lim_{x\to\infty} \frac{x\left(1-\frac{6}{x}-\frac{1}{x^2}\right)}{1+3/x}$$

$$= \frac{\infty\left(1-6/\infty-\frac{1}{\infty}\right)}{1+3/\infty} = \infty$$

$$\lim_{x\to\infty^-} \frac{-\infty\left(1-6/\infty-1/\infty\right)}{1+3/\infty} = -\infty$$

$\therefore$ The curve has no horizontal asymptote.

vertical asymptote:-

$$\lim_{x\to -3} \frac{x^2-6x-1}{x+3} = \frac{(-3)^2-6(-3)-1}{-3+3}$$

$$= \frac{9+18-1}{0} = \infty$$

$x=-3$ is the vertical asymptotes.

c) slant asymptote:.

$$\begin{array}{r|l} x+3 & x^2-6x-1 \;(x-9 \\ & x^2+3x \\ \hline & -9x-1 \\ & -9x-27 \\ \hline & 26 \end{array}$$

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$$\frac{x^2-6x-1}{x+3} = x-9+\frac{26}{x+3}$$

$\therefore$ $y = x-9$ is the slant asymptote of the curve.

(v) $f(x) = \dfrac{x^2+6x-4}{3x-6}$

a) Horizontal asymptote:-

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2+6x-4}{3x-6}$$

$$= \lim_{x\to\infty} \frac{x^2\left(1+\frac{6x}{x^2}-\frac{4}{x^2}\right)}{x\left(3-\frac{6}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{x\left(1+\frac{6}{x}-\frac{4}{x^2}\right)}{\left(3-\frac{6}{x}\right)}$$

$$= \frac{\infty\left(1+6/\infty-4/\infty^2\right)}{\left(3-6/\infty\right)} = \frac{\infty(1+0-0)}{3-0}$$

$$= \infty$$

$\therefore$ The curve has no horizontal asymptote.

Vertical asymptote:-

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2+6x-4}{3x-6}$$

$$= \frac{4+12-4}{6-6} = \frac{12}{0} = \infty.$$

$\therefore$ $x = 2$ is the vertical asymptote of the curve.

Slant asymptote:-

$$3x-6\ \big)\ x^2+6x-4\ \big(\ \frac{x}{3}+\frac{8}{3}$$

$$x^2-2x$$

$$8x-4$$

$$8x-16$$

$$12$$

$$\therefore \frac{x^2+6x-4}{3x-6} = \left(\frac{x}{3}+\frac{8}{3}\right)+\frac{12}{3x-6}$$

$\therefore$ $y = \frac{x}{3}+\frac{8}{3}$ is the slant asymptote.

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Eg 7.66 Find the asymptotes of the curve $f(x) = \frac{1}{x}$

a) **Horizontal asymptote:-**

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{1}{x} = \frac{1}{\infty} = 0$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1}{x} = \frac{1}{\infty} = 0$$

$\therefore y = 0$ is the horizontal asymptote

b) **Vertical asymptote:-**

$$\lim_{x \to 0} = \frac{1}{0} = \infty$$

$\therefore x = 0$ is the vertical asymptote.

Eg 7.67 Find the slant asymptote for the function $f(x) = \frac{x^2 - 6x + 7}{x + 5}$

$$\begin{array}{r} x - 11 \\ x+5 \overline{)\ x^2 - 6x + 7} \\ x^2 + 5x \\ \hline -11x + 7 \\ -11x - 55 \\ \hline 62 \end{array}$$

$$\frac{x^2 - 6x + 7}{x+5} = (x - 11) + \frac{62}{x+5}$$

$\therefore y = x - 11$ is the slant asymptote.

Eg 7.68 Find the asymptotes of the curve $f(x) = \frac{2x^2 - 8}{x^2 - 16}$

a) **Horizontal asymptote:-**

$$\lim_{x \to \infty -} f(x) = \lim_{x \to -\infty} \frac{2x^2 - 8}{x^2 - 16}$$

$$= \lim_{x \to -\infty} \frac{x^2\left(2 - \frac{8}{x^2}\right)}{x^2\left(1 - \frac{16}{x^2}\right)}$$

$$= \lim_{x \to -\infty} \frac{2 - 8/x^2}{1 - 16/x^2} = \frac{2 - \frac{8}{\infty}}{1 - \frac{16}{\infty}}$$

$$= \frac{2-0}{1-0}$$

$$= \boxed{2}$$

$\therefore y = 2$ is the horizontal asymptote.

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b) vertical asymptote:

$$\lim_{x \to 4} \frac{2x^2-8}{x^2-16} = \frac{2(16)-8}{16-16} = \frac{24}{0} = \infty$$

$$\lim_{x \to -4} \frac{2x^2-8}{x^2-16} = \frac{2(-4)^2-8}{(-4)^2-16} = \frac{24}{0} = \infty$$

∴ $x = 4$, $x = -4$ are the vertical asymptotes

SKETCHING of CURVES:-

Eg 7.69 Sketch the curve $y = f(x) = x^2 - x - 6$.

$$y = f(x) = x^2 - x - 6$$
$$= (x-3)(x+2)$$

1) Domain and Range

Domain : $(-\infty, \infty)$
Range : $(-\infty, \infty)$

2) Intercepts:-

To find x intercept put $y = 0$.

$\Rightarrow (x-3)(x+2) = 0$

$\Rightarrow x = 3, -2$

∴ x intercepts are $(-2, 0)$ and $(3, 0)$

To find y intercept put $x = 0$

$\Rightarrow y = -6$

∴ y intercept is $(0, -6)$

3) Critical numbers:-

$f(x) = x^2 - x - 6$

$f'(x) = 2x - 1$

$f'(x) = 0 \Rightarrow 2x - 1 = 0$

$\Rightarrow 2x = 1$

$\Rightarrow x = 1/2$

∴ The critical no is $x = \frac{1}{2}$

4) Local extrema:-

$f''(x) = 2 > 0$

∴ at $x = \frac{1}{2}$, the curve has local minimum.

∴ Local minimum value is $f(\frac{1}{2})$

$$f(\tfrac{1}{2}) = \frac{1}{4} - \frac{1}{2} - 6$$
$$= \frac{1-2-24}{4} = \frac{-25}{4}$$

5) Range of the function

The range of the function is $y \geq -\frac{25}{4}$.

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6) Concavity:-

$f''(x) = 2 > 0$

$\therefore$ The fun is concave upward for all $x \in R$.

7) point of inflection:-

$f''(x) = 2 \neq 0 \quad \forall x \in R$.

$\therefore$ The curve has no point of inflection.

8) Asymptotes:-

a) Horizontal asymptote:

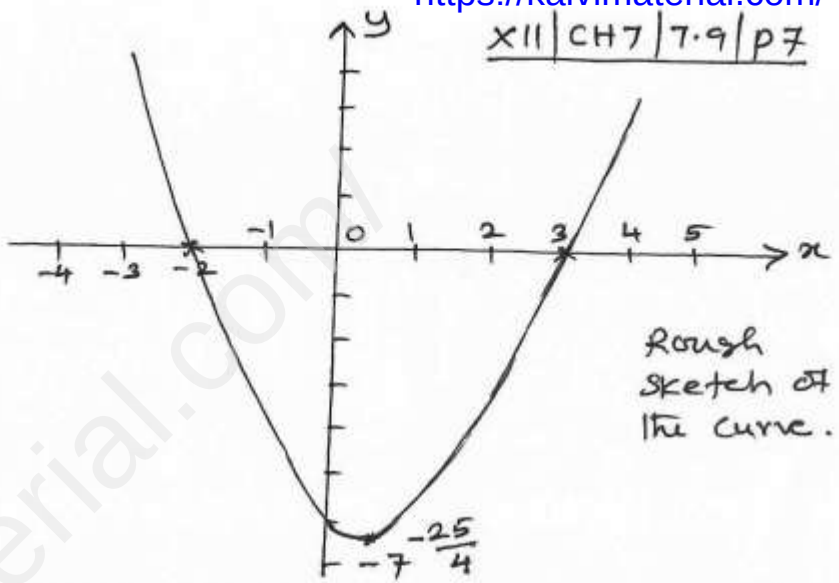
$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} x^2 - x - 6 = \infty - \infty - 6 = \infty.$$

$\therefore$ $f(x)$ has no horizontal asymptote.

b) vertical asymptote:

$$\lim_{x \to a} (x^2 - x - 6) = \pm\infty$$

$f(x)$ has ~~not~~ no vertical asymptote.



Rough sketch of the curve.

Eg 7.70 Sketch the curve

$y = f(x) = x^3 - 6x - 9$

Given $f(x) = x^3 - 6x - 9$

$f(1) = 1 - 6 - 9 = -14 \neq 0$

$f(-1) = -1 + 6 - 9 = -4 \neq 0$

$f(2) = 8 - 12 - 9 = -13 \neq 0$

$f(-2) = (-2)^3 + 12 - 9$

$= -8 + 12 - 9 = 12 - 17 \neq 0$

$f(3) = 27 - 18 - 9 = 27 - 27 = 0$

$\therefore$ $x = 3$ is a root of $f(x)$.

Hence $x - 3$ is a factor of $f(x)$.

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$$3 \;\Big|\; \begin{array}{cccc} 1 & 0 & -6 & -9 \\ 0 & 3 & 9 & 9 \\ \hline 1 & 3 & 3 & 0 \end{array}$$

$\therefore y = (x-3)(x^2+3x+3)$

1) **Domain and Range:-**

Domain : $(-\infty, \infty)$

Range : $(-\infty, \infty)$

2) **Intercepts:-**

x intercept, put $y = 0$

$\Rightarrow x-3=0$ or $x^2+3x+3=0$

$x = 3$ or Imaginary roots

$\therefore$ x intercept is $(3, 0)$

y intercept, put $x = 0$,

$\Rightarrow y = -9$

$\therefore$ y intercept is $(0, -9)$

3) **Critical no:-**

$f(x) = x^3 - 6x - 9$

$f'(x) = 3x^2 - 6$

$f'(x) = 0 \Rightarrow 3x^2 - 6 = 0$

$\Rightarrow 3x^2 = 6$

$\Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

$\therefore$ The critical nos are $x = \pm\sqrt{2}$

4) **Local extrema:-**

$f''(x) = 6x$

when $x = \sqrt{2}$, $f''(\sqrt{2}) = 6\sqrt{2} > 0$

$\therefore f(x)$ has local minimum at $x = \sqrt{2}$

$\therefore$ local minimum value is $f(\sqrt{2})$

$f(\sqrt{2}) = (\sqrt{2})^3 - 6\sqrt{2} - 9$

$= 2\sqrt{2} - 6\sqrt{2} - 9$

$= -4\sqrt{2} - 9$

when $x = -\sqrt{2}$, $f''(-\sqrt{2}) = 6(-\sqrt{2}) < 0$

$\therefore f(x)$ has local maximum at $x = -\sqrt{2}$

$\therefore$ local maximum value is

$f(-\sqrt{2}) = (-\sqrt{2})^3 - 6(-\sqrt{2}) - 9$

$= -2\sqrt{2} + 6\sqrt{2} - 9$

$= 4\sqrt{2} - 9$

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5) **Concavity:-**

$f''(x) = 6x$

For $x > 0$, the function is concave upward.

For $x < 0$, the function is concave downward.

6) **point of inflection:-**

$f''(x) = 0 \Rightarrow 6x = 0 \Rightarrow x = 0$.

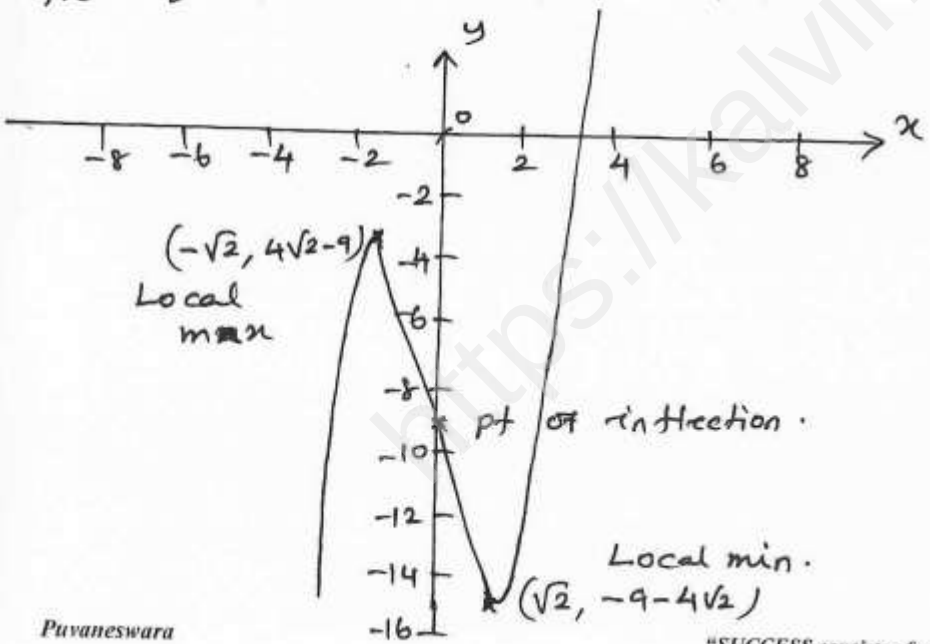
$f''(x)$ changes sign through $x = 0$.

∴ point of inflection is $(0, f(0))$

i.e. $(0, -9)$

7) **Asymptotes:-**

The given curve has no asymptotes.



Eg 7.71

Sketch the curve $y = \frac{x^2 - 3x}{x-1}$

Given $y = f(x) = \frac{x^2 - 3x}{x-1}$

$= \frac{x(x-3)}{x-1}$

1) **Domain:**

When $x = 1$, $f(x)$ is not defined

Hence the domain is $R - \{1\}$

Range : $(-\infty, \infty)$

2) **Intercepts:**

x intercept, $y = 0$,

$\Rightarrow x^2 - 3x = 0$

$\Rightarrow x(x-3) = 0$

$\Rightarrow x = 0$ or $x = 3$

∴ x intercepts are $(0,0)$ and $(3,0)$

y intercept, $x = 0$

$y = \frac{0(0-3)}{0-1} = \frac{0}{-1} = 0$

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∴ y intercept is (0,0)

3) Critical no:-

$$f(x) = \frac{x^2-3x}{x-1}$$

$$f'(x) = \frac{(x-1)(2x-3) - (x^2-3x)(1-0)}{(x-1)^2}$$

$$= \frac{2x^2-2x-3x+3-x^2+3x}{(x-1)^2}$$

$$= \frac{x^2-2x+3}{(x-1)^2}$$

$$f'(x)=0 \Rightarrow \frac{x^2-2x+3}{(x-1)^2}=0 \Rightarrow x^2-2x+3=0$$

$$x = \frac{4 \pm \sqrt{4-4(1)(3)}}{2}$$

$$= \frac{4 \pm \sqrt{-8}}{2}$$

∴ $x^2-2x+3=0$ has no real root.

when $x=1$, $f'(1) = \frac{1-2+3}{0} = \infty$

∴ $x=1$ is the only critical number.

4) Local extrema:-

When $x=1$ is not in the domain of the function and $f'(x) \neq 0$ $\forall x \in R-\{1\}$.

∴ There is no local maximum or local minimum.

5) Concavity:-

$$f'(x) = \frac{x^2-2x+3}{(x-1)^2}$$

$$f''(x) = \frac{(x-1)^2(2x-2) - (x^2-2x+3)2(x-1)}{(x-1)^4}$$

$$= \frac{(x^2-2x+1)(2x-2) - 2(x^2-2x+3)(x-1)}{(x-1)^4}$$

$$= \frac{2(x-1)\{x^2-2x+1-x^2+2x-3\}}{(x-1)^4}$$

$$= \frac{2(x-1)(-2)}{(x-1)^4} = \frac{-4}{(x-1)^3} \quad \forall x \in R-\{1\}$$

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when $x < 1$, $f''(x) > 0$

$\therefore$ The curve is concave upward in $(-\infty, 1)$

when $x > 1$, $f''(x) < 0$.

$\therefore$ The curve is concave downward in $(1, \infty)$.

6) pt of inflection:-

$f''(x) \neq 0 \quad \forall x \in R - \{1\}$

$\therefore$ There is no pt of inflection.

7) Asymptotes:-

a) Horizontal asymptote:

$$\lim_{x \to \infty} \frac{x^2 - 3x}{x - 1} = \lim_{x \to \infty} \frac{x^2\left(1 - \frac{3}{x}\right)}{x\left(1 - \frac{1}{x}\right)}$$

$$= \lim_{x \to \infty} \frac{x\left(1 - \frac{3}{x}\right)}{1 - \frac{1}{x}}$$

$$= \frac{\infty(1 - 0)}{1 - 0} = \infty.$$

$\therefore f(x)$ has no horizontal asymptote.

b) Vertical asymptote:-

$$\lim_{x \to 1} \frac{x^2 - 3x}{x - 1} = \frac{1 - 3}{1 - 1} = \frac{-2}{0} = \infty$$

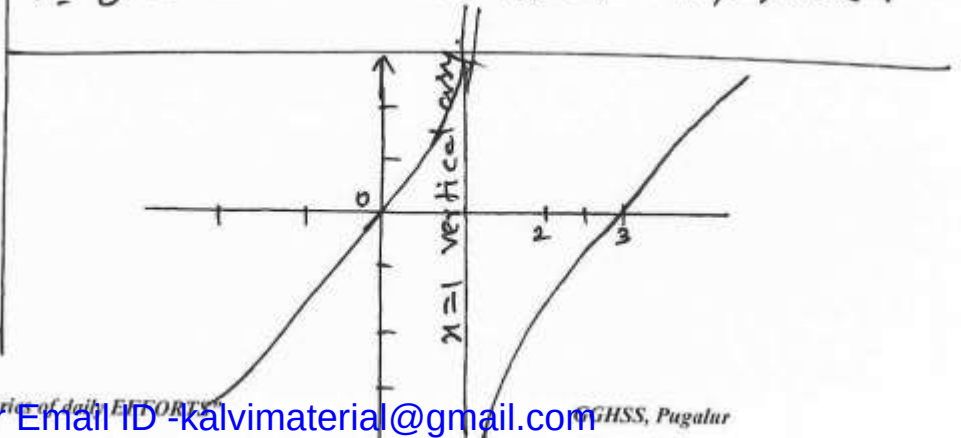
$\therefore x = 1$, is the vertical asymptote.

c) Slant asymptote:-

$$\begin{array}{r|l} x-1 & x^2 - 3x \;(\; x - 2 \\ & x^2 - x \\ \hline & -2x + 0 \\ & -2x + 2 \\ \hline & -2 \end{array}$$

$$\therefore \frac{x^2 - 3x}{x - 1} = (x - 2) - \frac{2}{x - 1}$$

$\therefore y = x - 2$ is the slant asymptote.



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Eg 7.72 Sketch the graph of the function $y = \frac{3x}{x^2-1}$

$y = f(x) = \frac{3x}{x^2-1}$

1) Domain : $f(x)$ is not defined for $x = 1, -1$

∴ The domain of $f(x)$ is $R - \{-1, 1\}$

2) Symmetry: $f(x, y) = y = \frac{3x}{x^2-1}$

$f(-x, -y) = -y = \frac{3(-x)}{(-x)^2-1}$

$= -y = \frac{-3x}{x^2-1}$

$y = \frac{3x}{x^2-1} = f(x, y)$

∴ The curve is symmetry about the origin.

3) Intercepts:

x intercept, $y = 0$,

$\Rightarrow \frac{3x}{x^2-1} = 0 \Rightarrow 3x = 0 \Rightarrow x = 0$.

∴ x intercept is $(0, 0)$

y-intercept, put $x = 0$

$\Rightarrow y = \frac{3(0)}{0-1} = \frac{0}{-1} = 0$.

∴ y intercept is $(0, 0)$

4) critical numbers:-

$f(x) = \frac{3x}{x^2-1}$

$f'(x) = \frac{(x^2-1)3 - 3x(2x)}{(x^2-1)^2}$

$= \frac{3x^2-3-6x^2}{(x^2-1)^2} = \frac{-3x^2-3}{(x^2-1)^2}$

$f'(x) = 0 \Rightarrow -3x^2 - 3 = 0$

$\Rightarrow -3x^2 = 3$

$x^2 = -1$

$x = i$

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Hence $f'(x)$ does not exists at $x = -1, 1$

$\therefore$ critical nos are $x = -1, 1$

$\therefore$ Intervals are $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$.

a) Sign of $f'(x)$ in $(-\infty, -1)$

Say $x = -2$, $f'(-2) = \dfrac{-3(4)-3}{(4-1)^2}$

$: \dfrac{-15}{9} < 0$

Hence $f'(x) < 0 \quad \forall \; x \in (-\infty, -1)$

b) $f'(x)$ in $(-1, 1)$ say $x = 0$ in $f'(x)$

$f'(0) = \dfrac{-3(0)-3}{(0-1)^2} = \dfrac{-3}{+1} = -3 < 0.$

$\therefore f'(x) < 0$ in $(-1, 1)$

c) $f'(x)$ in $(+1, \infty)$ say $x = 2$ in $f'(x)$

$f'(+2) = \dfrac{-3(4)-3}{(4-1)^2} = \dfrac{-12}{9} < 0$

$f'(x) < 0 \quad \forall \; x \in (+1, \infty)$

Intervals	Sign of $f'(x)$	function
$(-\infty, -1)$	< 0	strictly decreasing
$(-1, 1)$	< 0	strictly decreasing
$(1, \infty)$	< 0	strictly decreasing

5) Local extrema:-

no sign change for $f'(x)$ when passing through the critical nos.

$\therefore$ there is no local extrema.

6) concavity:-

$f'(x) = \dfrac{-3(x^2+1)}{(x^2-1)^2}$

$f''(x) = \dfrac{(x^2-1)^2(-3(2x)) + 3(x^2+1)\,2(x^2-1)\cdot 2x}{(x^2-1)^4}$

$= \dfrac{(x^2-1)^2(-6x) + 12x(x^2+1)(x^2-1)}{(x^2-1)^4}$

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$$= \frac{6x(x^2-1)\{-(x^2-1)+2(x^2+1)\}}{(x^2-1)^4}$$

$$= \frac{6x\{-x^2+1+2x^2+2\}}{(x^2-1)^3}$$

$$= \frac{6x(x^2+3)}{(x^2-1)^3}$$

$f''(x)=0 \Rightarrow 6x(x^2+3)=0$

$\Rightarrow x=0$ or $x^2+3=0$

$\Rightarrow x=0$ or $x^2=-3$

$\Rightarrow x=0$, which is imaginary

when $x=-1,1$, $f''(x)$ does not exist

∴ The intervals are $(-\infty,-1)(-1,0)(0,1)(1,\infty)$

In $(-\infty,-1)$ say $x=-2$

$$f''(-2) = \frac{6(-2)(4+3)}{(4-1)^3} = -\frac{84}{27} < 0$$

∴ f is concave down in $(-\infty,-1)$

In $(-1,0)$ say $x=-1/2$

$$f''(-\tfrac{1}{2}) = \frac{6(-\frac{1}{2})(\frac{1}{4}+3)}{(\frac{1}{4}-1)^3}$$

$$= \frac{-3(\frac{13}{4})}{(-\frac{3}{4})^3} = \frac{-\frac{39}{4}}{-\frac{27}{64}} > 0$$

∴ f is concave upward in $(-1,0)$

In $(0,1)$, say $x=1/2$

$$f''(\tfrac{1}{2}) = \frac{6(\frac{1}{2})(\frac{1}{4}+3)}{(\frac{1}{4}-1)^3}$$

$$= \frac{3(\frac{13}{4})}{(-\frac{3}{4})^3} = \frac{\frac{39}{4}}{-\frac{27}{64}} < 0$$

∴ f is concave down in $(0,1)$

In $(1,\infty)$, say $x=2$,

$$f''(2) = \frac{12(4+3)}{27} > 0$$

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$\therefore$ f is concave up in $(1, \infty)$

Interval	Sign of $f''(x)$	function (concavity)
$(-\infty, -1)$	−	concave down
$(-1, 0)$	+	concave up
$(0, 1)$	−	concave down
$(1, \infty)$	+	concave up

7) pt of inflection:-

$x = -1, 1$ are not in the domain of $f(x)$.

$\therefore x = -1, x = 1$ cannot be the pt of inflection.

$$f''(0) = \frac{6(0)(0+3)}{(0-1)^3} = 0$$

$f''(x)$ changes sign when passing through $x = 0$.

Hence the pt of inflection is $(0, f(0))$

$$f(x) = \frac{3x}{x^2-1}$$

$$f(0) = \frac{0}{0-1} = 0$$

$\therefore$ The pt of inflection is $(0,0)$

8) Asymptotes:-

a) Horizontal asymptote:-

$$f(x) = \frac{3x}{x^2-1}$$

$$\lim_{x\to\infty} f(x) = \frac{x(3)}{x^2\left(1-\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{3}{x\left(1-\frac{1}{x^2}\right)}$$

$$= \frac{3}{\infty(1-\frac{1}{\infty})} = \frac{3}{\infty(1-0)} = 0.$$

$$\left(\frac{3}{\frac{1}{0}(1)} = 0\right)$$

$\therefore y = 0$ is the horizontal asymptote.

b) vertical asymtote:

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{3(-1)}{(-1}} = \frac{-3}{0} = -\infty$$

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1} \frac{3(1)}{1-1} = \frac{3}{0} = \infty$$

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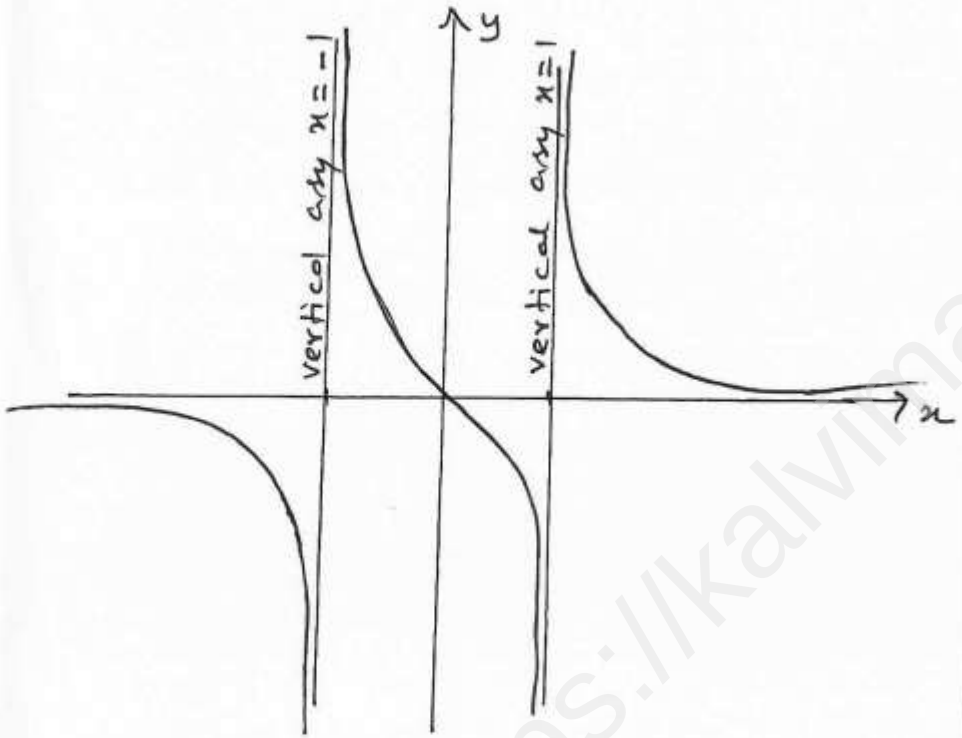
Hence $x = 1$ is the vertical asymtote

c) slant asymptote

There is no slant asymtote

(∵ degree of NR is < degree of DR)



EXERCISE 7.9

2) Sketch the graph

(i) $y = -\frac{1}{3}(x^3 - 3x + 2)$

$y = f(x) = -\frac{1}{3}(x^3 - 3x + 2)$

factorizing $x^3 - 3x + 2$

Let $g(x) = x^3 - 3x + 2$

$g(1) = 1 - 3 + 2 = 0$

$\therefore x - 1$ is a factor of $g(x)$

1	1	0	-3	2
		1	1	-2
	1	1	-2	0
1		1	2	
	1	2	0	

$\therefore (x^3 - 3x + 2)(x+2)(x-1)(x-1)$

when $y = 0$, $x = -2, 1, 1$.

(1) Domain and Range

Domain $= (-\infty, \infty)$

Range $= (-\infty, \infty)$

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2) x intercept, put $y=0$.

$(x+2)(x+1)(x+1)=0$

$\Rightarrow x=-2,+1,1$

$\therefore$ x intercepts are $(-2,0),(1,0),(1,0)$

y-intercept, put $x=0$,

$y = -\frac{1}{3}(0-0+2) = -\frac{2}{3}$

$\therefore$ y intercept is $(0,-\frac{2}{3})$

3) critical point:-

$f(x) = -\frac{1}{3}(x^3-3x+2)$

$f'(x) = -\frac{1}{3}(3x^2-3)$

$= -\frac{1}{3}\times 3(x^2-1)$

$= -(x^2-1)$

$f'(x)=0 \Rightarrow x^2-1=0 \Rightarrow (x+1)(x-1)=0$

$\Rightarrow x=-1,1$

$\therefore$ critical pts are $x=-1,1$

4) Local extrema:-

$f''(x) = -(2x) = -2x$

When $x=1$, $f''(1) = -2<0$

$\therefore f(x)$ is local maximum.

$\therefore$ local maximum value is $f(1)$

$f(1) = -\frac{1}{3}(1-3+2) = \boxed{0}$.

when $x=-1$, $f''(-1) = -2(-1) = \boxed{2} > 0$.

$\therefore f(x)$ is local minimum.

$\therefore$ local minimum value is $f(-1)$

$f(-1) = -\frac{1}{3}(-1+3+2)$

$= -\frac{1}{3}(4) = \boxed{-\frac{4}{3}}$

5) concavity:-

$f''(x) = -2x$

when $x<0$, $f''(x)$ is $+^{ve}$

$\therefore$ f is concave upward.

when $x>0$, $f''(x)$ is $-^{ve}$

$\therefore$ f is concave downward.

6) pt of inflection:-

$f''(x)=0 \Rightarrow -2x=0$

$\Rightarrow x=0$.

$f''(x)$ changes sign through $x=0$

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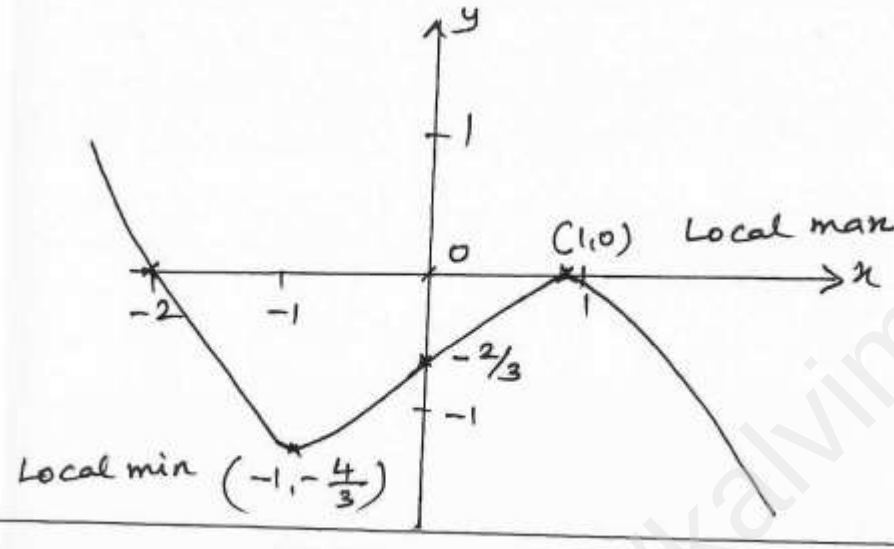
∴ The point of inflection is $(0, \text{from})$

$= (0, -\frac{2}{3})$

7) asymptotes:-

The curve has no asymptotes.



(11) $y = x\sqrt{4-x}$

$4 - x \geq 0 \Rightarrow x \leq 4$

∴ (1) Domain is $(-\infty, 4]$

2) Intercepts:-

x intercept, put $y = 0$

$x\sqrt{4-x} = 0 \Rightarrow x = 0$ or $4 - x = 0$

$x = 0$ or $x = 4$

∴ x intercept is $(0,0)$, and $(4,0)$

y-intercepts, put $x = 0$

$y = 0\sqrt{4-0} = 0\sqrt{2} = 0$

∴ y intercept as $(0,0)$

3) Critical no:-

$f(x) = x\sqrt{4-x}$

$f'(x) = x \frac{1}{2\sqrt{4-x}}(-1) + \sqrt{4-x} \cdot 1$

$= -\frac{x}{2\sqrt{4-x}} + \sqrt{4-x}$

$= \frac{-x + 2(4-x)}{2\sqrt{4-x}}$

$= \frac{-x + 8 - 2x}{2\sqrt{4-x}} = \frac{8-3x}{2\sqrt{4-x}}$

$f'(x) = 0 \Rightarrow 8 - 3x = 0 \Rightarrow x = 8/3$

∴ The critical no is $x = 8/3$

4) Monotonicity:

The intervals are $(-\infty, \frac{8}{3})$, $(\frac{8}{3}, 4)$

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In $(-\infty, 8/3)$, say $x = 0$.

$$f'(0) = \frac{8-3(0)}{2\sqrt{4-0}} = \frac{8}{2\times 2} = 8/4 = 2 > 0$$

Hence $f(x)$ is increasing in $(-\infty, 8/3)$

In $(8/3, 4)$ say $x = 3$,

$$f'(3) = \frac{8-3(3)}{2\sqrt{4-3}} = \frac{8-9}{2\sqrt{1}} = -\frac{1}{2} < 0.$$

$\therefore$ $f(x)$ is decreasing in $(8/3, 4)$

Intervals	sign of $f'(x)$	Monotonicity
$(-\infty, 8/3)$	+ve	increasing
$(8/3, 4)$	-ve	decreasing.

5) Local extrema:-

$$f'(x) = \frac{8-3x}{2\sqrt{4-x}}$$

$$f'(8/3) = \frac{8-3(8/3)}{2\sqrt{4-8/3}} = 0$$

$f'(x)$ changes sign from +ve to negative when it passes through $x = 8/3$.

$\therefore$ $f(x)$ has a local maximum at $x = 8/3$

$\therefore$ Local maximum value is

$$f(8/3) = \frac{8}{3}\sqrt{4-8/3}$$

$$= \frac{8}{3}\sqrt{\frac{4}{3}} = \frac{8}{3}\times\frac{2}{\sqrt{3}}$$

$$= \frac{16}{3\sqrt{3}}$$

6) Concavity:-

$$f'(x) = \frac{8-3x}{2\sqrt{4-x}}$$

$$f''(x) = \frac{1}{2}\,\frac{\sqrt{4-x}(-3) - (8-3x)\frac{1}{2\sqrt{4-x}}(-1)}{(\sqrt{4-x})^2}$$

$$= \frac{\dfrac{-6(4-x)+(8-3x)}{2\sqrt{4-x}}}{\dfrac{1}{2}(\sqrt{4-x})^2}$$

$$= \frac{1}{2}\,\frac{-24+6x+8-3x}{2(4-x)^{3/2}}$$

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$$= \frac{3x-16}{4(4-x)^{3/2}}$$

$f''(x) = 0 \Rightarrow 3x-16=0 \Rightarrow x = \frac{16}{3}$

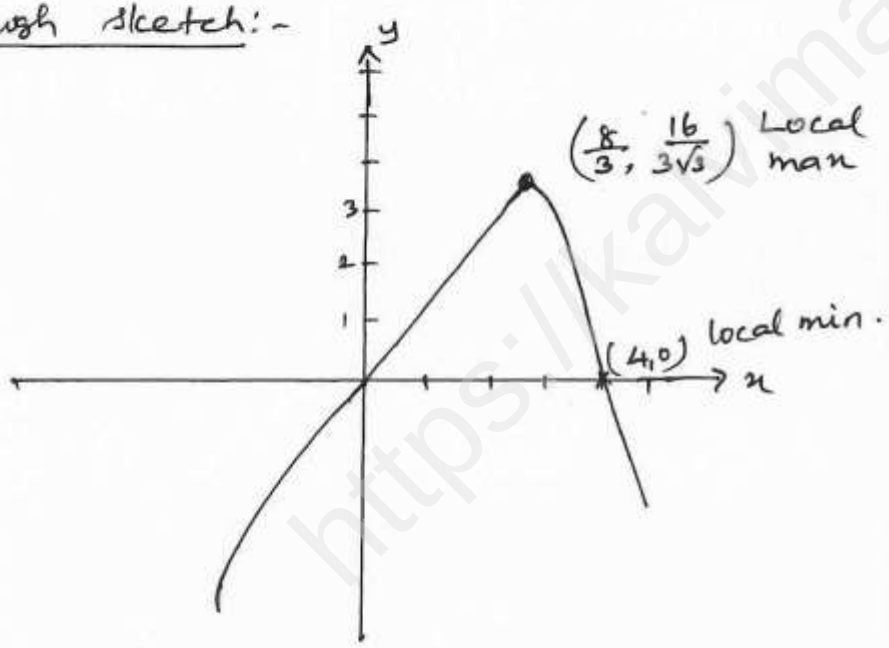
$x = \frac{16}{3} \notin (-\infty, 4]$

∴ Concavity and pt of inflection does not exist.

7) Asymptotes:-

$f(x)$ has no asymptotes.

Rough sketch:-



(iii) $y = \frac{x^2+1}{x^2-4}$

$y = f(x) = \frac{x^2+1}{x^2-4} = \frac{x^2+1}{(x+2)(x-2)}$

1) Domain:

$f(x)$ is not defined when $x=-2$ and $x=2$.

The domain of $f(x)$ is $R - \{-2, 2\}$

Range:-

$y = \frac{x^2+1}{x^2-4}$

$y(x^2-4) = x^2+1$

$yx^2 - 4y = x^2+1$

$yx^2 - x^2 = 1+4y$

$x^2(y-1) = 1+4y$

$x^2 = \frac{1+4y}{y-1}$

when $x=0$, $1+4y=0$

$4y = -1$

$y = -\frac{1}{4}$

when $y < -\frac{1}{4}$ say

$y = -\frac{1}{2}$

$x^2 = \frac{1+4(-\frac{1}{2})}{-\frac{1}{2}-1}$

$= \frac{1-2}{-3/2} = \frac{-1}{-3/2}$

$= \boxed{2/3 > 0}$

∴ x has real values

when $y > -\frac{1}{4}$ say $y=0$

$x^2 = \frac{1+0}{0-1} = \boxed{-1 < 0}$

x has imagi values

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when $y > 1$ say $y = 2$

$$x^2 = \frac{4 \times 2 + 1}{2 - 1} = \frac{9}{1} = 9 > 0$$

$\therefore$ x has real values

$\therefore$ for $-\frac{1}{4} < y \leq 1$, x^2 is negative

x has no real value.

$\therefore$ Range $= \left(-\infty, -\frac{1}{4}\right] \cup [1, \infty)$

2) Symmetry

$$f(-x, -y) = -y = \frac{x^2+1}{(-x+2)(-x-2)}$$

$$= -y = \frac{x^2+1}{[-(x-2)][-(x+2)]}$$

$$= -y = \frac{x^2+1}{(x-2)(x+2)}$$

$$= f(x, y)$$

$\therefore$ The curve is symmetrical about y-axis.

3) Intercepts:

x intercept, put $y = 0$

$\Rightarrow x^2 + 1 = 0 \Rightarrow x^2 = -1$

$x = \sqrt{-1}$

which is not possible.

$\therefore$ no x-intercept.

y intercept, put $x = 0$

$\Rightarrow y = -\frac{1}{4}$

$\therefore$ y intercept is $\left(0, -\frac{1}{4}\right)$

4) Critical no:-

$$f(x) = \frac{x^2+1}{x^2-4}$$

$$f'(x) = \frac{(x^2-4)(2x) - (x^2+1)(2x)}{(x^2-4)^2}$$

$$= \frac{2x^3 - 8x - 2x^3 - 2x}{(x^2-4)^2} = \frac{-10x}{(x^2-4)^2}$$

$f'(x) = 0 \Rightarrow -10x = 0$

$\Rightarrow x = 0$

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$f'(x)$ does not exist at $x = -2, 2$

$\therefore$ critical nos are $-2, 0, 2$.

6) Local extrema:-

$$f''(x) = \frac{(x^2-4)^2(-10) + 10x \cdot 2(x^2-4)(2x)}{[(x^2-4)^2]^2}$$

$$= \frac{2(x^2-4)[-5(x^2-4) + 20x^2]}{(x^2-4)^4}$$

$$= \frac{2[-5x^2+20+20x^2]}{(x^2-4)^3}$$

$$= \frac{2(15x^2+20)}{(x^2-4)^3} = \frac{10(3x^2+4)}{(x^2-4)^3}$$

$f''(x) = 0 \Rightarrow 3x^2+4 = 0$

$3x^2 = -4$

$x^2 = -4/3$

$x = \pm\sqrt{-4/3}$ not possible

No point of inflection.

7) Concavity:

$f''(x)$ does not exist at $x = -2, 2$.

In $(-\infty, -2)$ say $x = -3$

$$f''(-3) = \frac{10(27+4)}{(9-4)^3} > 0$$

$\therefore f(x)$ is concave upward

In $(-2, 2)$, say $x = 0$,

$$f''(0) = \frac{10(0+4)}{(0-4)^3} = \frac{40}{-64} < 0$$

$\therefore f(x)$ is concave downward.

In $(2, \infty)$, say $x = 3$,

$$f''(3) = \frac{10(27+4)}{(9-4)^3} > 0$$

$\therefore f(x)$ is concaveupward

Interval	$f''(x)$	concavity
$(-\infty, -2)$	+	upward
$(-2, 2)$	−	downward
$(2, \infty)$	+	upward.

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8) Asymptotes

Horizontal Asymptote

$$\lim_{x\to\infty} \frac{x^2+1}{x^2-4} = \lim_{x\to\infty} \frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2\left(1-\frac{4}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{1+\frac{1}{x^2}}{1-\frac{4}{x^2}}$$

$$= \frac{1+0}{1-0} = 1$$

$\therefore$ $y=1$ is the horizontal asymptote.

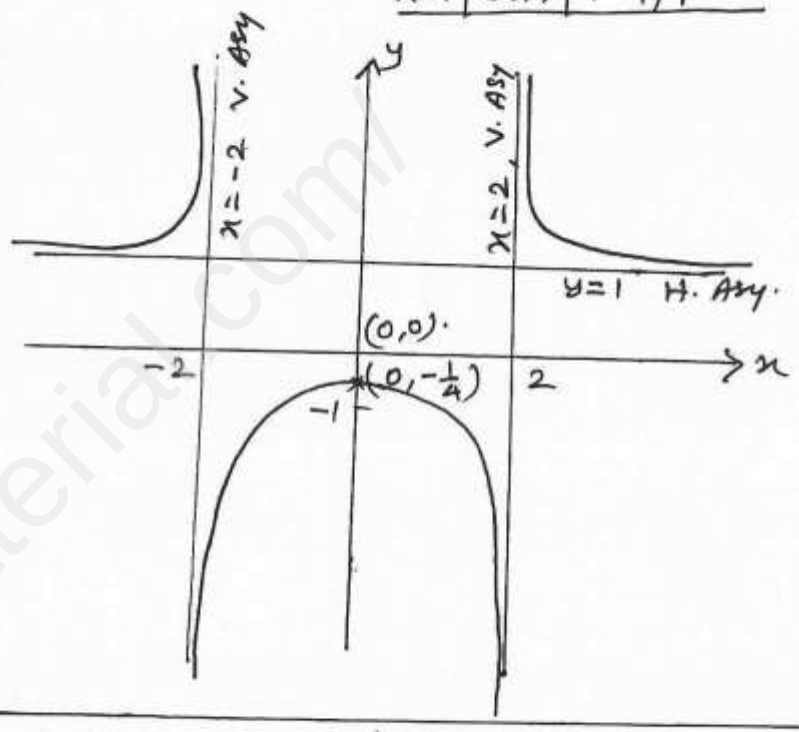
vertical asymptote:

$$\lim_{x\to\pm 2} \frac{x^2+1}{x^2-4} = \frac{4+1}{4-4} = \frac{5}{0} = \infty$$

$\therefore$ $x=\pm 2$ is the vertical asymptote.

Slant asymptote:

Degree of Nr is equal to degree of Dr. $\therefore$ Slant asymptote does not exist.



(IV) $y = \frac{1}{1+e^{-x}}$

$y = f(x) = \frac{1}{1+e^{-x}}$

1) Domain:- The domain of $f(x)$ is R.

Range : $(0,1)$

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2) Intercepts:-

x intercept, put $y=0$

$\frac{1}{1+e^{-x}} = 0 \Rightarrow$ x intercept does not exist.

y-intercept, put $x=0$

$y = \frac{1}{1+e^{-0}} = \frac{1}{1+1} = \frac{1}{2}$

$\therefore$ y intercept is $(0, \frac{1}{2})$.

3) critical point:-

$f(x) = \frac{1}{1+e^{-x}}$

$f'(x) = \frac{(1+e^{-x})0 - 1(e^{-x}(-1))}{(1+e^{-x})^2}$

$= \frac{e^{-x}}{(1+e^{-x})^2}$

$f'(x) = 0 \Rightarrow e^{-x} = 0 \Rightarrow x = -\infty$

$\therefore f(x)$ does not have critical points.

4) concavity:-

$f'(x) = \frac{e^{-x}}{(1+e^{-x})^2}$

$f''(x) = \frac{(1+e^{-x})^2 e^{-x}(-1) - e^{-x} 2(1+e^{-x})(e^{-x})(-1)}{[(1+e^{-x})^2]^2}$

$= \frac{-(1+e^{-x})^2 e^{-x} + e^{-x} e^{-x} 2(1+e^{-x})}{(1+e^{-x})^4}$

$= \frac{e^{-x}(1+e^{-x})[-(1+e^{-x}) + 2e^{-x}]}{(1+e^{-x})^4}$

$= \frac{e^{-x}[-1-e^{-x}+2e^{-x}]}{(1+e^{-x})^3}$

$= \frac{e^{-x}[e^{-x}-1]}{(1+e^{-x})^3}$

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$f''(x) = 0 \Rightarrow e^{-x}(e^{-x}-1) = 0$

$\Rightarrow e^{-x} = 0$ or $e^{-x} - 1 = 0$

$x = -\infty$ or $e^{-x} = 1$

$\Rightarrow x = 0$.

∴ The interval of concavity are $(-\infty, 0)$ and $(0, \infty)$

In $(-\infty, 0)$ say $x = -1$

$$f''(-1) = \frac{e^1(e^1-1)}{(1+e)^3} > 0$$

$\because 2 < e < 3$

∴ $f(x)$ is concave up in $(-\infty, 0)$

In $(0, \infty)$ say $x = 1$

$$f''(1) = \frac{e^{-1}(e^{-1}-1)}{(1+e^{-1})^3}$$

$$= \frac{\frac{1}{e}\left[\frac{1}{e}-1\right]}{\left(1+\frac{1}{e}\right)^3} = \frac{\frac{1}{e}\left[\frac{1-e}{e}\right]}{\left(\frac{e+1}{e}\right)^3}$$

$$= \frac{1-e}{e^2} \times \frac{e^3}{(e+1)^3} = \frac{e(1-e)}{(1+e)^3} < 0$$

$\because 2 < e < 3$

$1 - e < 0$.

∴ $f(x)$ is concave down in $(0, \infty)$

Interval	$f''(x)$	concavity
$(-\infty, 0)$	+	concave up
$(0, \infty)$	−	concave down

5) point of inflection:-

$f''(x)$ changes sign from +ve to −ve when passes through $x = 0$.

∴ The point of inflection is $x = 0$

point of inflection is $(0, f(0))$

i.e. $\left(0, \frac{1}{1+e^0}\right)$

$\left(0, \frac{1}{2}\right)$

6) Asymptotes:- (Horizontal asymptote)

$$\lim_{x \to \infty} \frac{1}{1+e^{-x}} = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

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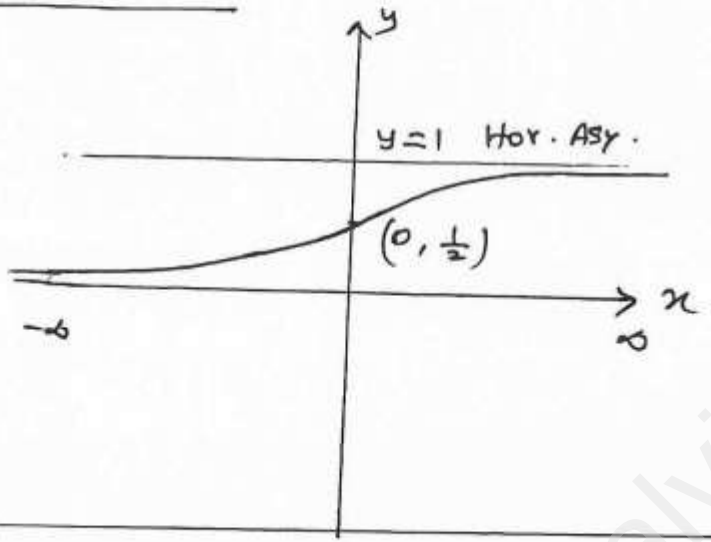
$y = 1$ is the horizontal asymptote

Vertical asymptote:

There is no vertical asymptote.

Rough Sketch:-



(v) $y = \frac{x^3}{24} - \log x$

1) $f(x)$ exists only for $x > 0$

∴ Domain is $(0, \infty)$ ∵ $\log x$ does not exist in $(-\infty, 0)$

2) Symmetry:- The curve is not symmetrical.

3) Intercepts:-

x - intercept, put $y = 0$

$\Rightarrow \frac{x^3}{24} - \log x = 0 \Rightarrow \frac{x^3}{24} = \log x$

∴ x - intercept does not exist.

y - intercept, put $x = 0$

$\Rightarrow y = \frac{0}{24} - \log 0 = \log 0$

$\Rightarrow y = -\infty$.

y intercept does not exist.

4) Critical no:-

$f(x) = \frac{x^3}{24} - \log x$

$f'(x) = \frac{1}{24} 3x^2 - \frac{1}{x}$

$= \frac{x^2}{8} - \frac{1}{x} = \frac{x^3 - 8}{8x}$

$f'(x) = 0 \Rightarrow x^3 - 8 = 0 \Rightarrow x = 2$

∴ The critical no is $x = 2$.

∴ The interval of monotonicity is $(0, 2)$ and $(2, \infty)$

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In $(0,2)$ say $x=1$

$f'(1) = \frac{1-8}{8} = -\frac{7}{8} < 0$

$\therefore$ f is decreasing in $(0,2)$

In $(2,\infty)$ say $x=3$

$f'(3) = \frac{3^3-8}{8(3)} = \frac{27-8}{24} = \frac{19}{24} > 0$

$\therefore$ f is increasing in $(2,\infty)$

Interval	$f'(x)$	monotonicity
$(0,2)$	-ve	Decreasing
$(2,\infty)$	+ve	increasing

6) Local extrema:-

$f'(x)$ changes sign from -ve to +ve when passing through $x=2$

$\therefore$ $f(x)$ is local minimum at $x=2$

local minimum value is $f(2) = \frac{2^3}{24} - 1092$

$= \frac{1}{4} - 1092$

$= 0.33 - 0.30$

$= 0.03$

7) concavity:-

$f'(x) = \frac{x^3-8}{8x} = \frac{1}{8}\left[\frac{x^3-8}{x}\right]$

$f''(x) = \frac{1}{8} \times \frac{x(3x^2) - (x^3-8)(1)}{x^2}$

$= \frac{3x^3 - x^3 + 8}{8x^2} = \frac{2x^3+8}{8x^2}$

$f''(x) = 0 \Rightarrow 2x^3 + 8 = 0$

$2x^3 = -8$

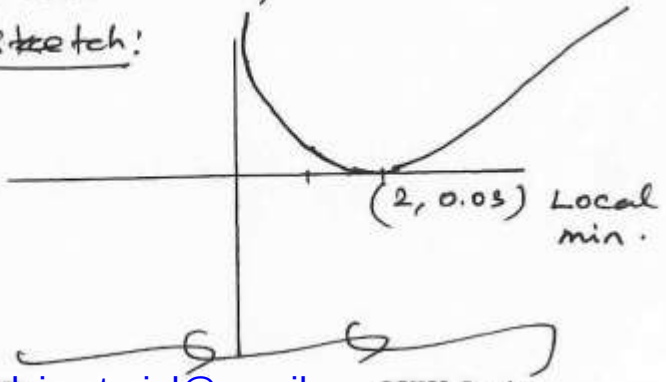
$x^3 = -4$

$x = \pm\sqrt{-4}$

not possible

$\therefore$ $f(x)$ has not point of inflection.

Rough sketch:



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Chapter 8 – Differentials & Partial Derivatives

* Exercise are hyperlink with page number. To get into particular exercise, please click exercise number

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CHAPTER - 8

Differentials and partial derivatives

Let $f:(a,b) \to R$ be a differentiable function and $x \in (a,b)$. Since f is differentiable at x, we have

$$f(x+\Delta x) = f(x) + f'(x)\Delta x$$

Linear approximation

Let $f:(a,b) \to R$ be a differentiable and $x_0 \in (a,b)$. we define the linear approximation L of f at x_0 by

$$L(x) = f(x_0) + f'(x_0)(x-x_0) \quad \forall x \in (a,b)$$

Eg 8.1 Find the linear approximation for $f(x) = \sqrt{1+x}$, $x \geq -1$ at $x_0 = 3$ use linear approximation to estimate $f(3.2)$

Given $f(x) = \sqrt{1+x}$ at $x_0 = 3$

$$f(x_0) = f(3) = \sqrt{1+3} = \sqrt{4} = 2$$

$$f'(x) = \frac{1}{2\sqrt{1+x}}$$

$$f'(x_0) = f'(3) = \frac{1}{2\sqrt{1+3}} = \frac{1}{2\sqrt{4}} = \boxed{\frac{1}{4}}$$

$\therefore$ Linear approximation is

$$L(x) = f(x_0) + f'(x_0)(x-x_0)$$

$$= 2 + \frac{1}{4}(x-3)$$

$$= 2 + \frac{x}{4} - \frac{3}{4}$$

$$= \frac{x}{4} + \frac{5}{4}$$

$$f(3.2) = \sqrt{1+3.2} = \sqrt{4.2} \simeq L(3.2) \quad \text{---(1)}$$

$$L(3.2) = \frac{3.2}{4} + \frac{5}{4}$$

$$= \frac{8.2}{4} = 2.05$$

From (1)

$\therefore$ $\boxed{f(3.2) \simeq 2.05}$

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Eg 8.2 Use linear approximation to find an approximate value of $\sqrt{9.2}$ without using calculator.

Let $f(x) = \sqrt{x}$, $x_0 = 9$, $\Delta x = 0.2$

$$f(x_0) = \sqrt{9} = 3$$

Also $f'(x_0) = \frac{1}{2\sqrt{x_0}} = \frac{1}{2\times 3} = \frac{1}{6}$

$$f(x+\Delta x) = f(x_0) + f'(x_0)\Delta x$$

$$f(9+0.2) = f(9.02)$$

$$= 3 + \frac{1}{6}(0.2)$$

$$= 3 + \frac{20}{60} \Rightarrow 3 + \frac{1}{3}$$

$$= 3.0333\ldots$$

$$= 3.0333$$

$\therefore f(9.2) = \sqrt{9.2} = 3.0333$

Eg 8.3 Let us assume that the shape of soap bubble is a sphere. Use linear approximation to approximate the increase in the surface area of a soap bubble as its radius increases from 5 cm to 5.2 cm. Also calculate the percentage of error.

Surface area of sphere is

$$S = 4\pi r^2$$

$$\therefore S' = 4\pi(2r)$$

$$= 8\pi r.$$

Approximate Value

$$S(r) - S(r_0) \simeq S'(r_0)\Delta r$$

Here $r = 5.2$, $r_0 = 5$, $\Delta r = 0.2$

$$S(5.2) - S(5) \simeq S'(5)(0.2)$$

$$\simeq 8\pi(5)(0.2)$$

$$= 8\pi$$

Surface area increases by $8\pi\ cm^2$

Actual Value:-

$$S(5.2) - S(5) = 4\pi(5.2)^2 - 4\pi(5)^2$$

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$= 4\pi [5.2^2 - 5^2]$

$= 4\pi (5.2+5)(5.2-5)$

$= 4\pi (10.2)(0.2)$

$= 4\pi \times 2.04$

$= 8.16\pi \text{ cm}^2$

percentage of error $= \dfrac{\text{actual value} - \text{approximate value}}{\text{actual value}} \times 100$

$= \dfrac{8.16\pi - 8\pi}{8.16\pi} \times 100$

$= \dfrac{0.16\pi}{8.16\pi} \times 100 \Rightarrow \dfrac{0.16}{8.16} = \dfrac{16}{816} \times 100$

$= \dfrac{1}{51} \times 100$

$= \dfrac{100}{51} = 1.9607$

Error:-

Absolute error = Actual error − Approximate error.

Relative error = $\dfrac{\text{Actual error} - \text{App. error}}{\text{actual error.}}$

percentage of error = $\dfrac{\text{Ac. error} - \text{App. error}}{\text{actual err.}} \times 100$

EXERCISE 8.1

1. Let $f(x) = \sqrt[3]{x}$. find the linear approximation at $x = 27$. use the linear approximation to approximate $\sqrt[3]{27.2}$

Given $f(x) = \sqrt[3]{x}$, $x_0 = 27$.

$f(x) = x^{\frac{1}{3}}$

$f(x_0) = 27^{\frac{1}{3}} = 3$

$f'(x) = \frac{1}{3} x^{\frac{1}{3}-1}$

$= \frac{1}{3} x^{-2/3} = \dfrac{1}{3x^{2/3}}$

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$f'(x_0) = f'(27)$

$= \frac{1}{3(27)^{2/3}} = \frac{1}{3(3^3)^{\frac{2}{3}}}$

$= \frac{1}{3(3^2)}$

$= \frac{1}{27}$

Linear approximation is

$L(x) = f(x_0) + f'(x_0)(x - x_0)$

$L(x) = 3 + \frac{1}{27}(x - 27)$

$= 3 + \frac{x}{27} - 1$

$= 2 + \frac{x}{27}$ ——— (1).

$f(27.2) = \sqrt[3]{27.2} \simeq L(27.2)$

$= 2 + \frac{27.2}{27}$

$= 2 + 1.0074$

$= 3.0074$ (app)

2) Use the linear approximation to find approximate values of

(i) $(123)^{2/3}$

Let $f(x) = x^{2/3}$, $x_0 = 125$, $\Delta x = -2$.

$f(x_0) = x_0^{2/3}$

$= (125)^{2/3} = (5^3)^{2/3} = 5^2$

$= 25$

Also $f'(x) = \frac{2}{3}x^{\frac{2}{3}-1}$

$= \frac{2}{3}x^{-\frac{1}{3}} = \boxed{\frac{2}{3x^{\frac{1}{3}}}}$

$f'(x_0) = f'(125)$

$= \frac{2}{3(125)^{\frac{1}{3}}} = \frac{2}{3(5)} = \boxed{\frac{2}{15}}$

$\therefore f(x + \Delta x) = f(x_0) + f'(x_0)\Delta x$

$(123)^{2/3} = 25 + \frac{2}{15}(-2)$

$= 25 - \frac{4}{15}$

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$= \frac{375-4}{15} = \frac{371}{15} = 24.733$ (app)

(ii) $\sqrt[4]{15}$

Let $f(x) = \sqrt[4]{x}$, $x_0 = 16$, $\Delta x = -1$

$f(x_0) = \sqrt[4]{16} = (2^4)^{\frac{1}{4}} = \boxed{2}$.

$f'(x) = \frac{1}{4} x^{\frac{1}{4}-1}$

$= \frac{1}{4} x^{-3/4}$

$= \frac{1}{4 x^{3/4}}$

$\therefore f'(x_0) = \frac{1}{4(16)^{3/4}} = \frac{1}{4(2^4)^{3/4}}$

$= \frac{1}{4(2^3)}$

$= \boxed{\frac{1}{32}}$

$f(x+\Delta x) = f(x_0) + f'(x_0)\Delta x$

$= 2 + \frac{1}{32}(-1)$

$= 2 - \frac{1}{32} = \frac{64-1}{32}$

$= \frac{63}{32}$

$= 1.9688$ (app)

(iii) $\sqrt[3]{26}$

Let $f(x) = \sqrt[3]{x}$, $x_0 = 27$, $\Delta x = -1$

$f(x) = x^{\frac{1}{3}}$

$f(x_0) = 27^{\frac{1}{3}} = (3^3)^{\frac{1}{3}} = \boxed{3}$

$f'(x) = \frac{1}{3} x^{\frac{1}{3}-1}$

$= \frac{1}{3 x^{2/3}}$

$f'(x_0) = f'(27)$

$= \frac{1}{3(27)^{2/3}} = \frac{1}{3(3^3)^{2/3}}$

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$= \frac{1}{3(3^2)} = \boxed{\frac{1}{27}}$

$f(x+\Delta x) = f(x_0) + f'(x_0)\Delta x$

$= 3 + \frac{1}{27}(-1)$

$= 3 - \frac{1}{27} = \frac{80}{27} = 2.9629$ (app)

3) Find a linear approximation for the following functions at the indicated points.

(i) $f(x) = x^3 - 5x + 12$, $x_0 = 2$

$f(x) = x^3 - 5x + 12$, $x_0 = 2$

$f(x_0) = f(2) = 2^3 - 5(2) + 12$

$= 8 - 10 + 12$

$= 20 - 10$

$= \boxed{10}$

Also, $f'(x) = 3x^2 - 5$

$f'(x_0) = f'(2) = 3(2)^2 - 5$

$= 3(4) - 5$

$= 12 - 5$

$= \boxed{7}$

Linear approximation is

$L(x) = f(x_0) + f'(x_0)(x - x_0)$

$= 10 + 7(x-2)$

$= 10 + 7x - 14$

$= \boxed{7x - 4}$

(ii) $g(x) = \sqrt{x^2+9}$, $x_0 = -4$

$g(x) = \sqrt{x^2+9}$

$g(x_0) = g(-4) = \sqrt{(-4)^2+9}$

$= \sqrt{16+9}$

$= \sqrt{25}$

$= \boxed{5}$

$g'(x) = \frac{1}{2\sqrt{x^2+9}}(2x) = \frac{x}{\sqrt{x^2+9}}$

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$g'(x_0) = g'(-4)$

$= \frac{-4}{\sqrt{16+9}} = \frac{-4}{\sqrt{25}} = -\frac{4}{5}$

∴ Linear approximation is

$L(x) = g(x_0) + g'(x_0)(x - x_0)$

$= 5 + \left(-\frac{4}{5}\right)(x+4)$

$= 5 - \frac{4x-16}{5}$

$= \frac{25-4x-16}{5} = \boxed{\frac{9-4x}{5}}$

(iii) $h(x) = \frac{x}{x+1}$, $x_0 = 1$

$h(x) = \frac{x}{x+1}$

$h(x_0) = \frac{1}{1+1} = \boxed{\frac{1}{2}}$

$h'(x) = \frac{x+1 - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$

$h'(x_0) = \frac{1}{(1+1)^2}$

$= \frac{1}{2^2} = \boxed{\frac{1}{4}}$

Linear approximation is

$L(x) = h(x_0) + h'(x_0)(x - x_0)$

$= \frac{1}{2} + \frac{1}{4}(x-1)$

$= \frac{1}{2} + \frac{x}{4} - \frac{1}{4}$

$= \frac{1}{4} + \frac{x}{4} = \boxed{\frac{x+1}{4}}$

---

4) The radius of a circular plate is measured as 12.65cm instead of the actual length 12.5cm. Find the following in calculating the area of the circular plate:-

a) Absolute error.

b) relative error

c) percentage error.

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Actual radius = 12.5cm

Area of the plate = πr^2

Actual area of the plate = πr^2

$= \pi (12.5)^2$

$= \pi\ 12.5 \times 12.5$

$= 156.25\pi\ cm^2$

Approximate radius = 12.65

Approximate area of the plate

$= \pi (12.65)^2$

$= \pi [12.65 \times 12.65]$

$= \pi [160.0225]\ cm^2$

Absolute value = Actual value − Approximate value

$= 12.65 - 12.50$

$= 0.15$

(i) Absolute error $= \pi (0.15)^2$

$= 0.0225\pi\ cm^2$

(ii) Relative error $= \dfrac{A(12.65) - A(12.5)}{A(12.5)}$

$= \dfrac{160.0225\pi - 156.25\pi}{156.25\pi}$

$= \dfrac{3.7725\pi}{156.25\pi} = \dfrac{3.7725}{156.25}$

$= 0.024144$

∴ Relative error = 0.024

(iii) percentage of error = Relative error × 100

$= 0.024 \times 100$

$= 2.4\ \%$.

5) A sphere is made of ice having radius 10cm. Its radius decreases from 10cm to 9.8cm find the approximate for the following

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(i) change in volume
(ii) change in surface area.

Volume of the sphere

$V = \frac{4}{3}\pi r^3$

$V' = \frac{4}{3}\pi(3r^2)$

$= 4\pi r^2$

(i) change in volume

$V(r) - V(r_0) \simeq V'(r_0)\Delta r$

$V(9.8) - V(10) \simeq V'(10)\Delta r$

Here $r = 9.8$, $r_0 = 10$, $\Delta r = -0.2$

$\therefore V(9.8) - V(10) \simeq 4\pi(10)^2(-0.2)$

$= 4\pi\, 100\,(-0.2)$

$= 400\pi(-0.2)$

$= -80\pi\ cm^3$

$\therefore$ change in volume decreases is $80\pi\ cm^3$.

(ii) To change in surface area

surface area of sphere $S = 4\pi r^2$

$S' = 4\pi(2r)$

$= 8\pi r$

change in surface area

$= S(r) - S(r_0) \simeq S'(r_0)\Delta r$

Here $r = 9.8$, $r_0 = 10$, $\Delta r = -0.2$

$S(9.8) - S(10) \simeq 8\pi(10)(-0.2)$

$= 80\pi(-0.2)$

$= -16\pi\ cm^2$

$\therefore$ surface area decreases $16\pi\ cm^2$

6) The time T, taken for a complete oscillation of a single pendulum with length l, is given by the eqn $T = 2\pi\sqrt{\frac{l}{g}}$, where g is constant. Find the approximate percentage error in the calculated value of T corresponding to an error of 2 percent in the value of l.

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$T(\ell) = 2\pi \frac{\sqrt{\ell}}{\sqrt{g}}$

$T'(\ell) = \frac{2\pi}{\sqrt{g}} \frac{1}{2\sqrt{\ell}} = \frac{\pi}{\sqrt{g}\sqrt{\ell}}$

$T(\ell+\Delta\ell) - T(\ell) \cong T'(\ell)\Delta\ell$.

Here $\Delta\ell$ = 2% of ℓ.

$\therefore T(\ell+\Delta\ell) - T(\ell) \cong \frac{\pi}{\sqrt{g}\sqrt{\ell}} \times \frac{2}{100} \times \ell$

$= \frac{2\pi\sqrt{\ell}}{100\sqrt{g}}$

$= \frac{1}{100} 2\pi\sqrt{\frac{\ell}{g}}$

$= \frac{1}{100} T(\ell)$

$\therefore \frac{T(\ell+\Delta\ell) - T(\ell)}{T(\ell)} = \frac{1}{100} = 1\%$.

$\therefore$ percentese of error = 1%.

Show that the percentage error in the nth root of a number is approximately $\frac{1}{n}$ times the percentage error in the number.

XII / CH 8 / 8.1 / P16

Let $f(x) = x^{\frac{1}{n}}$

$f'(x) = \frac{1}{n} x^{\frac{1}{n}-1} = \frac{1}{n} x^{\frac{1}{n}} \cdot x^{-1}$

$= \frac{1}{nx} x^{\frac{1}{n}}$

$f'(x) \cong \boxed{\frac{1}{nx} f(x)}$

$\therefore f(x+\Delta x) - f(x) \cong f'(x)\Delta x$

$= \frac{1}{nx} f(x)\Delta x$

$\frac{f(x+\Delta x) - f(x)}{f(x)} = \frac{1}{n}\left(\frac{\Delta x}{x}\right)$

$\frac{f(x+\Delta x) - f(x)}{f(x)} \times 100 = \frac{1}{n}\left(\frac{\Delta x}{x}\right) \times 100$

percentage error in the nth root of number $\cong \frac{1}{n}$ (percentage error in number)

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Differential :-

* $dy = f'(x)\,dx$
* $\Delta y = f(x+\Delta x) - f(x)$

Eg 8.6 Let $g(x) = x^2 + \sin x$ calculate the differential dg.

$g(x) = x^2 + \sin x$

$g'(x) = 2x + \cos x$

i.e. $\frac{dg}{dx} = 2x + \cos x$

$dg = (2x + \cos x)\,dx$

Ex 8.7 If the radius of the sphere, with radius 10cm, has to decrease by 0.1 cm, approximately how much will its volume decrease?

Volume of the sphere $V = \frac{4}{3}\pi r^3$

change in volume $dV = \frac{4}{3}\pi(3r^2)\,dr$

$= 4\pi r^2 dr$

Here $r = 10\text{cm}$, $dr = -0.1$

$\therefore dV = 4\pi(10)^2(-0.1)$

$= 400\pi(-0.1)$

$= 40\pi\ \text{cm}^3$

Hence the volume is decreased by $40\pi\ \text{cm}^3$

EXERCISE 8.2

1. Find differential dy for each of the following functions:-

(i) $y = \frac{(1-2x)^2}{3-4x}$

$$\frac{dy}{dx} = \frac{(3-4x)(2(1-2x)(-2) - (1-2x)^2(-4)}{(3-4x)^2}$$

$$= \frac{-4(1-2x)(3-4x) + 4(1-2x)^2}{(3-4x)^2}$$

$$= \frac{4(1-2x)[1-2x-(3-4x)]}{(3-4x)^2}$$

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$$= \frac{4(1-2x)[1-2x-3+4x]}{(3-4x)^2}$$

$$= \frac{4(1-2x)[2x-2]}{(3-4x)^2} = \frac{8(1-2x)(x-1)}{(3-4x)^2}$$

$$\therefore dy = \frac{8(1-2x)(x-1)}{(3-4x)^2}\,dx$$

(i) $y = \dfrac{(1-2x)^3}{3-4x}$

$$\frac{dy}{dx} = \frac{(3-4x)\,3(1-2x)^2(-2) - (1-2x)^3(-4)}{(3-4x)^2}$$

$$= \frac{-6(3-4x)(1-2x)^2 + 4(1-2x)^3}{(3-4x)^2}$$

$$= \frac{2(1-2x)^2[2(1-2x) - 3(3-4x)]}{(3-4x)^2}$$

$$= \frac{2(1-2x)^2[2-4x-9+12x]}{(3-4x)^2}$$

$$= \frac{2(1-2x)(8x-7)}{(3-4x)^2}$$

$$\therefore dy = \frac{2(1-2x)(8x-7)}{(3-4x)^2}\,dx$$

(ii) $y = [3+\sin(2x)]^{\frac{2}{3}}$

$$\frac{dy}{dx} = \frac{2}{3}[3+\sin(2x)]^{\frac{2}{3}-1}(0+\cos 2x)(2)$$

$$= \frac{4}{3}\cos 2x\,(3+\sin(2x))^{-1/3}$$

$$\therefore dy = \frac{4\cos 2x}{3(3+\sin 2x)^{1/3}}\,dx$$

(iii) $y = e^{x^2-5x+7}\cos(x^2-1)$

$$\frac{dy}{dx} = e^{x^2-5x+7}[-\sin(x^2-1)(2x)] + \cos(x^2-1)\,e^{x^2-5x+7}[2x-5]$$

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$= e^{x^2-5x+7}\left[\cos(x^2-1)(2x-5) - 2x \sin(x^2-1)\right]$

$\therefore dy = e^{x^2-5x+7}\left[\cos(x^2-1)(2x-5) - 2x \sin(x^2-1)\right]dx$

---

2) Find dt for $f(x) = x^2+3x$ and evaluate it for

(i) $x = 2$ and $dx = 0.1$

$f(x) = x^2+3x$

$\frac{dt}{dx} = 2x+3$

$dt = (2x+3)\,dx$

$dt_{(x=2,\, dx=0.1)} = [2(2)+3](0.1)$

$= (4+3)(0.1)$

$= 7(0.1)$

$\therefore dt = 0.7$

(ii) $x = 3$, and $dx = 0.02$

$f(x) = x^2+3x$

$dt = (2x+3)\,dx$

$dt_{(x=3,\, dx=0.02)}$

$= [2(3)+3]\,0.02$

$= (6+3)(0.02)$

$= 9(0.02)$

$= \boxed{0.18}$

---

3) Find Δt and dt for the function f for the indicated values of x, Δx and compare

(i) $f(x) = x^3 - 2x^2$, $x = 2$, $\Delta x = dx = 0.5$

$f(x) = x^3 - 2x^2$

Given $f(x) = x^3 - 2x^2$, $x = 2$,

$\Delta x = dx = 0.5$

$\Delta t = f(x+\Delta x) - f(x)$

$= f(2+0.5) - f(2)$

$= f(2.5) - f(2)$

$= f(5/2) - f(2)$

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$= \left(\frac{5}{2}\right)^3 - 2\left(\frac{5}{2}\right)^2$

$= \frac{125}{8} - \frac{50}{4}$

$= \frac{125 - 100}{8} = \frac{25}{8} = \boxed{3.125}$

Also $\frac{df}{dx} = 3x^2 - 4x$

$df = (3x^2 - 4x)\,dx$

$df\,(x=2, dx=0.5) = \left[3(2)^2 - 4(2)\right](0.5)$

$= (12-8)(0.5)$

$= 4(0.5)$

$= \boxed{2}$

Hence $\Delta f = 3.125$, $df = 2$

(ii) $f(x) = x^2 + 2x + 3$, $x = -0.5$, $\Delta x = dx = 0.1$

Given $f(x) = x^2 + 2x + 3$

$x = -0.5$, $\Delta x = dx = 0.1$

$\Delta f = f(x + \Delta x) - f(x)$

$= f(-0.5 + 0.1) - f(-0.5)$

$= f(-0.4) - f(-0.5)$

$= f\left(-\frac{2}{5}\right) - f\left(-\frac{1}{2}\right)$

$= \left[\left(-\frac{2}{5}\right)^2 + 2\left(-\frac{2}{5}\right) + 3\right] - \left[\left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right) + 3\right]$

$= \left(\frac{4}{25} - \frac{4}{5} + 3\right) - \left(\frac{1}{4} - 1 + 3\right)$

$= \left(\frac{4 - 20 + 75}{25}\right) - \left(\frac{1 - 4 + 12}{4}\right)$

$= \frac{59}{25} - \frac{9}{4} = \frac{236 - 225}{100}$

$= \frac{11}{100}$

$= \boxed{0.11}$

Also $\frac{df}{dx} = 2x + 2$

$df = (2x+2)\,dx$

$df_{at\,(x=-0.5,\,dx=0.1)} = \left[2(-0.5) + 2\right]0.1$

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$= +0.1[-1+2]$

$= 0.1(1) = \boxed{0.1}$

Hence $\Delta t = 0.11$, $dt = 0.1$

---

4) Assuming $\log_{10} e = 0.4343$ find an approximate value of $\log_{10} 1003$

Let $y = f(x) = \log_{10} x$

where $x = 1000$, $\Delta x = dx = 3$.

$$\frac{dy}{dx} = \frac{1}{x} d\log_{10} e$$

$$dy = \frac{1}{x} \log_{10} e \, dx$$

$$= \frac{0.4343}{1000} \cdot 3$$

$$= \frac{1.3029}{1000} = \boxed{0.0013029}$$

$f(x+\Delta x) \cong f(x) + dy$

$\log_{10} 1003 = \log_{10} 1000 + 0.0013029$

$= \log_{10} 10^3 + 0.0013029$

$= 3 \log_{10} 10 + 0.0013029$

$= 3.0013029$

---

5) The trunk of a tree has diameter 30cm. During the following year the circumference grew 6cm
(i) Approximately, how much did the tree's diameter grow?
(ii) What is the percentage increase in area of the tree's cross-section?

Given circumference change $dc = 6$cm

Diameter = 30cm.

circumferen of tree $= 2\pi r$

$= 2\pi \frac{D}{2}$

$= \pi D$

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$C = \pi D$

Diff w.r to D

$\frac{dC}{dD} = \pi$

$dD = \frac{dC}{\pi} = \boxed{\frac{6}{\pi}\ cm}$

(iii) Area of circle is $A = \pi r^2$

$= \pi \left(\frac{D}{2}\right)^2$

$= \boxed{\pi \frac{D^2}{4}}$

$\frac{dA}{dD} = \pi \frac{2D}{4} = \frac{\pi D}{2}$

$dA = \frac{\pi D}{2} dD$

percentage changing in Area

$= \frac{dA}{A} \times 100$

$= \frac{\frac{\pi D}{2} dD}{\frac{\pi D^2}{4}} \times 100$

$= \frac{\pi D}{2} \times \frac{4}{\pi D^2} \times 100\, dD$

$= \frac{200}{D} dD = \frac{20\cancel{0} \times 6}{\cancel{5}\cancel{0}}$

$= \frac{40}{\pi}\ cm^2$

6) A egg of a particular bird is very nearly spherical. If the radius to the inside of the shell is 5 mm and radius of the outside of the shell is 5.3 mm, find the volume of the shell approximately.

Volume of the sherical shell is

$V = \frac{4}{3} \pi r^3$

Diff w.r to r

$\frac{dV}{dr} = \frac{4}{3} \pi (3r^2)$

$= 4\pi r^2$

$dV = 4\pi r^2 dr$

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Given that $r = 5$, $\Delta r = 5.3 - 5$

$= 0.3 = dr$

$V(5.3) - V(5) \simeq dV \Delta r$

$= 4\pi(5^2)(0.3)$

$= 100\pi(0.3)$

$= 30\pi \text{ cm}^3$

---

7) Assume that the cross section of the artery of human is circular. A drug is given to a patient to dilate his arteries. If the radius of an artery is increased from 2mm to 2.1 mm, how much is cross sectional area increased approximately.

Area of the circle is $A = \pi r^2$

Diff w.r.to r

$\frac{dA}{dr} = \pi(2r)$

$\therefore dA = 2\pi r\, dr.$

Given that $\Delta r = dr = 0.1$, $r = 2$

$A(2.1) - A(2) \simeq dA.$

$= 2\pi(2)(0.1)$

$= 4\pi(0.1)$

$= 0.4\pi \text{ cm}^2$

Area is increased by $0.4\pi \text{ cm}^2$

---

8) In a newly developed city, it is estimated that the voting population will increase according to $V(t) = 30 + 12t^2 - t^3$, $0 \le t \le 8$ where t is the time in year. Find approximate change in voters for time change from 4 to $4\frac{1}{6}$ yrs.

Given $V(t) = 30 + 12t^2 - t^3$, $0 \le t \le 8$

$\frac{dV}{dt} = 24t - 3t^2$

$dV = (24t - 3t^2)\,dt.$

Also given that $t = 4$, $\Delta t = dt$

$= 4\frac{1}{6} - 4 = \frac{1}{6}$

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change in votes

$V(4\frac{1}{6}) - V(4) \simeq dV$

$= [24(4) - 3(4^2)] \frac{1}{6}$

$= (96 - 48)\frac{1}{6} = \frac{48}{6} = 8$ (Thous)

$\therefore$ change in voters = 8000.

---

9) The relation between the number of words y a person learns in x hours is given by $y = 52\sqrt{x}$, $0 \le x \le 9$. what is approximate number of words learned when x changes from

(i) 1 to 1.1 hrs (ii) 4 to 4.1 hr.

Given that $y = 52\sqrt{x}$, $0 \le x \le 9$

Diff w.r to x

$\frac{dy}{dx} = 52 \frac{1}{2\sqrt{x}} = \frac{26}{\sqrt{x}}$

$dy = \frac{26}{\sqrt{x}} dx$

(i) 1 to 1.1 hr

Here $x = 1$, $dx = 1.1 - 1 = 0.1$

$y(1.1) - y(1) \simeq dy$

$= \frac{26}{\sqrt{1}} (0.1) = 2.6$

$\therefore$ No of words changes = 3 word.

(ii) 4 to 4.1 hr.

$y(4.1) - y(4) \simeq dy$

$= \frac{26}{\sqrt{4}} (0.1)$

$= \frac{26}{2} (0.1)$

$= 13(0.1)$

$= 1.3$

$\therefore$ no of words learn = 1 word.

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10) A circular plate expands uniformly under the influence of heat. If it's radius increases from 10.5 cm to 10.75 cm, then find an approximate change in the area and the approximate percentage change in the area.

Area of the circular plate

$$A = \pi r^2$$

Diff w.r to r

$$\frac{dA}{dr} = \pi(2r)$$

$$\boxed{dA = 2\pi r\, dr}$$

Also given $r = 10.5$, $dr = \Delta r = 10.75 - 10.5$

$= 0.25$

change in area $= A(10.75) - A(10.5) \simeq dA$

$= 2\pi(10.5)(0.25)$

$= \pi(10.5)(0.5)$

$= 5.25\pi\ cm^2$

$\therefore$ change in Area increases by $5.25\pi\ cm^2$.

Also percentage of change in area $= \frac{dA}{A} \times 100$

$= \frac{2\pi r\, dr}{\pi r^2} \times 100$

$= \frac{2\, dr}{r} \times 100$

$= \frac{2(0.25)}{10.5} \times 100$

$= \frac{0.5 \times 100}{10.5}$

$= \frac{500}{105}$

$= \frac{100}{21} = 4.76\%$

$\therefore$ percentage change in Area

$= 4.76\%$

— x —

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11) A coat of paint of thickness 0.2 cm is applied to the faces of a cube whose edge is 10cm use the differentials to find approximately how much cubic centimeters of paint is used to paint this cube. Also calculate the exact amount of paint used to paint this cube.

Given that $a = 10$, $\Delta a = da = 0.2$

Volume of the cube is $V = a^3$

Diff w.r to a

$$\frac{dV}{da} = 3a^2$$

$$dV = 3a^2 da$$

$$= 3(10)^2(0.2)$$

$$= 3(100)(0.2)$$

$$= 0.6 \times 100$$

$$= 60 \text{ cm}^2$$

Exact amount of paint used is $V(10.2) - V(10) = (10.2)^3 - 10^3$

$$= 1061.208 - 1000$$

$$= 61.208 \text{ cm}^3$$

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Continuity :-

Suppose $A = \{(x,y) / a<x<b, c<y<d\} \subset R^2$

$f: A \to R$. we say that f is continuous at (u,v) if the following hold

a) f is defined at (u,v)

b) $\lim_{(x,y)\to(u,v)} f(x,y) = L$ exists.

c) $L = f(u,v)$

EXERCISE 8.3

1) Evaluate $\lim_{(x,y)\to(1,2)} g(x,y)$, if the limit exists, where $g(x,y) = \frac{3x^2-xy}{x^2+y^2+3}$

Given $g(x,y) = \frac{3x^2-xy}{x^2+y^2+3}$

$\lim_{(x,y)\to(1,2)} g(x,y) = \lim_{(x,y)\to(1,2)} \frac{3x^2-xy}{x^2+y^2+3}$

$= \frac{3(1)^2-1(2)}{1^2+2^2+3} = \frac{3-2}{1+4+3}$

$= \frac{1}{8}$

If $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ then the limit exists.

2) Evaluate $\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^3+y^2}{x+y+2}\right)$.

If the limit exists.

$\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^3+y^2}{x+y+2}\right)$

$= \cos\left[\lim_{(x,y)\to(0,0)} \frac{x^3+y^2}{x+y+2}\right]$

$= \cos\left(\frac{0}{2}\right) = \cos 0 = 1$

3) Let $f(x,y) = \frac{y^2-xy}{\sqrt{x}-\sqrt{y}}$ for $(x,y) \neq (0,0)$

Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

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Given $f(x,y) = \dfrac{y^2 - xy}{\sqrt{x} - \sqrt{y}}$

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \dfrac{y^2 - xy}{\sqrt{x} - \sqrt{y}}$

$\therefore \lim_{(x,y)\to(0,0)} \dfrac{y^2 - xy}{\sqrt{x} - \sqrt{y}} \times \dfrac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}$

$= \lim_{(x,y)\to(0,0)} \dfrac{(y^2 - xy)(\sqrt{x} + \sqrt{y})}{x - y}$

$= \lim_{(x,y)\to(0,0)} \dfrac{y(y - x)(\sqrt{x} + \sqrt{y})}{(x - y)}$

$= \lim_{(x,y)\to(0,0)} -y(\sqrt{x} + \sqrt{y})$

$= 0.$

Hence proved.

4) Evaluate $\lim_{(x,y)\to(0,0)} \cos\left(\dfrac{e^x \sin y}{y}\right)$, if $\frac{0}{0}$ form. The limit exists.

$\lim_{(x,y)\to(0,0)} \cos\left[\dfrac{e^x \sin y}{y}\right]$

$= \cos\left[\lim_{(x,y)\to(0,0)} e^x \cdot \lim_{(x,y)\to(0,0)} \left(\dfrac{\sin y}{y}\right)\right]$

$= \cos[e^0 . 1]$ $\because \lim_{(x,y)\to(0,0)} \dfrac{\sin y}{y} = 1$

$= \cos(1)$

$\therefore \lim_{(x,y)\to(0,0)} \cos\left(\dfrac{e^x \sin y}{y}\right) = \cos(1)$

---

5) Let $g(x,y) = \dfrac{x^2 y}{x^4 + y^2}$ for $(x,y) \neq (0,0)$

and $f(0,0) = 0$.

(i) Show that $\lim_{(x,y)\to(0,0)} g(x,y) = 0$ along every line $y = mx$, $m \in R$.

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Given $g(x,y) = \dfrac{x^2y}{x^4+y^2}$

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{x,y\to(0,0)} \frac{x^2y}{x^4+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2(mx)}{x^4+(mx)^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{mx^3}{x^2(x^2+m^2)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{mx}{x^2+m^2} = \frac{0}{m^2} = \boxed{0}$$

(ii) Show that $\lim_{(x,y)\to(0,0)} g(x,y) = \dfrac{k}{1+k^2}$

along every parabola $y = kx^2$, $k \in R-\{0\}$

$$g(x,y) = \frac{x^2y}{x^4+y^2}$$

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2(kx^2)}{x^4+k^2x^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{kx^4}{x^4(1+k^2)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{k}{1+k^2}$$

6) Show that $f(x,y) = \dfrac{x^2-y^2}{y^2+1}$ is continuous at every $(x,y) \in R^2$.

Given that

$$f(x,y) = \frac{x^2-y^2}{y^2+1}$$

Let $(a,b) \in R^2$

(i) $f(a,b) = \dfrac{a^2-b^2}{b^2+1}$ is defined.

(ii) $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} \dfrac{x^2-y^2}{y^2+1}$

$\lim_{(x,y)\to(a,b)} f(x,y) = \dfrac{a^2-b^2}{b^2+1}$ exists

if $b^2+1 \neq 0$.

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(iii) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

$\therefore f(x,y)$ is continuous for every $(x,y) \in R^2$.

---

7) Let $g(x,y) = \frac{e^y \sin x}{x}$, for $x \neq 0$ and $g(0,0) = 1$ Show that g is continuous at $(0,0)$.

Given $g(x,y) = \frac{e^y \sin x}{x}$

(i) $g(0,0) = 1$ is defined

(ii) $\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x}$

$= \lim_{(x,y)\to(0,0)} e^y \cdot \lim_{(x,y)\to(0,0)} \frac{\sin x}{x}$

$= e^0 \cdot (1) = 1 \cdot 1 = 1$

$\lim_{(x,y)\to(0,0)} g(x,y) = 1$

$\lim_{(x,y)\to(0,0)} g(x,y) = 1 = g(0,0)$

$\therefore g$ is continuous at $(0,0)$

Ex 8.8

Let $f(x,y) = \frac{3x-5y+8}{x^2+y^2+1}$ for all $(x,y) \in R^2$ Show that f is continuous on R^2

$f(x,y) = \frac{3x-5y+8}{x^2+y^2+1}$

Let $(a,b) \in R^2$

$f(a,b) = \frac{3a-5b+8}{a^2+b^2+1}$ is defined

$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} \frac{3x-5y+8}{x^2+y^2+1}$

$= \frac{\lim_{(x,y)\to(a,b)} 3x-5y+8}{\lim_{(x,y)\to(a,b)} x^2+y^2+1} = \frac{3a-5b+8}{a^2+b^2+1}$

limit exist.

(iii) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

$\therefore f$ is continuous. //

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8.9 consider $f(x,y) = \frac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that f is not continuous at $(0,0)$ and continuous at all other points of R^2.

Given $f(x,y) = \frac{xy}{x^2+y^2}$

Let $(a,b) \in R^2$ and $(a,b) \neq (0,0)$

(i) $f(a,b) = \frac{ab}{a^2+b^2}$ is defined.

(ii) $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} \frac{xy}{x^2+y^2}$

$= \frac{ab}{a^2+b^2}$ exists.

(iii) $\lim_{(x,y)\to(a,b)} f(x,y) = \frac{ab}{a^2+b^2} = f(a,b)$

$\therefore$ f is continuous at $(a,b) \in R^2$ and $(a,b) \neq (0,0)$

At $(0,0)$

(i) $f(0,0) = 0$ is defined.

(ii) $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

$= \lim_{(x,y)\to(0,0)} \frac{x(mx)}{x^2+(mx)^2}$

$= \lim_{(x\to y)\to(0,0)} \frac{mx^2}{x^2(1+m^2)}$

$= \lim_{(x,y)\to(0,0)} \frac{m}{1+m^2} = \frac{m}{1+m^2}$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \frac{m}{1+m^2}$ exists.

(iii) $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{m}{1+m^2} \neq f(0,0)$

$\therefore$ f is not continuous at $(0,0)$

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8.10 Consider $g(x,y) = \dfrac{2x^2y}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $g(0,0) = 0$ show that g is continuous on R^2

Given $g(x,y) = \dfrac{2x^2y}{x^2+y^2}$

Let $(a,b) \in R^2$ and $(a,b) \neq (0,0)$

(i) $g(a,b) = \dfrac{2a^2b}{a^2+b^2}$ is defined.

(ii) $\lim\limits_{(x,y)\to(a,b)} g(x,y) = \lim\limits_{(x,y)\to(a,b)} \dfrac{2x^2y}{x^2+y^2}$

$= \dfrac{2a^2b}{a^2+b^2}$ exists

(iii) $\lim\limits_{(x,y)\to(0,0)} g(x,y) = \dfrac{2a^2b}{a^2+b^2} = g(a,b)$

$\therefore$ g is continuous at $(a,b) \in R^2$ and $(a,b) \neq (0,0)$

At $(0,0)$

(i) $g(0,0) = 0$ is defined.

(ii) $\lim\limits_{(x,y)\to(0,0)} g(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^2+y^2}$

$= \lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2(mx)}{x^2+m^2x^2}$

$= \lim\limits_{(x,y)\to(0,0)} \dfrac{2mx^3}{x^2(1+m^2)}$

$= \lim\limits_{(x,y)\to(0,0)} \dfrac{2mx}{1+m^2} = 0$ exists.

(iii) $\lim\limits_{(x,y)\to(0,0)} g(x,y) = 0 = g(0,0)$

$\therefore$ g is continuous on $(0,0)$

$\therefore$ g is continuous every pt of R^2

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Partial Derivatives :-

Ex 8.12 Let $f(x,y) = x^3y + y^2x + 7$ for all $(x,y) \in R^2$. Calculate $\frac{\partial f}{\partial x}(-1,3)$ and $\frac{\partial f}{\partial y}(-2,1)$

Given $f(x,y) = x^3y + y^2x + 7$

$$\frac{\partial f}{\partial x} = y(3x^2) + y^2(1) + 0$$

$$= 3yx^2 + y^2$$

$$\frac{\partial f}{\partial x}(-1,3) = 3(3)(-1)^2 + 3^2$$

$$= 9 + 9 = \boxed{18}$$

$$\frac{\partial f}{\partial y} = x^3(1) + 2yx + 0$$

$$\frac{\partial f}{\partial y}(-2,1) = (-2)^3 + 2(-2)(1)$$

$$= -8 - 4 = \boxed{-12}$$

Ex 8.13

Let $f(x,y) = \sin(xy^2) + e^{x^3+5y}$ for all $(x,y) \in R^2$. calculate $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$

Given $f(x,y) = \sin(xy^2) + e^{x^3+5y}$

$$\frac{\partial f}{\partial x} = \cos(xy^2)(y^2) + e^{x^3+5y}(3x^2+0)$$

$$= \underline{\cos(xy^2)(y^2) + 3x^2 e^{x^3+5y}}$$

$$\frac{\partial f}{\partial y} = \cos(xy^2)(x\,2y) + e^{x^3+5y}(5)$$

$$= 2xy\cos(xy^2) + 5e^{x^3+5y}.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[2xy\cos(xy^2) + 5e^{x^3+5y}\right]$$

$$= \cos(xy^2)(2y) + 2xy(-\sin(xy^2)(y^2)) + 5e^{x^3+5y}(3x^2)$$

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$= 2y\cos(xy^2) - 2xy^3\sin(xy^2) + 15x^2e^{x^3+5y}$

$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$= \frac{\partial}{\partial y}\left[\cos(xy^2)(y^2) + 3x^2e^{x^3+5y}\right]$

$= \cos(xy^2)(2y) + y^2(-\sin(xy^2)(x\,2y) + 3\left[x^2e^{x^3+5y}(0+5) + e^{x^3+5y}(0)\right]$

$= 2y\cos(xy^2) - 2xy^3\sin(xy^2) + 15x^2e^{x^3+5y}$

8.14 Let $w(x,y) = xy + \frac{e^y}{y^2+1}$ for all $x, y \in R^2$. Calculate $\frac{\partial^2 w}{\partial x\partial y}$ and $\frac{\partial^2 w}{\partial y\partial x}$

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Given $w(x,y) = xy + \frac{e^y}{y^2+1}$

$\frac{\partial w}{\partial x} = y + \frac{(y^2+1)(0) + e^y(0)}{(y^2+1)^2} = y$

$\frac{\partial w}{\partial y} = x + \frac{(y^2+1)e^y - e^y(2y)}{(y^2+1)^2}$

$= x + \frac{e^y(y^2-2y+1)}{(y^2+1)^2}$

$\frac{\partial^2 w}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)$

$= \frac{\partial}{\partial x}\left[x + \frac{e^y(y^2-2y+1)}{(y^2+1)^2}\right]$

$= 1 + 0 = \boxed{1}$

$\frac{\partial^2 w}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)$

$= \frac{\partial}{\partial y}(y) = \boxed{1}$

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EXERCISE 8.4

1. Find the partial derivatives of the following

(i) $f(x,y) = 3x^2 - 2xy + y^2 + 5x + 2 \quad (2,-5)$

$\frac{\partial f}{\partial x} = 6x - 2y + 0 + 5$

$\frac{\partial f}{\partial x}(2,-5) = 6(2) - 2(-5) + 5$

$= 12 + 10 + 5 = \boxed{27}$

$\frac{\partial f}{\partial y} = 0 - 2x + 2y + 0 + 0$

$\frac{\partial f}{\partial y}(2,-5) = -2(2) + 2(-5)$

$= -4 - 10 = \boxed{-14}$

(ii) $g(x,y) = 3x^2 + y^2 + 5x + 2 \quad (1,-2)$

$\frac{\partial g}{\partial x} = 6x + 5$

$\frac{\partial g}{\partial x}(1,-2) = 6(1) + 5 = 6 + 5 = \boxed{11}$

$\frac{\partial g}{\partial y} = 0 + 2y + 0 + 0$

$\frac{\partial g}{\partial y}(1,-2) = 2(-2) = \boxed{-4}$

(iii) $h(x,y,z) = x \sin(xy) + z^2 x \quad (2, \pi/4, 1)$

$\frac{\partial h}{\partial x} = 1 \sin(xy) + x \cos(xy)(y) + z^2$

$\frac{\partial h}{\partial x} = \sin(xy) + xy \cos(xy) + z^2$

$\frac{\partial h}{\partial x}(2, \pi/4, 1) = \sin\left(2 \times \frac{\pi}{4}\right) + 2 \times \frac{\pi}{4} \cos\left(2 \times \frac{\pi}{4}\right) + 1^2$

$= \sin \pi/2 + \pi/2 \cos \pi/2 + 1$

$= 1 + \pi/2 (0) + 1$

$= 1 + 1 = \boxed{2}$

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$h(x,y,z) = x \sin(xy) + z^2 x$

$\frac{\partial h}{\partial y} = x \cos(xy)(x) + \sin(xy) \cdot 0 + z^2$

$= x^2 \cos(xy) + z^2$

$\frac{\partial h}{\partial y}(2, \pi/4, 1) = 2^2 \cos(2 \times \pi/4) + 1$

$= 4 \cos(\pi/2) + 1$

$= 4(0) + 1 = \boxed{1}$

$h(x,y,z) = x \sin(xy) + z^2 x$

$\frac{\partial h}{\partial z} = 0 + x(2z) = 2xz$

$\frac{\partial h}{\partial z}(2, \pi/4, 1) = 2(2)(1) = \boxed{4}$

(iv) $G_1(x,y) = e^{x+3y} \log(x^2+y^2), (-1,1)$

$\frac{\partial G}{\partial x} = e^{x+3y} \cdot \frac{1}{x^2+y^2}(2x) + \log(x^2+y^2) e^{x+3y}(1+0)$

$\frac{\partial h}{\partial x} = e^{x+3y} \frac{2x}{x^2+y^2} + \log(x^2+y^2) e^{x+3y}$

$\frac{\partial G}{\partial x}(-1,1) = e^{-1+3} \frac{2(-1)}{1+1} + \log(1+1) e^{-1+3(1)}$

$= e^2(-1) + \log 2 \, e^2$

$= e^2[\log 2 - 1]$

$\frac{\partial G}{\partial y} = e^{x+3y} \frac{1}{x^2+y^2}(2y) + \log(x^2+y^2) \cdot e^{x+3y}(3)$

$= e^{x+3y}\left[3\log(x^2+y^2) + \frac{2y}{x^2+y^2}\right]$

$\frac{\partial h}{\partial y}(-1,1) = e^2\left[3\log 2 + \frac{2}{1+1}\right]$

$= e^2[3\log 2 + 1]$

$= e^2[\log 2^3 + 1]$

$= e^2[\log 8 + 1]$ //

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2) Find f_x, f_y and show that $f_{xy} = f_{yx}$

a) $f(x,y) = \frac{3x}{y+\sin x}$

$f_x = \frac{\partial f}{\partial x} = \frac{(y+\sin x)3 - 3x(0+\cos x)}{(y+\sin x)^2}$

$= \frac{3y + 3\sin x - 3x\cos x}{(y+\sin x)^2}$

$f(y) = \frac{\partial f}{\partial y} = \frac{(y+\sin x)\cdot 0 - 3x(1+0)}{(y+\sin x)^2}$

$= \frac{-3x}{(y+\sin x)^2}$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$= \frac{\partial}{\partial x}\left(\frac{-3x}{(y+\sin x)^2}\right)$

$= \frac{(y+\sin x)^2(-3) - (-3x)\left[2(y+\sin x)\cdot (0+\cos x)\right]}{(y+\sin x)^4}$

$= \frac{(y+\sin x)^2(-3) + 3x\left[(2(y+\sin x)\cos x\right]}{(y+\sin x)^4}$

$= \frac{(y+\sin x)\left[6x\cos x - 3(y+\sin x)\right]}{(y+\sin x)^4}$

$= \frac{(y+\sin x)\left[6x\cos x - 3y - 3\sin x\right]}{(y+\sin x)^4}$

$= \frac{6x\cos x - 3y - 3\sin x}{(y+\sin x)^3}$ // ——— (1)

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$

$= \frac{\partial}{\partial y}\left[\frac{3y + 3\sin x - 3x\cos x}{(y+\sin x)^2}\right]$

$= \frac{(y+\sin x)^2(3) - \left[3y+3\sin x - 3x\cos x\right](2(y+\sin x)(1))}{(y+\sin x)^4}$

$= \frac{(y+\sin x)\left[3(y+\sin x) - 2(3y+3\sin x - 3x\cos x\right]}{(y+\sin x)^4}$

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$$= \frac{3y + 3\sin x - 6y - 6\sin x + 6x\cos x}{(y+\sin x)^3}$$

$$= \frac{6x\cos x - 3\sin x - 3y}{(y+\sin x)^3} \quad \text{——— (2)}$$

From (1) and (2)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

(ii) $f(x,y) = \tan^{-1}\left(\frac{x}{y}\right)$

Given $f(x,y) = \tan^{-1}\left(\frac{x}{y}\right)$

$$f(x) = f_x = \frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{1}{y}$$

$$= \frac{1}{\frac{x^2+y^2}{y^2}} \times \frac{1}{y}$$

$$= \frac{y^2}{x^2+y^2} \cdot \frac{1}{y} = \frac{y}{x^2+y^2}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{}{1+\left(\frac{x}{y}\right)^2} \cdot x\left(-\frac{1}{y^2}\right)$$

$$= \frac{1}{\frac{x^2+y^2}{y^2}} \left(-\frac{x}{y^2}\right)$$

$$= \frac{y^2}{x^2+y^2}\left(-\frac{x}{y^2}\right) = -\frac{x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[-\frac{x}{x^2+y^2}\right]$$

$$= -\frac{\left[x^2+y^2(1) - x(2x)\right]}{(x^2+y^2)^2}$$

$$= -\frac{\left[x^2+y^2-2x^2\right]}{(x^2+y^2)^2} = \frac{-x^2-y^2+2x^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2} \quad \text{——— (1)}$$

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$$\frac{\partial^2 f}{\partial y \partial x} = f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left[\frac{y}{x^2+y^2}\right]$$

$$= \frac{(x^2+y^2)\cdot 1 - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad \text{---} (2)$$

From (1) and (2) $f_{xy} = f_{yx}$.

(iii) $f(x,y) = \cos(x^2-3xy)$

$$\frac{\partial f}{\partial x} = f_x = -\sin(x^2-3xy)(2x-3y)$$

$$f_y = \frac{\partial f}{\partial y} = -\sin(x^2-3xy)(-3x)$$

$$= 3x\sin(x^2-3xy)$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}\left[3x\sin(x^2-3xy)\right]$$

$$= 3\left[x\cos(x^2-3xy)(2x-3y) + \sin(x^2-3xy)\cdot 1\right]$$

$$= 3x\cos(x^2-3xy)(2x-3y) + 3\sin(x^2-3xy) \quad \text{---} (1)$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$$

$$= \frac{\partial}{\partial y}\left[-(2x-3y)\sin(x^2-3xy)\right]$$

$$= -\left[(2x-3y)\cos(x^2-3xy)(-3x) + \sin(x^2-3xy)(0-3)\right]$$

$$= 3x(2x-3y)\cos(x^2-3xy) + 3\sin(x^2-3xy) \quad \text{---} (2)$$

From (1) and (2) $f_{xy} = f_{yx}$

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3) If $u(x,y,z) = \frac{x^2+y^2}{xy} + 3z^2y$, find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$.

Given $u(x,y,z) = \frac{x^2+y^2}{xy} + 3z^2y$

$$\frac{\partial u}{\partial x} = \frac{xy(2x) - (x^2+y^2)(y)}{(xy)^2} + 0$$

$$= \frac{2x^2y - x^2y - y^3}{(xy)^2} = \frac{x^2y - y^3}{(xy)^2}$$

$$= \frac{y(x^2-y^2)}{x^2y^2} = \frac{x^2-y^2}{x^2y} //$$

$$\frac{\partial u}{\partial y} = \frac{xy(2y) - (x^2+y^2)x}{(xy)^2} + 3z^2$$

$$= \frac{2xy^2 - x^3 - y^2x}{x^2y^2} + 3z^2$$

$$= \frac{xy^2 - x^3 + 3x^2y^2z^2}{x^2y^2}$$

$$= \frac{x(y^2 - x^2 + 3xy^2z^2)}{x^2y^2}$$

$$= \frac{y^2 - x^2 + 3xy^2z^2}{xy^2} //$$

$$\frac{\partial u}{\partial z} = \frac{xy(0) + (x^2+y^2)(0)}{(xy)^2} + 3y(2z)$$

$$= \underline{\underline{6yz}}$$

4) If $u(x,y,z) = \log(x^3+y^3+z^3)$, find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$.

Given $u(x,y,z) = \log(x^3+y^3+z^3)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3}(3x^2)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3} \cdot 3y^2$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3} 3z^2$$

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$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{1}{x^3+y^3+z^3}\left[3x^2+3y^2+3z^2\right]$$

$$=\frac{3(x^2+y^2+z^2)}{(x^3+y^3+z^3)}\ //$$

---

5) For each of the following functions find g_{xy}, g_{xx}, g_{yy} and g_{yx}.

(i) $g(x,y)=xe^y+3x^2y$

$$g_x=\frac{\partial g}{\partial x}=\boxed{e^y+6xy}$$

$$g_{xx}=\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right)=\boxed{6y}$$

$$g_y=\frac{\partial g}{\partial y}=\boxed{xe^y(1)+3x^2}$$

$$g_{yy}=\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial y}\right)=xe^y+0=\boxed{xe^y}$$

$$g_{xy}=\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right)=\frac{\partial}{\partial x}\left(xe^y+3x^2\right)$$

$$=\boxed{e^y+6x}$$

$$g_{yx}=\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right)$$

$$=\frac{\partial}{\partial y}\left(e^y+6xy\right)$$

$$=\boxed{e^y+6x}$$

(ii) $g(x,y)=\log(5x+3y)$

$$g_x=\frac{\partial g}{\partial x}=\frac{1}{5x+3y}\cdot 5=\frac{5}{5x+3y}$$

$$g_y=\frac{\partial g}{\partial y}=\frac{1}{5x+3y}\cdot 3=\frac{3}{5x+3y}$$

$$g_{xx}=\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{5}{5x+3y}\right)$$

$$=\frac{(5x+3y)\cdot 0-5(5)}{(5x+3y)^2}$$

$$=-\frac{25}{(5x+3y)^2}\ //$$

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$$g_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left[\frac{3}{5x+3y}\right]$$

$$= \frac{(5x+3y)\cdot 0 - 3(3)}{(5x+3y)^2}$$

$$= -\frac{9}{(5x+3y)^2} //$$

$$g_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{3}{5x+3y}\right]$$

$$= \frac{(5x+3y)\cdot 0 - 3(5)}{(5x+3y)^2}$$

$$= \frac{-15}{(5x+3y)^2} //$$

$$g_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial y}\left[\frac{5}{5x+3y}\right]$$

$$= \frac{(5x+3y)\cdot 0 - 5(3)}{(5x+3y)^2} = \frac{-15}{(5x+3y)^2} //$$

(iii) $g(x,y) = x^2 + 3xy - 7y + \cos(5x)$

$$\frac{\partial g}{\partial x} = 2x + 3y - \sin(5x)\cdot 5$$

$$= 2x + 3y - 5\sin(5x) //$$

$$\frac{\partial g}{\partial y} = 3x - 7 + 0 = 3x - 7 //$$

$$g_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial x}\left[2x+3y-5\sin(5x)\right]$$

$$= 2 - 5\cos(5x)\cdot 5$$

$$= 2 - 25\cos(5x) //$$

$$g_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left[3x-7\right] = 0 //$$

$$g_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial}{\partial x}\left[3x-7\right]$$

$$= 3 //$$

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$$g_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(2x + 3y - 5\sin(5x)\right)$$

$$= 3 \quad //$$

---

6) Let $w(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$, show that $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2} = 0$.

Given $w(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

$$= (x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$\frac{\partial w}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)$$

$$= -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left[-x(x^2+y^2+z^2)^{-3/2}\right]$$

$$= -\left[x\left(-\tfrac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) + (x^2+y^2+z^2)^{-3/2}(1)\right]$$

$$= -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2} \quad \text{---} (1)$$

III$_y$

$$\frac{\partial^2 w}{\partial y^2} = -(x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2} \quad \text{---} (2)$$

and $$\frac{\partial^2 w}{\partial z^2} = -(x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2} \quad \text{---} (3)$$

Now $$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

$$= (1) + (2) + (3)$$

$$= -3(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-5/2}(3x^2+3y^2+3z^2)$$

$$= -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{-\frac{5}{2}+1}$$

$$= -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{-3/2}$$

$$= 0 \quad //$$

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7) If $V(x,y) = e^x(x\cos y - y\sin y)$, then prove that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

Given

$V(x,y) = e^x(x\cos y - y\sin y)$

$\frac{\partial V}{\partial x} = e^x[\cos y] + [x\cos y - y\sin y]e^x$

$= e^x[\cos y + x\cos y - y\sin y]$

$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)$

$= e^x[\cos y] + [\cos y + x\cos y - y\sin y]e^x$

$= e^x[\cos y + \cos y + x\cos y - y\sin y]$

$= e^x[2\cos y + x\cos y - y\sin y]$ — ①

$\frac{\partial V}{\partial y} = e^x[x(-\sin y) - [y\cos y + \sin y]]$

$+(x\cos y - y\sin y).0$

$= -e^x[x\sin y + y\cos y + \sin y]$

$\frac{\partial^2 V}{\partial y^2} = \frac{\partial}{\partial y}\left[-e^x(x\sin y + y\cos y + \sin y)\right]$

$= -e^x[x\cos y + y(-\sin y) + \cos y + \cos y] +$

$(x\sin y + y\cos y + \sin y).0$

$= -e^x[x\cos y - y\sin y + 2\cos y]$ — ②

① + ② ⇒ $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$

$= (x\cos y - y\sin y + 2\cos y)(e^x - e^x)$

$= (x\cos y - y\sin y + 2\cos y).0$

$= 0$ //

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8) If $w(x,y) = xy + \sin(xy)$, then prove that $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$

Given $w(x,y) = xy + \sin(xy)$

$$\frac{\partial w}{\partial x} = y + \cos(xy)\cdot y$$

$$\frac{\partial w}{\partial y} = x + \cos(xy)\cdot x$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial x}\left[x + x\cos xy\right]$$

$$= 1 + x(-\sin xy)(y) + \cos(xy)\cdot 1$$

$$= 1 - xy\sin xy + \cos xy \quad \text{—— (1)}$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left[y + y\cos(xy)\right]$$

$$= 1 + y(-\sin(xy)\cdot x) + \cos xy \cdot 1]$$

$$= 1 - xy\sin(xy) + \cos xy \quad \text{— (2)}$$

From (1) and (2) $\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x}$

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9) If $V(x,y,z) = x^3 + y^3 + z^3 + 3xyz$ show that $\frac{\partial^2 V}{\partial y \partial z} = \frac{\partial^2 V}{\partial z \partial y}$

Given $V(x,y,z) = x^3 + y^3 + z^3 + 3xyz$

$$\frac{\partial V}{\partial y} = 3y^2 + 3xz$$

$$\frac{\partial V}{\partial z} = 3z^2 + 3xy$$

$$\frac{\partial^2 V}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z}\right)$$

$$= \frac{\partial}{\partial y}\left[3z^2 + 3xy\right] = \boxed{3x} \quad \text{—— (1)}$$

$$\frac{\partial^2 V}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial y}\right)$$

$$= \frac{\partial}{\partial z}\left(3y^2 + 3xz\right) = \boxed{3x} \quad \text{— (2)}$$

From (1) and (2).

$$\frac{\partial^2 V}{\partial y \partial z} = \frac{\partial^2 V}{\partial z \partial y} \quad //$$

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16) A firm produces two types of calculators each week, x numbers of type A and y number of each type B. The weekly revenue and cost functions in rupees are $R(x,y) = 80x + 90y + 0.04xy - 0.05x^2 - 0.05y^2$ and $C(x,y) = 8x + 6y + 2000$ respectively.

(i) Find the profit function $p(x,y)$

(ii) Find $\frac{\partial p}{\partial x}(1200, 1800)$, $\frac{\partial p}{\partial y}(1200, 1800)$

Given $R(x,y) = 80x + 90y + 0.04xy - 0.05x^2 - 0.05y^2$

$C(x,y) = 8x + 6y + 2000$.

(i) profit = Revenue − Cost

$= 80x + 90y + 0.04xy - 0.05x^2 - 0.05y^2 - [8x + 6y + 2000]$

$= 72x + 84y + 0.04xy - 0.05x^2 - 0.05y^2 - 2000$.

(ii) $\frac{\partial p}{\partial x} = 72 + 0.04y - 0.05(2x)$

$\frac{\partial p}{\partial x}(1200, 1800) = 72 + 0.04(1800) - 0.05(2 \times 1200)$

$= 72 + 72 - 0.05(24 \times 100)$

$= 72 + 72 - 120$

$= 144 - 120$

$= \boxed{24}$

$\frac{\partial p}{\partial y} = 84 + 0.04(x) - 0.05(2y)$

$\frac{\partial p}{\partial y}(1200, 1800) = 84 + 0.04(1200) - 0.05(2 \times 18 \times 100)$

$= 84 + 48 - 180$

$= 132 - 180 = \boxed{-48}$

$\therefore \frac{\partial p}{\partial x}(1200, 1800) = 24$

$\frac{\partial p}{\partial y}(1200, 1800) = -48$

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Linear Approximation and Differential of a function of several variable.

Defn:-

i) The linear approximation of f at $(x_0, y_0) \in A$ is defined to be

$$f(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

(ii) The differential of f is defined to be

$$df = \frac{\partial f}{\partial x}(x,y)\,dx + \frac{\partial f}{\partial y}(x,y)\,dy$$

(iii) The differential of f is defined to be

$$df = \frac{\partial f}{\partial x}(x,y)\,dx + \frac{\partial f}{\partial y}(x,y)\,dy$$

where $dx = \Delta x$ and $dy = \Delta y$.

Defn:- The linear approximation of f at $(x_0, y_0, z_0) \in A$ is defined to be

$$f(x,y,z) = f(x_0,y_0,z_0) + \frac{\partial f}{\partial x}(x_0,y_0,z_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0,z_0)(y-y_0) + \frac{\partial f}{\partial z}(x_0,y_0,z_0)(z-z_0)$$

(ii) The differential of f is defined to be

$$df = \frac{\partial f}{\partial x}(x,y,z)\,dx + \frac{\partial f}{\partial y}(x,y,z)\,dy + \frac{\partial f}{\partial z}(x,y,z)\,dz.$$

where $\Delta x = dx$, $dy = \Delta y$, $dz = \Delta z$.

Ex 8.16 If $w(x,y,z) = x^2y + y^2z + z^2x$, $x, y, z \in R$, find the differential dw.

Given

$$w(x,y,z) = x^2y + y^2z + z^2x$$

$$w_x = \frac{\partial w}{\partial x} = 2xy + z^2$$

$$w_y = \frac{\partial w}{\partial y} = x^2 + 2yz$$

$$w_z = \frac{\partial w}{\partial z} = y^2 + 2zx$$

WKT, $dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$

$$= (2xy + z^2)\,dx + (x^2 + 2yz)\,dy + (y^2 + 2zx)\,dz.$$

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Ex 8.17 If $u(x,y,z) = x^2 - xy + 3\sin z$, $x,y,z \in R$. find the linear approximation for u at $(2,-1,0)$

Given $u(x,y,z) = x^2 - xy + 3\sin z$.

linear approximation is

$$L(x,y,z) = u(x_0,y_0,z_0) + \left(\frac{\partial u}{\partial x}\right)_{(x_0,y_0,z_0)} (x-x_0) + \left(\frac{\partial u}{\partial y}\right)_{(x_0,y_0,z_0)} (y-y_0) + \frac{\partial u}{\partial z}(x_0,y_0,z_0)(z-z_0) \quad \text{—— (1)}$$

Now $u(x_0,y_0,z_0) = u(2,-1,0)$

$= 2^2 - 2(-1) + 3\sin(0)$

$= 4 + 2 + 0 = \boxed{6}$

$\frac{\partial u}{\partial x} = 2x - y$

$\frac{\partial u}{\partial x}(2,-1,0) = 2(2) - (-1) = 4 + 1 = \boxed{5}$

$\frac{\partial u}{\partial y} = -x$

$\frac{\partial u}{\partial y}(2,-1,0) = \boxed{-2}$

$\frac{\partial u}{\partial z} = 3\cos z$

$\frac{\partial u}{\partial z}(2,-1,0) = 3\cos(0) = 3(1) = \boxed{3}$

∴ from (1)

linear approximation is

$6 + 5(x-2) + (-2)(y+1) + 3(z-0)$

$= 6 + 5x - 10 - 2y - 2 + 3z$

$= 5x - 2y + 3z - 6$

× ——— ×

EXERCISE 8.5

1) If $w(x,y) = x^3 - 3xy + 2y^2$, $x,y \in R$ find the linear approximation for w at $(1,-1)$

Given $w(x,y) = x^3 - 3xy + 2y^2$

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Linear approximation is

$$L(x,y) = w(x_0,y_0) + \frac{\partial w}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial w}{\partial y}(x_0,y_0)(y-y_0) \quad \text{—①}$$

$$w(x_0,y_0) = w(1,-1)$$
$$= 1 - 3(1)(-1) + 2(-1)^2$$
$$= 1+3+2 = \boxed{6}$$

$$\frac{\partial w}{\partial x} = 3x^2 - 3y$$

$$\frac{\partial w}{\partial x}(1,-1) = 3(1)^2 - 3(-1) = 3+3 = \boxed{6}$$

$$\frac{\partial w}{\partial y} = -3x + 4y$$

$$\frac{\partial w}{\partial y}(1,-1) = -3(1) + 4(-1) = -3-4 = \boxed{-7}$$

∴ linear approximation is

$$L(x,y) = 6 + 6(x-1) + (-7)(y+1)$$
$$= 6 + 6x - 6 - 7y - 7$$
$$= \boxed{6x - 7y - 7}$$

2) Let $z(x,y) = x^2y + 3xy^4$, $x,y \in R$ find the linear approximation for z at $(2,-1)$

Given $z(x,y) = x^2y + 3xy^4$, $x,y \in R$

∴ The linear approximation is

$$L(x,y) = z(x_0,y_0) + \frac{\partial z}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial z}{\partial y}(x_0,y_0)(y-y_0) \quad \text{—①}$$

$$z(x_0,y_0) = z(2,-1)$$
$$= 2^2(-1) + 3(2)(-1)^4$$
$$= -4 + 6 = \boxed{2}$$

$$\frac{\partial z}{\partial x} = 2xy + 3y^4$$

$$\frac{\partial z}{\partial x}(2,-1) = 2(2)(-1) + 3(-1)^4$$
$$= -4 + 3 = \boxed{-1}$$

$$\frac{\partial z}{\partial y} = x^2 + 12xy^3$$

$$\frac{\partial z}{\partial y}(2,-1) = 4 + 12(2)(-1)^3$$
$$= 4 - 24 = \boxed{-20}$$

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$\therefore$ ① $\Rightarrow$ $2 + (-1)(x-2) + (-20)(y+1)$

i.e. $L(x,y) = 2 - x + 2 - 20y - 20$

$= -x - 20y - 16$

$= -(x + 20y + 16)$

---

3) If $V(x,y) = x^2 - xy + \frac{1}{4}y^2 + 7$, $x, y \in \mathbb{R}$, find the differential dV.

Given $V(x,y) = x^2 - xy + \frac{1}{4}y^2 + 7$

WKT, $dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy$ ——— ①

$\frac{\partial V}{\partial x} = 2x - y$

$\frac{\partial V}{\partial y} = -x + \frac{1}{4}(2y) = -x + \frac{y}{2}$

$\therefore$ ① $\Rightarrow$ $dV = (2x - y)dx + \left(-x + \frac{1}{2}y\right)dV$

---

4) Let $w(x,y,z) = x^2 - xy + 3\sin z$, $x, y, z \in \mathbb{R}$. Find the linear approximation at $(2,-1,0)$

Given,

$w(x,y,z) = x^2 - xy + 3\sin z$.

Linear approximation is

$L(x,y,z) = w(x_0,y_0,z_0) + \frac{\partial w}{\partial x}(x_0,y_0,z_0)(x - x_0) + \frac{\partial w}{\partial y}(x_0,y_0,z_0)(y - y_0) + \frac{\partial w}{\partial z}(x_0,y_0,z_0)(z - z_0)$. ——— ①.

$\frac{\partial w}{\partial x} = 2x - y$

$\frac{\partial w}{\partial x}(2,-1,0) = 2(2) - (-1) = 4 + 1 = \boxed{5}$

$\frac{\partial w}{\partial y}(2,-1,0) = -x = \boxed{-2}$

$\frac{\partial w}{\partial z}(2,-1,0) = 3\cos z = 3\cos 0 = \boxed{3}$

$w(x_0,y_0,z_0) = w(2,-1,0)$

$= 4 - 2(-1) + 3\sin(0)$

$= 4 + 2 + 0$

$= \boxed{6}$

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① ⇒ $L(x,y,z)$

$= 6 + 5(x-2) + (-2)(y+1) + 3(z-0)$

$= 6 + 5x - 10 - 2y - 2 + 3z$

$= 5x - 2y + 3z - 6$

---

5) Let $V(x,y,z) = xy + yz + zx$, $x,y,z \in R$ Find the differential dV

Given $V(x,y,z) = xy + yz + zx$

$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$ ——①

$\frac{\partial V}{\partial x} = y + z$,

$\frac{\partial V}{\partial y} = x + z$

$\frac{\partial V}{\partial z} = y + x$

∴ ① ⇒ $dV = (y+z)dx + (x+z)dy + (y+x)dz$

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Function of Function Rule (chain rule)

If $w(x,y)$ and $x = f(t)$, $y = g(t)$, Then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

Ex 8.18 Verify the above theorem for $f(x,y) = x^2 - 2y^2 + 2xy$ and $x(t) = \cos t$, $y(t) = \sin t$, $t \in [0, 2\pi]$

Given $f(x,y) = x^2 - 2y^2 + 2xy$

$x(t) = \cos t$, $y(t) = \sin t$.

$\therefore f(x,y) = \cos^2 t - 2\sin^2 t + 2\sin t \cos t$.

$$\frac{df}{dt} = 2\cos t(-\sin t) - 2(2)\sin t(\cos t) + 2[\sin t(-\sin t) + \cos t(\cos t)]$$

$$= -2\sin t\cos t - 4\sin t\cos t - 2\sin^2 t + 2\cos^2 t.$$

$$= 2\cos^2 t - 2\sin^2 t - 6\sin t\cos t \quad \text{—— (1)}$$

WKT $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$.

$F(x,y) = x^2 - 2y^2 + 2xy$

$x = \cos t$ | $y = \sin t$.

$\frac{\partial f}{\partial x} = 2x + 2y$ | $\frac{\partial f}{\partial y} = -4y + 2x$

$\frac{dx}{dt} = -\sin t$ | $\frac{dy}{dt} = \cos t$.

$\therefore \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$.

$= (2x+2y)(-\sin t) + (-4y+2x)\cos t$.

$= (2\cos t + 2\sin t)(-\sin t) + (-4\sin t + 2\cos t)\cos t$.

$= -2\sin t\cos t - 2\sin^2 t - 4\sin t\cos t + 2\cos^2 t$.

$= 2\cos^2 t - 2\sin^2 t - 6\sin t\cos t \quad \text{—— (2)}$.

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From ① and ②

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}.$$

Hence proved.

Ex 8.19 Let $g(x,y) = x^2 - yx + \sin(x+y)$, $x(t) = e^{3t}$, $y(t) = t^2$, $t \in R$. Find $\frac{dy}{dt}$.

Given $g(x,y) = x^2 - yx + \sin(x+y)$

$x(t) = e^{3t}$ $\quad y(t) = t^2$

WKT $\frac{dg}{dt} = \frac{\partial g}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial g}{\partial y}\cdot\frac{dy}{dt}$.

$\frac{\partial g}{\partial x} = 2x - y + \cos(x+y)$ (1)

$= 2x - y + \cos(x+y)$

$\frac{dx}{dt} = e^{3t}\cdot(3)$

$= 3e^{3t}$.

$\frac{\partial g}{\partial y} = -x + \cos(x+y)$ (1)

$= -x + \cos(x+y)$

$\frac{dy}{dt} = 2t$.

Now $\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\cdot\frac{dy}{dt}$.

$= [2x - y + \cos(x+y)]\,3e^{3t} + [-x + \cos(x+y)]\,2t$.

$= [2e^{3t} - t^2 + \cos(e^{3t}+t^2)]\,3e^{3t} + [-e^{3t} + \cos(e^{3t}+t^2)]\,2t$

$= 6e^{6t} - 3t^2e^{3t} + 3\cos(e^{3t}+t^2)e^{3t} - 2te^{3t} + 2t\cos(e^{3t}+t^2)$

Ex 8.20 Let $g(x,y) = 2y + x^2$, $x = 2r - s$, $y = r^2 + 2s$, $r, s \in R$. Find $\frac{\partial g}{\partial r}$, $\frac{\partial g}{\partial s}$.

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WKT $\frac{\partial g}{\partial r} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial r}$

$g(x,y) = 2y + x^2$ | $g(x,y) = 2y + x^2$

$x = 2r - s$ | $y = r^2 + 2s$

$\frac{\partial g}{\partial x} = 2x$ | $\frac{\partial g}{\partial y} = 2$

$\frac{\partial x}{\partial r} = 2$ | $\frac{\partial y}{\partial r} = 2r$

$\therefore \frac{\partial g}{\partial r} = 2x(2) + 2(2r)$

$= 4x + 4r$

$= 4(2r - s) + 4r$

$= 8r - 4s + 4r$

$= \underline{\underline{12r - 4s}}$.

WKT $\frac{\partial g}{\partial s} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial s}$

$\frac{\partial g}{\partial x} = 2x$ | $\frac{\partial g}{\partial y} = 2$

$\frac{\partial x}{\partial s} = -1$ | $\frac{\partial y}{\partial s} = 2$

$\therefore \frac{\partial g}{\partial s} = 2x(-1) + 2(2)$

$= -2x + 4$

$= -2(2r - s) + 4$

$= -4r + 2s + 4$

EXERCISE 8.6

1) If $u(x,y) = x^2y + 3xy^4$, $x = e^t$, $y = \sin t$, find $\frac{du}{dt}$ and evaluate it at $t = 0$.

$u(x,y) = x^2y + 3xy^4$

$x = e^t$, $y = \sin t$

WKT $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$ ——— (1)

$\frac{\partial u}{\partial x} = 2xy + 3y^4$ | $\frac{\partial u}{\partial y} = x^2 + 12xy^3$

$\frac{dx}{dt} = e^t$ | $\frac{dy}{dt} = \cos t$.

$\therefore$ (1) $\Rightarrow (2xy + 3y^4)e^t + (x^2 + 12xy^3)\cos t$

i.e. $(2e^t \sin t + 3\sin^4 t)e^t + (e^{2t} + 12e^t \sin^3 t)\cos t$.

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$$\frac{du}{dt}(t=0) = \left[2e^0 \sin 0 + 3\sin^4(0)\right] e^0 + \left[e^0 + 12e^0 \sin^3(0)\right] \cos 0$$

$$= \left[2(1)(0) + 3(0)\right](1) + \left[1 + 12(1)(0)\right] 1$$

$$= (0+0) + (1+0)(1)$$

$$= \boxed{1}$$

---

2) If $u(x,y) = xy^2z^3$, $x = \sin t$, $y = \cos t$, $z = 1+e^{2t}$, find $\frac{du}{dt}$.

Given $u(x,y) = xy^2z^3$

$x = \sin t$

$y = \cos t$

WKT $$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} \quad \text{——(1)}$$

$$\frac{\partial u}{\partial x} = y^2z^3, \quad \frac{\partial u}{\partial y} = 2xyz^3, \quad \frac{\partial u}{\partial z} = 3xy^2z^2.$$

$$\frac{dx}{dt} = \cos t, \quad \frac{dy}{dt} = -\sin t, \quad \frac{dz}{dt} = 2e^{2t}.$$

$\therefore$ (1) $\Rightarrow$

$$\frac{du}{dt} = (y^2z^3)\cos t + (2xyz^3)(-\sin t) + (3xy^2z^2)\,2e^{2t}.$$

$$= \cos^2 t(1+e^{2t})^3 \cos t + 2\cos t \sin t (1+e^{2t})^3(-\sin t) + 3\sin t \cos^2 t (1+e^{2t})^2 \cdot 2e^{2t}.$$

$$= \cos^3 t(1+e^{2t})^3 - 2\sin^2 t \cos t (1+e^{2t})^3 + 6e^{2t}\sin t \cos^2 t (1+e^{2t})^2$$

$$= (1+e^{2t})^2\left[\cos^3 t(1+e^{2t}) - \sin t \cdot \sin 2t (1+e^{2t}) + 3e^{2t}\sin 2t \cos t\right]$$

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3) If $w(x,y,z) = x^2+y^2+z^2$,
$x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$
find $\frac{dw}{dt}$.

Given $w(x,y,z) = x^2+y^2+z^2$
$x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$.

WKT,
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt} \quad \text{---------- (1)}$$

$\frac{\partial w}{\partial x} = 2x$, $\frac{\partial w}{\partial y} = 2y$, $\frac{\partial w}{\partial z} = 2z$.

$\frac{dx}{dt} = e^t$, $\frac{dy}{dt} = e^t \cos t + \sin t\, e^t$.

$\frac{dz}{dt} = e^t(-\sin t) + \cos t\, e^t$

$\therefore$ (1) $\Rightarrow$ $\frac{dw}{dt} = 2x e^t + 2y[e^t \cos t + e^t \sin t] + 2z[e^t \cos t - e^t \sin t]$

$= 2e^t \cdot e^t + 2e^t \sin t[e^t \cos t + e^t \sin t] + 2e^t \cos t[e^t \cos t - e^t \sin t]$

$= 2e^{2t} + 2e^{2t}\sin t \cos t + 2e^{2t}\sin^2 t + 2e^{2t}\cos^2 t - 2e^{2t}\sin t \cos t$.

$= 2e^{2t} + 2e^{2t}(\sin^2 t + \cos^2 t)$

$= 2e^{2t} + 2e^{2t}(1)$

$= 4e^{2t}$

---

4) Let $u(x,y,z) = xyz$,
$x = e^{-t}$, $y = e^{-t}\cos t$, $z = \sin t$
Find $\frac{du}{dt}$.

Given $u(x,y,z) = xyz$
$x = e^{-t}$, $y = e^{-t}\cos t$, $z = \sin t$

WKT $\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$. ---------- (1)

$\frac{\partial u}{\partial x} = yz$

$\frac{\partial u}{\partial y} = xz$

$\frac{\partial u}{\partial z} = xy$

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$\frac{dx}{dt} = e^{-t}(-1)$, $\frac{dy}{dt} = e^{-t}(-\sin t) + \cos t\, e^{-t}(-1)$

$\frac{dz}{dt} = \cos t$.

① $\Rightarrow \frac{du}{dt} = yz(-e^{-t}) + xz\left[-e^{-t}\sin t + \cos t\, e^{-t}\right]$

$+ xy\cos t$

$= e^{-t}\cos t \sin t(-e^{-t}) + e^{-t}\sin t\left[-e^{-t}\sin t + e^{-t}\cos t\right]$

$+ e^{-t}e^{-t}\cos t \cdot \cos t$

$= -e^{-2t}\sin t\cos t - e^{-2t}\sin^2 t + e^{-2t}\sin t\cos t$

$+ e^{-2t}\cos^2 t$.

$= -2e^{-2t}\sin t\cos t + e^{-2t}(\cos^2 t - \sin^2 t)$

$= -e^{-2t}\sin 2t + e^{-2t}\cos 2t$

$= e^{-2t}(\cos 2t - \sin 2t)$

5) If $w(x,y) = 6x^3 - 3xy + 2y^2$, $x = e^s$, $y = \cos s$, $s \in R$, find $\frac{dw}{ds}$ and evaluate at $s = 0$.

Given $w(x,y) = 6x^3 - 3xy + 2y^2$

$x = e^s$ $\quad y = \cos s$.

WKT $\frac{dw}{ds} = \frac{\partial w}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial w}{\partial y}\cdot\frac{dy}{ds}$ ——①

$\frac{\partial w}{\partial x} = 18x^2 - 3y$

$\frac{\partial w}{\partial y} = -3x + 4y$

$\frac{dx}{ds} = e^s$

$\frac{dy}{ds} = -\sin s$.

Now ① $\Rightarrow (18x^2 - 3y)e^s + (-3x + 4y)(-\sin s)$

$= (18e^{2s} - 3\cos s)e^s + (-3e^s + 4\cos s)(-\sin s)$

$= 18e^{3s} - 3e^s\cos s + 3e^s\sin s - 4\sin s\cos s$ //

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$$\frac{dw}{ds}(s=0) = 18e^0 - 3e^0\cos(0) + 3e^0\sin(0) - 4\sin(0)\cos(0)$$

$$= 18 - 3(1)(1) + 3(1)0 - 4(0)(1)$$

$$= 18 - 3 = \boxed{15}$$

---

b) If $z(x,y) = x\tan^{-1}(xy)$, $x = t^2$, $y = se^t$, $t \in R$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at $s=t=1$.

Given $z(x,y) = x\tan^{-1}(xy)$

$x = t^2$, $y = se^t$.

WKT $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial z}{\partial y}\cdot\frac{dy}{ds}$ —— (1)

$$\frac{\partial z}{\partial x} = x\left(\frac{1}{1+(xy)^2}\right)(y) + \tan^{-1}(xy)\cdot 1$$

$$= \frac{xy}{1+(xy)^2} + \tan^{-1}(xy)$$

$$\frac{\partial z}{\partial y} = \frac{x}{1+(xy)^2}\cdot x + \tan^{-1}(xy)(0)$$

$$= \frac{x^2}{1+(xy)^2}$$

$$\frac{dx}{ds} = 0$$

$$\frac{dy}{ds} = e^t(1) = e^t.$$

$$\therefore \frac{dz}{ds} = \frac{xy}{1+(xy)^2} + \tan^{-1}(xy)\cdot 0 + \frac{x^2}{1+(xy)^2}e^t$$

$$= \frac{x^2}{1+(xy)^2}e^t = \frac{t^4e^t}{1+t^4s^2e^{2t}}$$

$$\frac{dz}{ds}(s=t=1) = \frac{1\cdot e^1}{1+1\cdot 1\cdot e^2} = \frac{e}{1+e^2}$$ //

(ii) $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$ —— (2)

$$\frac{\partial z}{\partial x} = \frac{xy}{1+(xy)^2} + \tan^{-1}(xy)$$

$$\frac{\partial z}{\partial y} = \frac{x^2}{1+(xy)^2}$$

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$\frac{\partial x}{\partial t} = 2t, \quad \frac{\partial y}{\partial t} = se^t$

(1) $\Rightarrow \frac{\partial z}{\partial t} = \left[\frac{xy}{1+(xy)^2} + \tan^{-1}(xy)\right] 2t + \frac{x^2}{1+(xy)^2} se^t.$

$= \frac{x}{1+(xy)^2}\left[2yt + xse^t\right] + 2t\tan^{-1}(xy)$

$= \frac{t^2}{1+t^4s^2e^{2t}}\left[2se^t \cdot t + t^2se^t\right] + 2t\tan^{-1}(t^2se^t)$

$\left.\frac{\partial z}{\partial t}\right|_{(s=t=1)} = \frac{1}{1+e^2}\left[2e + e\right] + 2\tan^{-1}(e)$

$= \frac{1}{1+e^2}\left[3e\right] + 2\tan^{-1}(e)$

$= \frac{3e}{1+e^2} + 2\tan^{-1}(e)$

//.

7) let $U(x,y) = e^x \sin y$, where $x = st^2$, $y = s^2t$. find $\frac{\partial u}{\partial s}$, $\frac{\partial u}{\partial t}$ and evaluate them at $s=t=1$.

Given $U(x,y) = e^x \sin y$

$x = st^2, \quad y = s^2t.$

WKT $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$ — (1)

$\frac{\partial u}{\partial x} = e^x(0) + \sin y\, e^x = \sin y\, e^x$

$\frac{\partial u}{\partial y} = e^x \cos y + \sin y\,(0) = e^x \cos y$

$\frac{\partial x}{\partial s} = t^2, \quad \frac{\partial y}{\partial s} = 2st.$

$\therefore$ (1) $\Rightarrow e^x \sin y\, t^2 + e^x \cos y \cdot 2st.$

$= e^x t\left[t\sin y + 2s\cos y\right]$

$= e^{st^2} t\left[t\sin(s^2t) + 2s\cos(s^2t)\right]$

$\left.\frac{\partial u}{\partial s}\right|_{(s=t=1)} = e\left[1\cdot\sin(1) + 2\cos(1)\right]$

//

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$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t} \quad \text{— (2)}.$$

$$\frac{\partial u}{\partial x} = e^x \sin y, \quad \frac{\partial u}{\partial y} = e^x \cos y$$

$$\frac{\partial x}{\partial t} = 2st, \quad \frac{\partial y}{\partial t} = s^2$$

$$(2) \Rightarrow \frac{\partial u}{\partial t} = e^x \sin y \cdot 2st + e^x \cos y \cdot s^2$$

$$= e^{st^2}\sin(s^2t)\,2st + e^{st^2}\cos(s^2t)\,s^2$$

$$= e^{st^2}\left[2st\sin(s^2t) + s^2\cos(s^2t)\right]$$

$$\frac{\partial u}{\partial t}(s=t=1) = e\left[2\sin(1) + \cos(1)\right] \;//$$

---

8) $z(x,y) = x^3 - 3x^2y^3$, $x = se^t$, $y = se^{-t}$, Find $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$.

Given, $z(x,y) = x^3 - 3x^2y^3$, $x = se^t$, $y = se^{-t}$

$$\frac{\partial z}{\partial x} = 3x^2 - 6xy^3 \qquad \frac{\partial x}{\partial s} = e^t$$

$$\frac{\partial z}{\partial y} = -9x^2y^2 \qquad \frac{\partial y}{\partial s} = e^{-t}$$

$$\frac{\partial x}{\partial t} = se^t, \quad \frac{\partial y}{\partial t} = se^{-t}(-1)$$

$$\frac{\partial x}{\partial s} = e^t, \quad \frac{\partial y}{\partial s} = e^{-t}$$

WKT $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$

$$= (3x^2 - 6xy^3)\cdot e^t + (-9x^2y^2)e^{-t}$$

$$= \left[3s^2e^{2t} - 6se^t s^3 e^{-3t}\right]e^t + \left[-9s^2e^{2t}s^2e^{-2t}\right]e^{-t}$$

$$= 3s^2e^{3t} - 6s^4e^{-t} - 9s^4e^{-t}$$

$$= 3s^2\left[e^{3t} - 5s^4e^{-t}\right]$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$= (3x^2 - 6xy^3)\cdot se^t + (-9x^2y^2)(-se^{-t})$$

$$= 3\left[s^2e^{2t} - 6se^t s^3 e^{-3t}\right]se^t + 9s^2e^{2t}s^2e^{-2t}(se^{-t})$$

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$$\frac{\partial z}{\partial t} = 3s^3e^{3t} - 18s^5e^{-t} + 9s^5e^{-t}$$
$$= 3s^3e^{3t} - 9s^5e^{-t}$$
$$= 3s^3\left(e^{3t} - 3s^2e^{-t}\right)$$

---

a) $w(x,y,z) = xy+yz+zx$, $x = u-v$, $y = uv$, $z = u+v$, find $\frac{\partial w}{\partial u}$, $\frac{\partial w}{\partial v}$

Given $w(x,y,z) = xy+yz+zx$

$x = u-v$, $y = uv$, $z = u+v$

$\frac{\partial w}{\partial x} = y+z$	$\frac{\partial x}{\partial u} = 1$	$\frac{\partial x}{\partial v} = -1$
$\frac{\partial w}{\partial y} = x+z$	$\frac{\partial y}{\partial u} = v$	$\frac{\partial y}{\partial v} = u$
$\frac{\partial w}{\partial z} = y+x$	$\frac{\partial z}{\partial u} = 1$	$\frac{\partial z}{\partial v} = 1$

WKT $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$

$= (y+z)\cdot 1 + (x+z)v + (y+x)\cdot 1$

$= uv+u+v + (u-v+u+v)v + uv + u - v$

$= uv+u+v+2uv+uv+u-v$

$= 4uv+2u = 2u(2v+1)$

$\frac{\partial w}{\partial u}\left(\frac{1}{2},1\right) = 2\left(\frac{1}{2}\right)[2(1)+1]$

$= 1[2+1] = \boxed{3}$

$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$

$= (y+z)(-1) + (x+z)u + (y+x)\cdot 1$

$= -(uv+u+v) + (u-v+u+v)u + (uv+u-v)\cdot 1$

$= -uv-u-v+2u^2+uv+u-v$

$= 2u^2-2v = 2(u^2-v)$

$\frac{\partial w}{\partial v}\left(\frac{1}{2},1\right) = 2\left[\frac{1}{4}-1\right]$

$= 2\left[-\frac{3}{4}\right] = \boxed{-\frac{3}{2}}$

Homogeneous functions and Euler's Theorem:-

8.21 Show that $f(x,y) = \frac{x^2+5xy-10y^2}{3x+7y}$ is a homogeneous function of degree 1.

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Given, $f(x,y) = \dfrac{x^2+5xy-10y^2}{3x+7y}$

Replacing x by λx, y by λy,

$$f(\lambda x, \lambda y) = \frac{(\lambda x)^2 + 5(\lambda x)(\lambda y) - 10(\lambda y)^2}{3(\lambda x) + 7(\lambda y)}$$

$$= \frac{\lambda^2[x^2+5xy-10y^2]}{\lambda[3x+7y]} = \lambda f(x,y)$$

$\therefore$ f is a homogeneous function of degree 1.

Eg 8.22 If $u = \sin^{-1}\left[\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\tan u$.

Given $u = \sin^{-1}\left[\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

Replace x by λx, y by λy

$$\therefore \sin u = \frac{\lambda x + \lambda y}{\sqrt{\lambda}\sqrt{x}+\sqrt{y}\sqrt{\lambda}} = \frac{\lambda[x+y]}{\sqrt{\lambda}[\sqrt{x}+\sqrt{y}]}$$

$$= \sqrt{\lambda}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$$

$$= \sqrt{\lambda}\sin u$$

$\therefore$ $\sin u$ is homogeneous function of degree $\frac{1}{2}$.

By Euler's Th,

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$\cos u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \frac{1}{2}\sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\sin u}{\cos u}$$

$$= \frac{1}{2}\tan u$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$

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Chapter 9 – Applications of Integration

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CHAPTER 9 APPLICATIONS OF INTEGRATION

Definite integral as the limit of a sum.

1) $\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f\left(a+(b-a)\frac{r}{n}\right)$

2) $\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)$

Eg 9.1 Estimate the value of $\int_0^{0.5} x^2\,dx$ using the Riemann sums corresponding to 5 subintervals of equal width and applying (i) left-end rule (ii) right-end rule (iii) the midpt rule.

Here $a=0$, $b=0.5$, $n=5$, $f(x)=x^2$

width $h = \Delta x = \frac{b-a}{n} = \frac{0.5-0}{5} = 0.1$

The partition of intervals is

$x_0 = 0$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$x_3 = x_2 + h = 0.2 + 0.1 = 0.3$

$x_4 = x_3 + h = 0.3 + 0.1 = 0.4$

$x_5 = x_4 + h = 0.4 + 0.1 = 0.5$

The left end rule with equal width Δx is

$S = [f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4)]\,\Delta x$

$= [x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2]\,(0.1)$

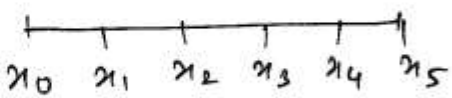
$= [0^2 + (0.1)^2 + (0.2)^2 + (0.3)^2 + (0.4)^2]\,(0.1)$

$= [0.01 + 0.04 + 0.09 + 0.16]\,0.1$

$= (0.30)(0.1)$

$= (0.030)$

$\therefore \int_0^{0.5} x^2\,dx$ is approximately 0.03.



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(ii) The right-end rule for with equal width Δx.

$$S = [f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]\,\Delta x$$

$$= (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2)(0.1)$$

$$= [(0.1)^2 + (0.2)^2 + (0.3)^2 + (0.4)^2 + (0.5)^2](0.1)$$

$$= (0.01 + 0.04 + 0.09 + 0.16 + 0.25)(0.1)$$

$$= (0.55)(0.1)$$

$$= 0.055$$

$$\therefore \int_0^{0.5} f(x)\,dx = \int_0^{0.5} x^2\,dx = 0.055$$

(iii) The mid point rule with equal width Δx

$$S = \left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + f\left(\frac{x_3+x_4}{2}\right) + f\left(\frac{x_4+x_5}{2}\right)\right]\Delta x$$

$$= [f(0.05) + f(0.15) + f(0.25) + f(0.35) + f(0.45)](0.1)$$

$$= (0.0025 + 0.0225 + 0.0625 + 0.1225 + 0.2025)(0.1)$$

$$= 0.04125$$

$\therefore \int_0^{0.5} x^2\,dx$ is approximately 0.04125

EXERCISE 9.1

1) Find an approximate value of $\int_1^{1.5} x\,dx$ by applying the left end rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$

Here $a = 1$, $b = 1.5$, $n = 5$, $f(x) = x$.

$\therefore$ The width $h = \Delta x = \frac{b-a}{n}$

$$= \frac{1.5-1}{5} = \frac{0.5}{5} = 0.1$$

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Here $x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$

$x_3 = 1.3$, $x_4 = 1.4$, $x_5 = 1.5$

The left-end rule for Riemnn sum

$S = [f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4)] \Delta x$.

$= [f(1) + f(1.1) + f(1.2) + f(1.3) + f(1.4)] (0.1)$

$= (1 + 1.1 + 1.2 + 1.3 + 1.4) (0.1)$

$= (6.0) (0.1) = 0.6$

---

2) Find an approximate value of $\int_1^{1.5} x^2 dx$ by applying the right end rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$

Here $a = 1$, $b = 1.5$, $n = 5$, $f(x) = x^2$

width $h = \Delta x = \frac{b-a}{n} = \frac{1.5-1}{5}$

$= \frac{0.5}{5} = 0.1$

Here $x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$

$x_3 = 1.3$, $x_4 = 1.4$, $x_5 = 1.5$

The right hand Riemnn sum is

$S = [f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)] \Delta x$

$= [f(1.1) + f(1.2) + f(1.3) + f(1.4) + f(1.5)] \Delta x$

$= ((1.1)^2 + (1.2)^2 + (1.3)^2 + (1.4)^2 + (1.5)^2)(0.1)$

$= (1.21 + 1.44 + 1.69 + 1.96 + 2.25)(0.1)$

$(8.55)(0.1) = 0.855$

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3) Find the approximate value of $\int_1^{1.5} (2-x)\,dx$ by applying the mid point rule with partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$

Here $a=1$, $b=1.5$, $n=5$, $f(x) = 2-x$

Width of the interval

$$h = \Delta x = \frac{b-a}{n} = \frac{1.5-1}{5} = \frac{0.5}{5} = (0.1).$$

$x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$, $x_3 = 1.3$

$x_4 = 1.4$, $x_5 = 1.5$

The mid pt rule for Reimnn sum □

$$S = \left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + f\left(\frac{x_3+x_4}{2}\right) + f\left(\frac{x_4+x_5}{2}\right)\right]\Delta x$$

$$S = \left[f\left(\frac{1+1.1}{2}\right) + f\left(\frac{1.1+1.2}{2}\right) + f\left(\frac{1.2+1.3}{2}\right) + f\left(\frac{1.3+1.4}{2}\right) + f\left(\frac{1.4+1.5}{2}\right)\right](0.1)$$

$$= \left[f\left(\frac{2.1}{2}\right) + f\left(\frac{2.3}{2}\right) + f\left(\frac{2.5}{2}\right) + f\left(\frac{2.7}{2}\right) + f\left(\frac{2.9}{2}\right)\right](0.1)$$

$$= \left[f(1.05) + f(1.15) + f(1.25) + f(1.35) + f(1.45)\right](0.1)$$

$$= \left[(2-1.05) + (2-1.15) + (2-1.25) + (2-1.35) + (2-1.45)\right](0.1)$$

$$= (0.95 + 0.85 + 0.75 + 0.65 + 0.55)(0.1)$$

$$= (3.75)(0.1)$$

$$= 0.375$$

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Eg 9.2 Evaluate $\int_0^1 x\,dx$, as the limit of a sum.

Here $a=0$, $b=1$, $f(x)=x$.

WKT $\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$

$\therefore \int_0^1 x\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{r}{n}$

$= \lim_{n\to\infty} \frac{1}{n^2}\sum_{r=1}^{n} r$

$= \lim_{n\to\infty} \frac{1}{n^2}[1+2+3+\cdots+n]$

$\lim_{n\to\infty} \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right)$

$= \lim_{n\to\infty} \left(\frac{n+1}{2n}\right)$

$= \frac{1}{2}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)$

$= \frac{1}{2}\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right]$

$= \frac{1}{2}\left[\lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{1}{n}\right]$

$= \frac{1}{2}(1+0) = \frac{1}{2}$ //.

Eg 9.3 Evaluate $\int_0^1 x^3\,dx$, as limit of a sum.

Here $a=0$, $b=1$, $f(x)=x^3$.

WKT $\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$

$\therefore \int_0^1 x^3\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n}\left(\frac{r^3}{n^3}\right)$

$= \lim_{n\to\infty} \frac{1}{n^4}[1^3+2^3+\cdots+n^3]$

$= \lim_{n\to\infty} \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]$

$= \frac{1}{4}\lim_{n\to\infty}\frac{1}{n^4}\left(n^2(n+1)^2\right)$

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$$= \frac{1}{4} \lim_{n\to\infty} \frac{(n+1)^2}{n^2}$$

$$= \frac{1}{4} \lim_{n\to\infty} \left(\frac{n^2+2n+1}{n^2}\right)$$

$$= \frac{1}{4} \lim_{n\to\infty} \left[1 + \frac{2}{n} + \frac{1}{n}\right]$$

$$= \frac{1}{4} \left[\lim_{n\to\infty} 1 + 2 \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{1}{n}\right]$$

$$= \frac{1}{4} [1 + 2(0) + (0)]$$

$$= \frac{1}{4}(1) = 1/4.$$

Eg 9.4 Evaluate $\int_1^4 (2x^2+3)\,dx$, as the limit of sum !

Here $a=1$, $b=4$, $f(x) = 2x^2+3$

WKT $\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f\left(a+(b-a)\frac{r}{n}\right)$

Now $f\left(a+(b-a)\frac{r}{n}\right) =$

$f\left(1+(4-1)\frac{r}{n}\right)$

$= f\left(1+\frac{3r}{n}\right)$

$= 2\left[1+\frac{3r}{n}\right]^2 + 3$

$= 2\left[1+\frac{6r}{n}+\frac{9r^2}{n^2}\right] + 3$

$= 2 + \frac{12r}{n} + \frac{18r^2}{n^2} + 3$

$= 5 + 12\frac{r}{n} + 18\frac{r^2}{n^2}$

$$\int_1^4 (2x^2+3)\,dx = \lim_{n\to\infty} \frac{4-1}{n} \sum_{r=1}^{n} \left(5+\frac{12r}{n}+\frac{18r^2}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{r=1}^{n} \left(5+\frac{12r}{n}+\frac{18r^2}{n^2}\right)$$

$$= \lim_{n\to\infty} \left[\sum_{r=1}^{n} \frac{15}{n} + \sum_{r=1}^{n} \frac{36r}{n^2} + \sum_{r=1}^{n} \frac{54r^2}{n^3}\right]$$

$$= \lim_{n\to\infty} \left[\frac{15}{n}(n) + \frac{36}{n^2}(1+2+\cdots+n) + \frac{54}{n^3}(1^2+2^2+\cdots+n^2)\right]$$

$$= \lim_{n\to\infty} \left[15 + \frac{36}{n^2}\frac{n(n+1)}{2} + \frac{54}{n^3}\frac{n(n+1)(2n+1)}{6}\right]$$

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$$= \lim_{n\to\infty}\left[15 + \frac{9(n+1)(2n+1)}{n^2} + 18\left(\frac{n+1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[15 + \frac{9(2n^2+n+2n+1)}{n^2} + 18\left(1+\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[15 + 9\left(\frac{2n^2}{n^2} + \frac{3n}{n^2} + \frac{1}{n^2}\right) + 18\left(1+\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[15 + 9\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 18\left(1+\frac{1}{n}\right)\right]$$

$$= 15 + 9(2+0+0) + 18(1+0)$$

$$= 15 + 18 + 18 = 51 //$$

EXERCISE 9.2

Evaluate $\int_0^1 (5x+4)\,dx$ as the limit of sums.

WKT $\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)$

$\therefore\therefore \int_0^1 (5x+4)\,dx$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{5r}{n}+4\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{5}{n}\sum_{r=1}^{n} r + \sum_{r=1}^{n} 4\right]$$

$$= \lim_{n\to\infty}\left[\frac{5}{n^2}(1+2+\cdots+n) + \frac{1}{n}4n\right]$$

$$= \lim_{n\to\infty}\left[\frac{5}{n^2}\frac{n(n+1)}{2} + 4\right]$$

$$= \lim_{n\to\infty}\left[\frac{5(n+1)}{2n} + 4\right]$$

$$= \lim_{n\to\infty}\left[\frac{5}{2}\left(1+\frac{1}{n}\right) + 4\right]$$

$$= \lim_{n\to\infty}\left[\frac{5}{2} + \frac{5}{2n} + 4\right] = \frac{5}{2} + 0 + 4$$

$$= \frac{5}{2} + 4$$

$$= \frac{5+8}{2} = \boxed{\frac{13}{2}}$$

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ii) $\int_1^2 (4x^2-1)\,dx$

WKT $\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f\left(a+(b-a)\frac{r}{n}\right)$

Here $a=1$, $b=2$, $f(x) = 4x^2-1$

$\therefore \int_1^2 (4x^2-1)\,dx = \lim_{n\to\infty} \frac{2-1}{n} \sum_{r=1}^{n} f\left(1+(2-1)\frac{r}{n}\right)$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(1+\frac{r}{n}\right)$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left[4\left(1+\frac{r}{n}\right)^2-1\right]$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left[4\left(1+\frac{r^2}{n^2}+\frac{2r}{n}\right)-1\right]$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left[4+\frac{4r^2}{n^2}+\frac{8r}{n}-1\right]$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left[3+\frac{4r^2}{n^2}+\frac{8r}{n}\right]$

$= \lim_{n\to\infty} \frac{1}{n} \left[3n+\frac{4}{n^2}(1^2+2^2+\cdots+n^2)+\frac{8}{n}(1+2+\cdots+n)\right]$

$= \lim_{n\to\infty} \left[3+\frac{4}{n^3}\times\frac{n(n+1)(2n+1)}{6} + \frac{8}{n^2}\times\frac{n(n+1)}{2}\right]$

$\frac{12|C+9}{9.2|P4}$

$= \lim_{n\to\infty} \left[3+\frac{2}{3n^2}(n+1)(2n+1)+\frac{4(n+1)}{n}\right]$

$= \lim_{n\to\infty} \left[3+\frac{2}{3}\left(\frac{2n^2+n+2n+1}{n^2}\right)+4\left(1+\frac{1}{n}\right)\right]$

$= \lim_{n\to\infty} \left[3+\frac{2}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)+4\left(1+\frac{1}{n}\right)\right]$

$= 3+\frac{2}{3}(2+0+0)+4(1+0)$

$= 3+\frac{4}{3}+4$

$= \frac{9+4+12}{3} = \boxed{\frac{25}{3}}$

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Properties of definite Integrals:

a) $\int_a^b f(x)\,dx = \int_a^b f(u)\,du$

b) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

c) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

d) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

e) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

f) $\int_0^{2a} f(x)\,dx = \int_0^a f(x) + f(2a-x)\,dx$

g) If $f(x)$ is even function, Then

$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

h) If $f(x)$ is odd function

Then $\int_{-a}^{a} f(x)\,dx = 0$.

i) If $f(2a-x) = f(x)$, Then

$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

j) If $f(2a-x) = -f(x)$, Then $\int_0^{2a} f(x)\,dx = 0$

k) $\int_0^a x\, f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$ If

$f(a-x) = f(x)$.

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Fundamental Theorems of Integral calculus and their applications.

First Fundamental Theorem of Integral calculus.

If $f(x)$ be a continuous function defined on a closed interval $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$ $a < x < b$ then, $\frac{d}{dx}(F(x)) = f(x)$

In otherwards, $F(x)$ is anit derivative of $f(x)$.

Second fundamental Th.

If $f(x)$ be a continuous function defined on a closed Interval $[a,b]$ and $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Eg 9.5 Evaluate $\int_0^3 (3x^2 - 4x + 5)\,dx$

$$\int_0^3 (3x^2 - 4x + 5)\,dx$$

$$= \int_0^3 3x^2\,dx - 4\int_0^3 x\,dx + 5\int_0^3 dx$$

$$= 3\left[\frac{x^3}{3}\right]_0^3 - 4\left[\frac{x^2}{2}\right]_0^3 + 5[x]_0^3$$

$$= 3\left[\frac{3^3}{3} - \frac{0^3}{3}\right] - 4\left[\frac{3^2}{2} - \frac{0^2}{2}\right] + 5[3-0]$$

$$= 3[9-0] - 4\left[\frac{9}{2} - 0\right] + 15$$

$$= 27 - 18 + 15 = 24$$

Eg 9.6 Evaluate $\int_0^1 \frac{2x+7}{5x^2+9}\,dx$.

$$\int_0^1 \frac{2x+7}{5x^2+9}\,dx = \int_0^1 \frac{2x}{5x^2+9}\,dx + \int_0^1 \frac{7}{5x^2+9}\,dx$$

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$$= \frac{1}{5}\left[\log(5x^2+9)\right]_0^1 + \frac{7}{5}\int_0^1 \frac{dx}{x^2+\left(\frac{3}{\sqrt{5}}\right)^2}$$

$$= \frac{1}{5}\left[\log(5x^2+9)\right]_0^1 + \frac{7}{5}\times\frac{1}{3/\sqrt{5}}\left[\tan^{-1}\frac{x}{3/\sqrt{5}}\right]_0^1$$

$$= \frac{1}{5}\left[\log 14 - \log 9\right] + \frac{7}{5}\times\frac{\sqrt{5}}{3}\left[\tan^{-1}\left(\frac{x\sqrt{5}}{3}\right)\right]_0^1$$

$$= \frac{1}{5}\left[\log\frac{14}{9}\right] + \frac{7}{3\sqrt{5}}\left[\tan^{-1}\frac{\sqrt{5}}{3}\right]$$

$$\int \frac{1}{ax+b} = \frac{1}{a}\log(ax+b)$$

$$\int \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

9.7 Evaluate $\int_0^1 [2x]\,dx$ where $[\cdot]$ is the greatest Integer fun.

$$\int_0^1 [2x]\,dx = \int_0^{1/2} [2x]\,dx + \int_{1/2}^1 [2x]\,dx$$

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Eg 9.8 Evaluate $\int_0^{\pi/3} \frac{\sec x \tan x}{1+\sec^2 x}\, dx$

put $\sec x = u$

$du = \sec x \tan x\, dx$.

$\therefore I = \int_1^2 \frac{du}{1+u^2}$ | when $x = 0$, $u = \sec 0 = 1$; when $x = \pi/3$, $u = \sec \pi/3 = 2$

$= \left[\tan^{-1}(u)\right]_1^2 = \tan^{-1}(2) - \tan^{-1}(1)$

$= \tan^{-1}(2) - \pi/4$.

Eg 9.9 Evaluate $\int_0^9 \frac{1}{x+\sqrt{x}}\, dx$.

Let $I = \int_0^9 \frac{1}{x+\sqrt{x}}\, dx$.

put $u = \sqrt{x} \Rightarrow u^2 = x$

$du = \frac{1}{2\sqrt{x}}\, dx$

$2du = \frac{1}{\sqrt{x}}\, dx$

$2u\,du = dx$.

put $x = 0$, $u = 0$

$x = 9$, $u = \sqrt{9} = 3$

$\therefore I = \int_0^3 \frac{2u\,du}{u^2+u} = \int_0^3 \frac{2u\,du}{u(u+1)}$

$= \int_0^3 \frac{2\,du}{1+u} = 2\left[\log(1+u)\right]_0^3$

$= 2\left[\log 4 - \log 1\right]$

$= 2\left[\log 4\right] = \log 4^2$

$= \log 16$.

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Eg 9.10 Evaluate $\int_1^2 \frac{x}{(x+1)(x+2)}\,dx$

Let $I = \int_1^2 \frac{x}{(x+1)(x+2)}\,dx$

$$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$x = A(x+2) + B(x+1)$$

put $x = -2$, $-B = -2 \Rightarrow B = 2$

put $x = -1$, $A = -1$

$$\therefore I = \int_1^2 \left[\left(\frac{-1}{x+1}\right) + \left(\frac{2}{x+2}\right)\right] dx.$$

$$= \int_1^2 \frac{-1}{x+1}\,dx + \int_1^2 \frac{2}{x+2}\,dx$$

$$= -\left[\log(x+1)\right]_1^2 + 2\left[\log(x+2)\right]_1^2$$

$$= -\left[\log 3 - \log 2\right] + 2\left[\log 4 - \log 3\right]$$

$$= (\log 2 - \log 3) + 2\log 4 - 2\log 3$$

$$= \log\left(\frac{2}{3}\right) + \log 4^2 - \log 3^2$$

$$= \log\left(\frac{2}{3}\right) + (\log 16 - \log 9)$$

$$= \log\left(\frac{2}{3}\right) + \log\left(\frac{16}{9}\right)$$

$$= \log \frac{2}{3} \times \frac{16}{9} = \log \frac{32}{27}$$

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9.11 Evaluate $\int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)}\, d\theta$

put $u = 1+\sin\theta$

$\frac{du}{d\theta} = \cos\theta$

$du = \cos\theta\, d\theta$

$\theta = 0$,	$u = 1+\sin 0$
	$u = 1$
$\theta = \pi/2$,	$u = 1+\sin\pi/2$
	$= 1+1$
	$= 2$

$\therefore \int_0^{\pi/2} \frac{\cos\theta\, d\theta}{(1+\sin\theta)(2+\sin\theta)}$

$= \int_1^2 \frac{du}{u(1+u)} = \int_1^2 \frac{(1+u)-u}{u(1+u)}\, du$

$= \int_1^2 \frac{1+u}{u(1+u)}\, du - \int_1^2 \frac{u\, du}{u(1+u)}$

$= \int_1^2 \frac{du}{u} - \int_1^2 \frac{du}{1+u}$

$= [\log u]_1^2 - [\log(1+u)]_1^2$

$= (\log 2 - \log 1) - (\log 3 - \log 2)$

$= \log 2 - \log 1 - \log 3 + \log 2$

$= 2\log 2 - \log 3$

$= \log 2^2 - \log 3 \Rightarrow \log 4 - \log 3$

$= \log(4/3)$

9.12 Evaluate $I = \int_0^{1/\sqrt{2}} \frac{\sin^{-1}x}{(1-x^2)^{3/2}}\, dx$

put $u = \sin^{-1}x \Rightarrow x = \sin u$

Then $du = \frac{1}{\sqrt{1-x^2}}\, dx$

$x = 0$, then $u = 0$

$x = \frac{1}{\sqrt{2}}$ then $u = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$u = \pi/4$

$\therefore I = \int_0^{\pi/4} \frac{u}{(1-x^2)\sqrt{(1-x^2)}}\, dx = \int_0^{\pi} \frac{u\, du}{1-x^2}$ —

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$\therefore I = \int_0^{\pi/4} \frac{u\,du}{1-\sin^2 u} = \int_0^{\pi/4} \frac{u\,du}{\cos^2 u}$

$= \int_0^{\pi/4} u \sec^2 u\,du$

$= [u \tan u]_0^{\pi/4} - \int_0^{\pi/4} \tan u\,du$ (*).

$= [u \tan u]_0^{\pi/4} + [\log \cos u]_0^{\pi/4}$

$= \left(\frac{\pi}{4} \tan \frac{\pi}{4} - 0\right) + \left[\log \cos \frac{\pi}{4} - \log \cos 0\right]$

$= \pi/4 + \log \frac{1}{\sqrt{2}} - \log 1$

$= \frac{\pi}{4} + \log 2^{-\frac{1}{2}} - 0$

$= \frac{\pi}{4} - \frac{1}{2} \log 2$

(*) $\int u\,dv = uv - \int v\,du$ Here

$dv = \sec^2 u\,du$

$\int dv = \int \sec^2 u\,du$

$= \tan u$

XII / AIE 9.3 / P7

9.13 Evaluate

$\int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x})\,dx$

Let $I = \int_0^{\pi/2} \sqrt{\tan x} + \sqrt{\cot x}\,dx$

$= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\,dx$.

$= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\,dx$.

$= \sqrt{2} \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}}\,dx$.

$= \sqrt{2} \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}}\,dx$.

put $\sin x - \cos x = u$

$(\cos x + \sin x)\,dx = du$

$x = 0$,	$u = -1$
$x = \pi/2$,	$1 - 0$
$x = \pi/2$	$u = 1$

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$\therefore I = \sqrt{2}\int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} = \sqrt{2}\left[\sin^{-1}u\right]_{-1}^{1}$

$= \sqrt{2}\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$

$= \sqrt{2}\left[\sin^{-1}(1) + \sin^{-1}(1)\right]$

$= \sqrt{2}\left[\pi/2 + \pi/2\right] = \pi\sqrt{2}$

9.16 (X) Show that $\int_0^{\pi/2} \frac{dx}{4+5\sin x} = \frac{1}{3}\log 2$.

put $u = \tan\frac{x}{2}$

$du = \sec^2\frac{x}{2}\cdot\frac{1}{2}\cdot dx$

$du = \left(1+\tan^2\frac{x}{2}\right)\frac{1}{2}\,dx$

$du = \frac{(1+u^2)}{2}\,dx$

$\therefore dx = \frac{2du}{1+u^2}$

$x=0,\ u=\tan 0 = 0$

$x=\pi/2,\ u=\tan\pi/4$

$u=1$

$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$= \frac{2u}{1+u^2}$

XII/AIE 9.3/P8

$\therefore I = \int_0^1 \frac{\frac{2du}{1+u^2}}{4+5\left(\frac{2u}{1+u^2}\right)}$

$= \int_0^1 \frac{\frac{2du}{1+u^2}}{\frac{4(1+u^2)+5(2u)}{1+u^2}} = \int_0^1 \frac{2du}{4u^2+10u+4}$

$= \int_0^1 \frac{2du}{2(2u^2+5u+2)} = \int_0^1 \frac{du}{2u^2+5u+2}$

$= \int_0^1 \frac{du}{2(u^2+\frac{5}{2}u+1)} = \frac{1}{2}\int_0^1 \frac{du}{u^2+\frac{5}{2}u+1}$

$u^2+\frac{5}{2}u+1 = u^2+\frac{5}{2}u+\left(\frac{5}{4}\right)^2 = -1+\frac{25}{16}$

$= \left(u+\frac{5}{4}\right)^2 = \frac{9}{16}$

$\left(u+\frac{5}{4}\right)^2 - \left(\frac{3}{4}\right)^2$

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$$I = \frac{1}{2}\int_0^1 \frac{du}{\left(u+\frac{5}{4}\right)^2 - \left(\frac{3}{4}\right)^2}$$

$$= \frac{1}{2} \times \frac{1}{2\left(\frac{3}{4}\right)} \left[\log \frac{u+\frac{5}{4}-\frac{3}{4}}{u+\frac{5}{4}+\frac{3}{4}}\right]_0^1 \quad \circledast$$

$$= \frac{1}{2} \times \frac{2}{3} \left[\log \frac{u+\frac{2}{4}}{u+\frac{8}{4}}\right]_0^1 = \frac{1}{3}\log\left(\frac{u+\frac{1}{2}}{u+2}\right)_0^1$$

$$= \frac{1}{3}\log\left(\frac{2u+1}{2(u+2)}\right)_0^1$$

$$= \frac{1}{3}\left[\log\frac{3}{6} - \log\frac{1}{4}\right]$$

$$= \frac{1}{3}\log\left(\frac{3}{6}\times\frac{4}{1}\right)$$

$$= \frac{1}{3}\log 2$$

$$\circledast \quad \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$$

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9.17 prove that $\int_0^{\pi/4} \frac{\sin 2x\,dx}{\sin^4 x + \cos^4 x} = \pi/4$

let $I = \int_0^{\pi/4} \frac{\sin 2x\,dx}{\sin^4 x + \cos^4 x}$

$= \int_0^{\pi/4} \frac{\sin 2x\,dx}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x}$

$= \int_0^{\pi/4} \frac{\sin 2x\,dx}{1 - 2\sin^2 x \cos^2 x}$

$= \int_0^{\pi/4} \frac{\sin 2x\,dx}{1 - \frac{1}{2}(2\sin x \cos x)^2}$

$= \int_0^{\pi/4} \frac{\sin 2x\,dx}{1 - \frac{1}{2}\sin^2 2x}$

$= \int_0^{\pi/4} \frac{\sin 2x\,dx}{\frac{2 - \sin^2 2x}{2}} = \int_0^{\pi/4} \frac{2\sin 2x\,dx}{2 - \sin^2 2x}$

$= \int_0^{\pi/4} \frac{2\sin 2x\,dx}{2 - (1 - \cos^2 2x)}$

$= \int_0^{\pi/4} \frac{2\sin 2x\,dx}{2 - 1 + \cos^2 2x}$

$= \int_0^{\pi/4} \frac{2\sin 2x}{1 + \cos^2 2x}$

put $u = \cos 2x$, $du = -2\sin 2x\,dx$

when $x = 0$, $u = \cos 0 = 1$

when $x = \pi/4$, $u = \cos 2\pi/4 = \cos \pi/2 = 0$

$\therefore \int_0^{\pi/4} \frac{2\sin 2x\,dx}{1 + \cos^2 2x} = \int_1^0 \frac{-du}{1 + u^2}$

$= \int_0^1 \frac{du}{1 + u^2} = \left[\tan^{-1}(u)\right]_0^1$

$= \tan^{-1}(1) - \tan^{-1}(0)$

$= \pi/4.$

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9.18 Evaluate $\int_0^{\pi/4} \frac{dx}{a^2\sin^2x + b^2\cos^2x}$

$= \frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\right)$.

Let $I = \int_0^{\pi/4} \frac{dx}{a^2\sin^2x + b^2\cos^2x}$.

$= \int_0^{\pi/4} \frac{1}{\cos^2x}\left(\frac{dx}{a^2\frac{\sin^2x}{\cos^2x} + b^2}\right)$

$= \int_0^{\pi/4} \frac{\sec^2x\,dx}{a^2\tan^2x + b^2}$

- put $u = \tan x$

$du = \sec^2x\,dx$

when $x = 0$, $u = \tan 0 = 0$

when $x = \pi/4$ $u = \tan\pi/4 = 1$

$\therefore I = \int_0^1 \frac{du}{a^2u^2 + b^2}$

$= \int_0^1 \frac{du}{a^2\left(u^2 + \frac{b^2}{a^2}\right)}$

$= \frac{1}{a^2}\int_0^1 \frac{du}{u^2 + \frac{b^2}{a^2}}$

$= \frac{1}{a^2} \times \frac{1}{\frac{b}{a}}\left[\tan^{-1}\left(\frac{u}{b/a}\right)\right]_0^1$

$= \frac{1}{a^2} \times \frac{a}{b}\left[\tan^{-1}\frac{au}{b}\right]_0^1$ $\left[\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right.$

$= \frac{1}{ab}\left[\tan^{-1}\frac{a}{b} - \tan^{-1}(0)\right]$

$= \frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\right)$ //

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9.19 Evaluate $\int_0^{\pi/4} \frac{1}{\sin x + \cos x}\, dx$

$$I = \int_0^{\pi/4} \frac{1}{\sin x + \cos x}\, dx.$$

$$= \int_0^{\pi/4} \frac{dx}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{dx}{\cos\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{dx}{\cos\left(\frac{\pi}{4} - x\right)} \quad (*)$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{dx}{\cos x}.$$

$$\left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \sec x\, dx$$

$$= \frac{1}{\sqrt{2}} \left[\log(\sec x + \tan x)\right]_0^{\pi/4} \quad (*1)$$

$$= \frac{1}{\sqrt{2}} \left[\log\left(\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}\right) - \log(\sec 0 + \tan 0)\right]$$

$$= \frac{1}{\sqrt{2}} \left[\log(\sqrt{2}+1) - \log(1+0)\right]$$

$$= \frac{1}{\sqrt{2}} \left[\log(\sqrt{2}+1)\right]$$

(*1) $\frac{d}{dx}\log(\sec x + \tan x) = \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}$

$$= \frac{\sec x(\tan x + \sec x)}{(\tan x + \sec x)} = \sec x.$$

(*) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$

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9.20 Show that

$$\int_0^{\pi} g(\sin x)\,dx = 2\int_0^{\pi/2} g(\sin x)\,dx.$$

where $g(\sin x)$ is a function of $\sin x$

We know that

$$\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$$

If $f(2a-x) = f(x)$

$$\int_0^{\pi} g(\sin x)\,dx$$

Here $2a = \pi$ and $f(x) = g(\sin x)$

$$f(2a-x) = g(\sin(\pi - x))$$
$$= g(\sin x) = f(x)$$

$$\therefore \int_0^{\pi} g(\sin x)\,dx = 2\int_0^{\pi/2} g(\sin x)\,dx$$

//

9.21 Evaluate $\int_0^{\pi} \frac{x}{1+\sin x}\,dx$ XII AIE 9.3/P13

Let $I = \int_0^{\pi} \frac{x}{1+\sin x}\,dx$.

$$= \int_0^{\pi} x \cdot \frac{1}{1+\sin x}\,dx.$$

Let $f(x) = \frac{1}{1+\sin x}$

$$f(\pi - x) = \frac{1}{1+\sin(\pi - x)} = \frac{1}{1+\sin x} = f(x)$$

$$\therefore I = \frac{\pi}{2}\int_0^{\pi} \frac{1}{1+\sin x}\,dx$$ (*).

(*) $\int_0^{a} x f(x)\,dx = \frac{a}{2}\int_0^{a} f(x)\,dx$ If $f(a-x) = f(x)$

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$$= \frac{\pi}{2} \times 2 \int_0^{\pi/2} \frac{1}{1+\sin x}\, dx$$

$$\left[\int_0^{\pi} g(\sin x)\, dx = 2 \int_0^{\pi/2} g(\sin x)\, dx \right]$$

$$= \pi \int_0^{\pi/2} \frac{1}{1+\sin\left(\frac{\pi}{2}-x\right)}\, dx \quad \left[\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx \right]$$

$$= \pi \int_0^{\pi/2} \frac{1}{1+\cos x}\, dx$$

$$= \pi \int_0^{\pi/2} \frac{1}{2\cos^2 \frac{x}{2}}\, dx$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \sec^2 \frac{x}{2}\, dx$$

$$= \frac{\pi}{2} \left[\tan \frac{x}{2} \right]_0^{\pi/2} = \frac{\pi}{2} \left[\tan \pi/4 - \tan 0 \right]$$

$$= \pi/2 \,(1-0) = \pi/2.$$

9.22 Show that $\int_0^{2\pi} g(\cos x)\, dx$ XII/ TD2E 9.3/p14

$$= 2 \int_0^{\pi} g(\cos x)\, dx.$$

Take $2a = 2\pi$, $f(x) = g(\cos x)$.

Then $f(2a-x) = g(\cos(2\pi - x))$

$$= g(\cos x)$$

$$= f(x)$$

$$\therefore \int_0^{2\pi} g(\cos x)\, dx = 2 \int_0^{\pi} g(\cos x)\, dx.$$

9.23 If $f(x) = f(a+x)$, then

$$\int_0^{2a} f(x)\, dx = 2 \int_0^{a} f(x)\, dx$$

we write $\int_0^{2a} f(x)\, dx = \int_0^{a} f(x)\, dx + \int_a^{2a} f(x)\, dx$

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Consider $\int_a^{2a} f(x)\,dx$

Substitute $x = a+u$, $dx = du$ | $x=a$, then $u=0$; $x=2a$ then $u=a$

$$\therefore \int_a^{2a} f(x)\,dx = \int_0^a f(a+u)\,du.$$

$$= \int_0^a f(u)\,du \quad [\because f(x) = f(a+x)]$$

$$= \int_0^a f(x)\,dx$$

$$\therefore \text{(1)} \Rightarrow \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx$$

$$= 2\int_0^a f(x)\,dx.$$

XII | AIE 9.3 | P15

9.24 Evaluate $\int_{-\pi/2}^{\pi/2} x\cos x\,dx$

Let $f(x) = x\cos x$

Then $f(-x) = -x\cos(-x)$

$= -x\cos x = -f(x)$

So $f(x)$ is an odd function

Hence $\int_{-\pi/2}^{\pi/2} x\cos x\,dx = 0$.

9.25) Evaluate $\int_{-\log 2}^{\log 2} e^{-|x|}\,dx$.

Let $f(x) = e^{-|x|}$ Then

$f(-x) = e^{-|-x|} = e^{-|x|} = f(x)$

$f(-x) = f(x)$, so $f(x)$ is even function

Hence $\int_{-\log 2}^{\log 2} e^{-|x|}\,dx = 2\int_0^{\log 2} e^{-x}\,dx$

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$$= 2\left(-e^{-x}\right)_0^{\log 2}$$

$$= 2\left(-e^{-\log 2} - (-e^{-0})\right)$$

$$= 2\left[-e^{-\log 2} + e^{0}\right]$$

$$= 2\left[-e^{\log \frac{1}{2}} + 1\right]$$

$$= 2\left[-\frac{1}{2} + 1\right] = 2\left(\frac{1}{2}\right) = 1.$$

Ex 9.26 Evaluate $\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx$

Let $I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx$ ——— (1)

Applying the formula $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$I = \int_0^a \frac{f(a-x)}{f(a-x)+f(a-(a-x))}\,dx$$

$$= \int_0^a \frac{f(a-x)}{f(a-x)+f(x)} \quad \text{——— (2)}$$

(1)+(2) $\Rightarrow$ $2I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx + \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}\,dx$

$$= \int_0^a \frac{f(x)+f(a-x)}{f(a-x)+f(x)}\,dx$$

$$= \int_0^a dx = a$$

Hence $2I = a \Rightarrow I = a/2$

i.e. $\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = a/2.$

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Eg 9.27 prove that

$$\int_0^{\pi/4} \log(1+\tan x)\,dx = \pi/8 \log 2$$

let $I = \int_0^{\pi/4} \log(1+\tan x)\,dx$ — ①

Applying the property

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \text{ in ①}$$

$$I = \int_0^{\pi} \log\left[1+\tan(\pi/4 - x)\right]dx$$

$$= \int_0^{\pi/4} \log\left[1+\frac{\tan\pi/4 - \tan x}{1+\tan\pi/4\tan x}\right]dx$$

$$= \int_0^{\pi/4} \log\left[1+\frac{1-\tan x}{1+\tan x}\right]dx$$

$$= \int_0^{\pi/4} \log\left[\frac{1+\tan x+1-\tan x}{1+\tan x}\right]dx.$$

$$= \int_0^{\pi/4} \log\left[\frac{2}{1+\tan x}\right]dx.$$

$$= \int_0^{\pi/4} \left[\log 2 - \log(1+\tan x)\right]dx.$$

$$= \log 2 \int_0^{\pi/4} dx - \int_0^{\pi/4} \log(1+\tan x)\,dx.$$

$$= \log 2 \int_0^{\pi/4} dx - I$$

$$= 2I = \log 2\,[x]_0^{\pi/4}$$

$$= \log 2 \cdot (\pi/4 - 0) = \pi/4 \log 2$$

$$\therefore I = \frac{\pi}{8}\log 2$$ //

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XII | 9.3
P18

9.28 Show that $\int_0^1 (\tan^{-1}x + \tan^{-1}(1-x))\,dx = \frac{\pi}{2} - \log 2$

Let $I = \int_0^1 [\tan^{-1}x + \tan^{-1}(1-x)]\,dx$

$= \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}(1-x)\,dx$

$= \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}x\,dx$

$= 2\int_0^1 \tan^{-1}x\,dx$

$= [2\int u\,dv]_0^1$ where $u = \tan^{-1}x$, $dv = dx$

$= 2[uv - \int v\,du]_0^1$ $u' = \frac{1}{1+x^2}dx$, $\int dv = \int dx = x$

$= 2\left[x\tan^{-1}x - \int x\,\frac{dx}{1+x^2}\right]_0^1$

$= 2\left[(\tan^{-1}(1) + \tan^{-1}(0))\right.$

$= 2\left[x\tan^{-1}x - \frac{1}{2}\log(1+x^2)\right]_0^1$

$= 2\left[\tan^{-1}(1) - \frac{1}{2}[\log(1+1) - \log(1)]\right]$

$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right]$

$= \frac{\pi}{2} - \log 2$ //

$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

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9.29 Evaluate $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx$

put $I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx$ — ①

Applying the formula $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

in ①

$$I = \int_2^3 \frac{\sqrt{2+3-x}}{\sqrt{5-(2+3)-x}+\sqrt{(2+3)-x}}\,dx$$

$$= \int_2^3 \frac{\sqrt{5-x}}{\sqrt{5-5+x}+\sqrt{5-x}}\,dx$$

$$= \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}\,dx \quad \text{— ②}$$

① + ② $\Rightarrow$

$$2I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx + \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}\,dx$$

$$2I = \int_2^3 \frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}\,dx$$

$$= \int_2^3 dx = [x]_2^3 = (3-2) = 1$$

$\therefore 2I = 1 \Rightarrow I = \frac{1}{2}$

$\therefore \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx = \frac{1}{2}$

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9.30 Evaluate $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx$

Let $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx$ ——— (1)

Using $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$.

$\therefore I = \int_{-\pi}^{\pi} \frac{\cos^2(-\pi+\pi-x)}{1+a^{(-\pi+\pi-x)}}\,dx$.

$= \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+a^{-x}}\,dx$

$= \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+\frac{1}{a^x}}\,dx$

$= \int_{-\pi}^{\pi} \frac{\cos^2 x}{\frac{a^x+1}{a^x}}\,dx$

$= \int_{-\pi}^{\pi} a^x \frac{\cos^2 x}{a^x+1}\,dx$ ——— (2)

(1) + (2) $\Rightarrow 2I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx + \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{a^x+1}\,dx$

$= \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}(1+a^x)\,dx$

$= \int_{-\pi}^{\pi} \cos^2 x\,dx = 2\int_0^{\pi} \cos^2 x\,dx$ $\because \cos^2 x$ is an even function

$= 2\int_0^{\pi} \frac{1+\cos 2x}{2}\,dx$

$=$

$\cos 2\theta = 2\cos^2\theta - 1$

$2\cos^2\theta = 1+\cos 2\theta$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

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$$\int_0^{\pi} (1+\cos 2x)\,dx = \int_0^{\pi} dx + \int_0^{\pi} \cos 2x\,dx$$

$$= [x]_0^{\pi} + \left[\frac{\sin 2x}{2}\right]_0^{\pi} = (\pi - 0) + \frac{\sin 2\pi}{2} - \frac{\sin 0}{2}$$

$$= \pi + 0 - 0$$

$$= \pi$$

i.e. $2I = \pi \Rightarrow I = \pi/2$.

× ——— ×

EXERCISE 9.3

1) Evaluate $\int_3^4 \frac{dx}{x^2-4}$

$$\int_3^4 \frac{dx}{x^2-2^2} = \left[\frac{1}{2(2)} \log\left(\frac{x-2}{x+2}\right)\right]_3^4 \quad *$$

$$= \frac{1}{4}\left[\log \frac{2}{6} - \log \frac{1}{5}\right]$$

$$= \frac{1}{4}\left[\log \frac{1}{3} \times \frac{5}{1}\right]$$

$$= \frac{1}{4} \log 5/3 \;//$$

(*) $\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right)$

2) $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

Let $I = \int_{-1}^{1} \frac{dx}{x^2+2x+5} = \int_{-1}^{1} \frac{dx}{x^2+2x+1+4}$

$$= \int_{-1}^{1} \frac{dx}{(x+1)^2+2^2}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{x+1}{2}\right)\right]_{-1}^{1} \quad (*)$$

$$= \frac{1}{2}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= \frac{1}{2} \tan^{-1}(1)$$

$$= \frac{1}{2} \times \frac{\pi}{4} = \frac{\pi}{8} \;//$$

(*) $\frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

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3) Evaluate $\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$

Let $I = \int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$

put $x = \cos 2\theta$

$dx = -\sin 2\theta\,(2)\,d\theta$

$= -2\sin 2\theta\,.$

when $x = 0$, $\cos 2\theta = 0$, $1 - 2\sin^2\theta = 0$, $2\sin^2\theta = 1$, $\sin^2\theta = \frac{1}{2}$, $\sin\theta = \frac{1}{\sqrt{2}}$, $\theta = \pi/4$

when $x = 1$, $\cos 2\theta = 1$, $1 - 2\sin^2\theta = 1$, $2\sin^2\theta = 0$, $\sin^2\theta = 0$, $\sin\theta = 0$, $\theta = 0$

$\therefore I = \int_{\pi/4}^{0} \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\,(-2\sin 2\theta)\,.d\theta$

R11 / AIE 9.3 / P23

$I = \int_{\pi/4}^{0} (-2)\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\,.\,\sin 2\theta\,d\theta\,.$

$= 2\int_0^{\pi/4} \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\,.2.\sin\theta\cos\theta\,d\theta$

$= 2\int_0^{\pi/4} \frac{\sin\theta}{\cos\theta}\,.\,2\sin\theta\cos\theta\,.\,d\theta$

$= 2\int_0^{\pi/4} 2\sin^2\theta\,d\theta = 2\int_0^{\pi/4} 2\left(\frac{1-\cos 2\theta}{2}\right)d\theta$

$= 2\int_0^{\pi/4} (1-\cos 2\theta)\,d\theta$

$= 2\left[\int_0^{\pi/4} d\theta - \int_0^{\pi/4} \cos 2\theta\,d\theta\right]$

$= 2\left[\pi/4 - \left(\frac{\sin 2\theta}{2}\right)_0^{\pi/4}\right]$

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$$= 2\left[\frac{\pi}{4} - \left(\frac{\sin\frac{\pi}{2}}{2} - \frac{\sin 0}{0}\right)\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \frac{\pi}{2} - 1 \quad //.$$

4) $\int_0^{\frac{\pi}{2}} e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx$

Let $I = \int_0^{\frac{\pi}{2}} e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx$

$$= \int_0^{\frac{\pi}{2}} e^x \left(\frac{1+\sin x}{2\cos^2\frac{x}{2}}\right) dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} e^x \left(\frac{1}{\cos^2\frac{x}{2}} + \frac{\sin x}{\cos^2\frac{x}{2}}\right) dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} e^x \left[\sec^2\frac{x}{2} + \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}}\right] dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} e^x \left[\sec^2\frac{x}{2} + 2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right] dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} e^x \left[\sec^2\frac{x}{2} + 2\tan\frac{x}{2}\right] dx.$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} e^x \left(f'(x) + f(x)\right)$$

where $f(x) = 2\tan\frac{x}{2}$

$$= \frac{1}{2}\left[e^x f(x)\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left[e^x 2\tan\frac{x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[e^{\frac{\pi}{2}} 2\tan\frac{\pi}{4} - e^0 2\tan 0\right]$$

$$= e^{\frac{\pi}{2}} \quad //$$

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(v) $\int_0^{\pi/2} \sqrt{\cos\alpha}\, \sin^3\alpha\, d\alpha$

Let $I = \int_0^{\pi/2} \sqrt{\cos\alpha}\, \sin^3\alpha\, d\alpha$

$= \int_0^{\pi/2} \sqrt{\cos\alpha}\, \sin^2\alpha . \sin\alpha . d\alpha$.

put $t = \cos\alpha$ $dt = -\sin\alpha\, d\alpha$

when $\alpha = 0$, $t = \cos 0 = 1$

$\alpha = \pi/2$, $t = \cos \pi/2 = 0$

$\therefore I = \int_1^0 \sqrt{t}\,(1-t^2)(-dt)$

$= \int_0^1 \sqrt{t}\,(1-t^2)\,dt$

$= \int_0^1 t^{1/2}\,dt - \int_0^1 t^{3/2}\,dt$

$= \left[\frac{t^{3/2}}{3/2}\right]_0^1 - \left[\frac{t^{7/2}}{7/2}\right]_0^1$

$= \frac{2}{3} - \frac{2}{7} = \frac{14-6}{21} = \frac{8}{21}$

(vi) $\int_0^1 \frac{1-x^2}{(1+x^2)^2}\,dx$

Let $I = \int_0^1 \frac{1-x^2}{(1+x^2)^2}\,dx$

$= \int_0^1 \frac{1+1-1-x^2}{(1+x^2)^2}\,dx = \int_0^1 \frac{2-(1+x^2)}{(1+x^2)^2}\,dx$

$= \int_0^1 \frac{2}{(1+x^2)^2}\,dx - \int_0^1 \frac{1+x^2}{(1+x^2)^2}\,dx$

$= 2\int_0^1 \frac{1}{(1+x^2)^2}\,dx - \int \frac{1}{1+x^2}\,dx$ — (1)

$I_1 - I_2$

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Let $I_1 = 2\int_0^1 \frac{1}{(1+x^2)^2}\,dx$

put $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$.

$x=0$, $\tan\theta = 0 \Rightarrow \theta = 0$ | $x=1$, $\tan\theta = 1 \Rightarrow \theta = \pi/4$

$\therefore I_1 = 2\int_0^{\pi/4} \frac{1}{(1+\tan^2\theta)^2}\,d\theta$

$= 2\int_0^{\pi/4} \frac{1}{\sec^2\theta}\,d\theta$

$= 2\int_0^{\pi/4} \cos^2\theta\,d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$2\cos^2\theta = 1+\cos 2\theta$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$= 2\int_0^{\pi/4} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$

$= \int_0^{\pi/4} d\theta + \int_0^{\pi/4} \cos 2\theta\,d\theta$

$= [\theta]_0^{\pi/4} + \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$ XII | 9.3 p26

$= \pi/4 + \left[\frac{\sin\frac{\pi}{2}}{2} - \frac{\sin 0}{2}\right]$

$= \pi/4 + \frac{1}{2} - 0 = \pi/4 + \frac{1}{2}$

$I_2 = \int_0^1 \frac{1}{1+x^2}\,dx = [\tan^{-1}(x)]_0^1$

$= \tan^{-1}(1) - \tan^{-1}(0)$

$= \pi/4$

① $\Rightarrow \pi/4 + 1/2 - \pi/4 = \frac{1}{2}$

$\therefore I = \frac{1}{2}$ //

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2 (i) $\int_{-5}^{5} x \cos\left(\frac{e^x-1}{e^x+1}\right) dx$

Let $I = \int_{-5}^{5} x \cos\left(\frac{e^x-1}{e^x+1}\right) dx$.

$f(x) = x \cos\left(\frac{e^x-1}{e^x+1}\right)$

$f(-x) = -x \cos\left(\frac{e^{-x}-1}{e^{-x}+1}\right)$

$= -x \cos\left(\frac{\frac{1}{e^x}-1}{\frac{1}{e^x}+1}\right)$

$= -x \cos\left(\frac{\frac{1-e^x}{e^x}}{\frac{1+e^x}{e^x}}\right)$

$= -x \cos\left(\frac{1-e^x}{1+e^x}\right)$

$= -x \cos\left[-\frac{(e^x-1)}{e^x+1}\right]$

$= -x \cos\left(\frac{e^x-1}{e^x+1}\right)$ $\because \cos(-\alpha) = \cos\alpha$.

$= -f(x)$

$\therefore f(x)$ is an odd function

$\therefore \int_{-5}^{5} x \cos\left(\frac{e^x-1}{e^x+1}\right) dx = 0$.

(ii) $\int_{-\pi/2}^{\pi/2} (x^5 + x\cos x + \tan^3 x + 1)\, dx$

$= \int_{-\pi/2}^{\pi/2} x^5\, dx + \int_{-\pi/2}^{\pi/2} x\cos x\, dx + \int_{-\pi/2}^{\pi/2} \tan^3 x\, dx + \int_{-\pi/2}^{\pi/2} dx$ ——— (1)

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Let $f(x) = x^5$

$f(-x) = (-x)^5 = -x^5$

$= -f(x)$

$\therefore f$ is an odd function.

Hence $\int_{-\pi/2}^{\pi/2} x^5 \, dx = 0$

Let $g(x) = x \cos x$

$g(-x) = -x \cos(-x) = -x \cos x$

$= -g(x)$

Hence g is an odd function

$\therefore \int_{-\pi/2}^{\pi/2} x \cos x \, dx = 0$

Let $h(x) = \tan^3 x$

$h(-x) = \tan^3(-x) = -\tan^3 x$

$= -h(x)$

$\therefore \int_{-\pi/2}^{\pi/2} \tan^3 x \, dx = 0.$

XII / AIE 9.3 / p 28

$\therefore$ (I) $\Rightarrow 0 + 0 + 0 + \int_{-\pi/2}^{\pi/2} dx.$

$= [x]_{-\pi/2}^{\pi/2}$

$= \pi/2 - (-\pi/2) = \pi/2 + \pi/2 = \pi$

(iii) $\int_{-\pi/4}^{\pi/4} \sin^2 x \, dx$

Let $f(x) = \sin^2 x$

$f(-x) = \sin^2(-x)$

$= (\sin(-x))^2$

$= (-\sin x)^2 = \sin^2 x$

$= f(x)$

$f(-x) = f(x) \Rightarrow f$ is an even function

$\therefore \int_{-\pi/4}^{\pi/4} \sin^2 x \, dx = 2 \int_0^{\pi/4} \sin^2 x \, dx$

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$\cos 2\alpha = 1 - 2\sin^2\alpha$

$2\sin^2\alpha = 1 \mp \cos 2\alpha$

$\sin^2\alpha = \frac{1 \mp \cos 2\alpha}{2}$

$\therefore \quad 2\int_0^{\pi/4} \sin^2 x\,dx = 2\int_0^{\pi/4} \frac{1 \mp \cos 2x}{2}\,dx$

$= \frac{2}{2}\int_0^{\pi/4} [1 - \cos 2x]\,dx.$

$= \int_0^{\pi/4} dx - \int_0^{\pi/4} \cos 2x\,dx.$

$= [x]_0^{\pi/4} - \left[\frac{+\sin 2x}{2}\right]_0^{\pi/4}.$

$= [\pi/4 - 0] \mp \left[\frac{\sin \pi/2}{2} - \frac{\sin 0}{2}\right]$

$= \pi/4 \mp \frac{1}{2}$ //

(IV) $\int_0^{2\pi} x \log\left(\frac{3+\cos x}{3-\cos x}\right) dx$ $\frac{\text{XII}|9.3}{\text{P29}}$

Let $I = \int_0^{2\pi} x \log\left(\frac{3+\cos x}{3-\cos x}\right) dx.$

Let $f(x) = x \log\left(\frac{3+\cos x}{3-\cos x}\right)$

$f(2\pi - x) = (2\pi - x)\log\left[\frac{3+\cos(2\pi - x)}{3-\cos(2\pi - x)}\right]$

$= (2\pi - x)\log\left[\frac{3+\cos x}{3-\cos x}\right]$

$= 2\pi \log\left[\frac{3+\cos x}{3-\cos x}\right] - \log\left[\frac{3+\cos x}{3-\cos x}\right]$

$= 2\pi \log\left[\frac{3+\cos x}{3-\cos x}\right] - f(x)$

$f(2\pi - x) + f(x) = 2\pi \log\left[\frac{3+\cos x}{3-\cos x}\right]$

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$= 0$ as $f(2\pi - x) + f(x) = 0$.

$\therefore \int_0^{2\pi} x \log\left(\frac{3+\cos x}{3-\cos x}\right) dx = 0$

(v) $\int_0^{2\pi} \sin^4 x \cos^3 x \, dx$.

$f(x) = \sin^4 x \cos^3 x$.

$f(2\pi - x) = \sin^4(2\pi - x) \cos^3(2\pi - x)$

$= \sin^4 x \cos^3 x = f(x)$

$\therefore \int_0^{2\pi} \sin^4 x \cos^3 x \, dx = 2\int_0^{\pi} \sin^4 x \cos^3 x \, dx$

$\int_0^{2a} f(x) dx = 0$ if $f(2a - x) = -f(x)$

$= 2\int_0^{a} f(x) dx$ if $f(2a - x) = f(x)$

Again $f(x) = \sin^4 x \cos^3 x$

$f(\pi - x) = \sin^4(\pi - x) \cos^3(\pi - x)$

$= \sin^4 x [-\cos^3 x]$

$= -\sin^4 x \cos^3 x$

$= -f(x)$

$= 0$.

(vi) $\int_0^1 |5x - 3| dx$

$$|5x-3| = \begin{cases} 5x-3 & \text{if } x \geq 3/5 \\ 3-5x & x < 3/5 \end{cases}$$

$= \int_0^{3/5} (3 - 5x) dx + \int_{3/5}^1 (5x - 3) dx$

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$$= \int_0^{3/5} (3-5x)\,dx + \int_{3/5}^{1} (5x-3)\,dx$$

$$= \left[3x - \frac{5x^2}{2}\right]_0^{3/5} + \left[\frac{5x^2}{2} - 3x\right]_{3/5}^{1}$$

$$= \left[3\left(\frac{3}{5}\right) - \frac{5}{2}\left(\frac{3}{5}\right)^2\right] + \left[\left(\frac{5}{2} - 3\right) - \left(\frac{5}{2}\left(\frac{3}{5}\right)^2 - 3\left(\frac{3}{5}\right)\right)\right]$$

$$= \frac{9}{5} - \frac{5}{2}\left(\frac{9}{25}\right) + \left[-\frac{1}{2} - \left(\frac{5}{2}\times\frac{9}{25} - \frac{9}{5}\right)\right]$$

$$= \frac{9}{5} - \frac{9}{10} + \left[-\frac{1}{2} - \left(\frac{9}{10} - \frac{9}{5}\right)\right]$$

$$= \frac{9}{5} - \frac{9}{10} - \frac{1}{2} - \frac{9}{10} + \frac{9}{5}$$

$$\frac{18-9-5-9+18}{10} = \boxed{\frac{13}{10}}$$

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VII

$$\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt.$$

Let $I = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt$ ——— (1)

Consider $I_1 = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt$.

put $t = \sin^2 x$

$dt = 2\sin x \cos x\,dx$.

when $t = 0$, $\sin^2 x = 0 \Rightarrow x = 0$

when $t = \sin^2 x$, $x = x$

$$\therefore I_1 = \int_0^{x} \sin^{-1}\sqrt{\sin^2 x}\,(2\sin x \cos x\,dx)$$

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$$= \int_0^x \sin^{-1}(\sin x) \cdot 2 \sin x \cos x \, dx$$

$$= \int_0^x 2x \sin x \cos x \, dx = \int_0^x x \sin 2x \, dx$$

$$I_2 = \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt$$

put $t = \cos^2 x$ $\quad dt = -2\cos x \sin x \, dx$

$\therefore -\sin 2x \, dx$

when $t = 0$, $\cos^2 x = 0$

$\cos x = 0$

$x = \pi/2$

when $t = \cos^2 x$, $t = x$

$$\therefore I_2 = \int_{\pi/2}^{x} \cos^{-1}\sqrt{\cos^2 x}\,(-\sin 2x)\, dx$$

$$= -\int_{\pi/2}^{x} \cos^{-1}(\cos x)(\sin 2x)\, dx$$

$$= -\int_{\pi/2}^{x} x \sin 2x \, dx = \int_{x}^{\pi/2} x \sin 2x \, dx$$

$\therefore$ (1) $\Rightarrow \int_0^x x \sin 2x \, dx + \int_x^{\pi/2} x \sin 2x \, dx$

$$= \int_0^{\pi/2} x \sin 2x \, dx$$

$u = x$ $\quad V = \sin 2x$

$u' = 1$ $\quad V_1 = -\frac{\cos 2x}{2}$

$V_2 = -\frac{\sin 2x}{4}$

Bernoulli's formula

$$\int uv\, dx = uV_1 - u'V_2$$

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$$\therefore \int_0^{\pi/2} x \sin 2x \, dx$$

$$= \left[x \left(-\frac{\cos 2x}{2} \right) - \left(-\frac{\sin 2x}{4} \right) \right]_0^{\pi/2}$$

$$= \left[-\frac{x \cos 2x}{2} + \frac{\sin 2x}{4} \right]_0^{\pi/2}$$

$$= \left(-\frac{\pi/2 \cos \pi}{2} + \frac{\sin \pi}{4} \right) - (0 + 0)$$

$$= -\frac{\pi}{4}(-1) + \frac{0}{4}$$

$$= \pi/4 \text{ //.}$$

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(VIII) $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$.

Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$

put $x = \tan\theta$ | $x=0,\ \tan\theta = 0,\ \theta = 0$

$dx = \sec^2\theta\,d\theta$ | $x=1,\ \tan\theta = 1,\ \theta = \pi/4$

$\therefore I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\cdot \sec^2\theta\,d\theta$

$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta}\cdot \sec^2\theta\cdot d\theta$.

$= \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$.

$=$

Applying the property

$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

$\therefore I = \int_0^{\pi/4} \log\left[1+\tan(\pi/4-\theta)\right]d\theta$.

$= \int_0^{\pi/4} \log\left[1+\frac{\tan\pi/4 - \tan\theta}{1+\tan\pi/4\cdot\tan\theta}\right]d\theta$

$= \int_0^{\pi/4} \log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta$

$= \int_0^{\pi/4} \log\left[\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right]d\theta$

$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right)d\theta$.

$=$

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$$\int_0^{\pi/4} [\log 2 - \log(1+\tan\theta)]\, d\theta$$

$$= \int_0^{\pi/4} \log 2\, d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta .$$

$$= \log 2 \int_0^{\pi/4} d\theta - I$$

$$\therefore 2I = \log 2 \int_0^{\pi/4} d\theta = \log 2 [\pi/4 - 0]$$

$$= \pi/4 \log 2$$

$$\therefore I = \frac{\pi}{8} \log 2$$

(IX) $\int_0^{\pi} \frac{x \sin x}{1+\sin x}\, dx$

Let $I = \int_0^{\pi} \frac{x \sin x}{1+\sin x}\, dx$ — ①

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Applying the property

$$\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$$

$$\therefore I = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)}\, dx .$$

$$= \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\sin x} \quad \text{—— ②}$$

① + ② $\Rightarrow$ $2I = \int_0^{\pi} \frac{x \sin x}{1+\sin x}\, dx + \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\sin x}$

$$= \int_0^{\pi} \frac{x \sin x + (\pi-x)\sin x}{1+\sin x}\, dx$$

$$= \int_0^{\pi} \frac{\sin x (x+\pi-x)}{1+\sin x}\, dx$$

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$$= \pi \int_0^{\pi} \frac{\sin x}{1+\sin x}\, dx .$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\, dx$$

$$= \pi \int_0^{\pi} \frac{\sin x (1-\sin x)}{1-\sin^2 x}\, dx .$$

$$= \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{\cos^2 x}\, dx$$

$$= \pi \left[\int_0^{\pi} \frac{\sin x}{\cos^2 x}\, dx - \int_0^{\pi} \frac{\sin^2 x}{\cos^2 x}\, dx \right]$$

$$= \pi \left[\int_0^{\pi} \sec x \tan x\, dx - \int_0^{\pi} \tan^2 x\, dx \right]$$

$$= \pi \left[\sec x \right]_0^{\pi} - \pi \int_0^{\pi} \tan^2 x\, dx .$$

$$= \pi \left[\sec \pi - \sec 0 \right] - \pi \left[\int_0^{\pi} (\sec^2 x - 1)\, dx \right]$$

$$= \pi \left[-1 - 1 \right] - \pi \left[\int_0^{\pi} \sec^2 x\, dx - \int_0^{\pi} dx \right]$$

$$= -2\pi - \pi \left[\tan x - x \right]_0^{\pi}$$

$$= -2\pi - \pi \left[(\tan \pi - \pi) - (\tan 0 - 0) \right]$$

$$= -2\pi - \pi \left[0 - \pi \right]$$

$$= -2\pi + \pi^2$$

$$2I = \pi^2 - 2\pi$$

$$I = \frac{\pi^2}{2} - \pi$$

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X) $\int_{\pi/8}^{3\pi/8} \frac{1}{1+\sqrt{\tan x}}\,dx$

Let $I = \int_{\pi/8}^{3\pi/8} \frac{1}{1+\sqrt{\tan x}}\,dx$ — ①

WKT $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$.

$\therefore I = \int_{\pi/8}^{3\pi/8} \frac{1}{1+\sqrt{\tan\left(\frac{\pi}{8}+\frac{3\pi}{8}-x\right)}}\,dx$

$= \int_{\pi/8}^{3\pi/8} \frac{1}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}\,dx$

$= \int_{\pi/8}^{3\pi/8} \frac{1}{1+\sqrt{\cot x}}$

$= \int_{\pi/8}^{3\pi/8} \frac{1}{1+\frac{1}{\sqrt{\tan x}}}\,dx$

$= \int_{\pi/8}^{3\pi/8} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\,dx$ — ②

① + ② ⇒ $2I = \int_{\pi/8}^{3\pi/8} \frac{1}{1+\sqrt{\tan x}}\,dx + \int_{\pi/8}^{3\pi/8} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\,dx$

$= \int_{\pi/8}^{3\pi/8} \left(\frac{1+\sqrt{\tan x}}{1+\sqrt{\tan x}}\right)dx = \int_{\pi/8}^{3\pi/8} dx$

$= [x]_{\pi/8}^{3\pi/8} = 3\pi/8 - \pi/8 = 2\pi/8 = \pi/4$

$2I = \pi/4 \Rightarrow I = \pi/8$ //.

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XI $\int_0^{\pi} x\left[\sin^2(\sin x)+\cos^2(\cos x)\right]dx$

Let $I=\int_0^{\pi} x\left[\sin^2(\sin x)+\cos^2(\cos x)\right]dx$ — (1)

WKT $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$.

$\therefore I=\int_0^{\pi}(\pi-x)\left[\sin^2(\sin(\pi-x))+\cos^2(\cos(\pi-x))\right]dx$

$=\int_0^{\pi}(\pi-x)\left[\sin^2(\sin x)+\cos^2(\cos x)\right]dx$ — (2)

(1)+(2) $\Rightarrow \int_0^{\pi}\left[\sin^2(\sin x)+\cos^2(\cos x)\right]\left[x+\pi-x\right]dx$

$2I=\pi\int_0^{\pi}\left[\sin^2(\sin x)+\cos^2(\cos x)\right]dx$

$\therefore I=\frac{\pi}{2}\int_0^{\pi}\left[\sin^2(\sin x)+\cos^2(\cos x)\right]dx$

WKT $\int_0^{2a} f(2a-x)\,dx=2\int_0^a f(x)\,dx$

$\therefore I=\frac{\pi}{2}\times 2\int_0^{\pi/2}\left[\sin^2(\sin x)+\cos^2(\cos x)\right]dx$

$=\pi\int_0^{\pi/2}\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$ — (3)

$=\pi\int_0^{\pi/2}\left[\sin^2\left(\sin\left(\frac{\pi}{2}-x\right)\right)+\cos^2\left(\cos\left(\frac{\pi}{2}-x\right)\right)\right]dx$

$=\pi\int_0^{\pi/2}\left(\sin^2(\cos x)+\cos^2(\sin x)\right)dx$ — (4)

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(3) + (4) ⇒ 2I

$$\pi \int_0^{\pi/2} [\sin^2(\sin x) + \cos^2(\cos x) + \sin^2(\cos x) + \cos^2(\sin x)]\, dx$$

$$= \pi \int_0^{\pi/2} [1+1]\, dx = \pi \int_0^{\pi/2} 2\, dx$$

$$= 2\pi [x]_0^{\pi/2}$$

$$= 2\pi \left(\frac{\pi}{2}\right)$$

$$= \pi^2$$

$$2I = \pi^2$$

$$I = \frac{\pi^2}{2}$$

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Bernoulli's formula.

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

Eg 9.31

Evaluate $\int_0^{\pi} x^2 \cos nx\,dx$.

Taking $u = x^2$ | $v = \cos nx$.

$u' = 2x$ | $v_1 = +\frac{\sin nx}{n}$

$u'' = 2$ | $v_2 = -\frac{\cos nx}{n^2}$

$v_3 = -\frac{(\sin nx)}{n^3}$

Bernoulli's formula

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 + \cdots$

$\therefore \int_0^{\pi} x^2 \cos nx\,dx$

$= \left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$

$= \left(\pi^2 \frac{\sin n\pi}{n} + 2\pi \frac{\cos n\pi}{n^2} - 2\frac{\sin n\pi}{n^3} - 0 + 2(0) - 2(0)\right)$

$= 0 + \frac{2\pi}{n^2}(-1)^n - 0$

$= \frac{2\pi}{n^2}(-1)^n \quad \because \cos n\pi = (-1)^n$

$\sin n\pi = 0$.

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9.32 Evaluate $\int_0^1 e^{-2x}(1+x-2x^3)\,dx$

$u = 1+x-2x^3$

$u' = 1-6x^2$

$u'' = -12x$

$u''' = -12$

$v = e^{-2x}$

$v_1 = \frac{e^{-2x}}{-2}$

$v_2 = \frac{e^{-2x}}{4}$

$v_3 = \frac{e^{-2x}}{-8}$

$v_4 = \frac{e^{-2x}}{16}$

Bernoulli's formula,

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

$\therefore \int_0^1 e^{-2x}(1+x-2x^3)\,dx$

$= \left[(1+x-2x^3)\frac{e^{-2x}}{-2} - (1-6x^2)\frac{e^{-2x}}{4} + (-12x)\frac{e^{-2x}}{-8} - (-12)\frac{e^{-2x}}{16}\right]_0^1$

$= \left[\frac{e^{-2x}}{16}\left[-8-8x+16x^3-4+24x^2+24x+12\right]\right]_0^1$

$= \left[\frac{e^{-2x}}{16}\left[16x^3+24x^2+16x\right]\right]_0^1$

$= \left[\frac{e^{-2}}{16}(16+24+16) - \frac{e^{-0}}{16}(0)\right]$

$= \frac{e^{-2}}{16} \times 56 = \frac{7}{2}e^{-2} = \frac{7}{2e^2}$

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Eg 9.33 Evaluate $\int_0^{2\pi} x^2 \sin nx \, dx$

$u = x^2$ $\quad v = \sin nx$

$u' = 2x$ $\quad v_1 = -\frac{\cos nx}{n}$

$u'' = 2$ $\quad v_2 = -\frac{\sin nx}{n^2}$

$v_3 = -\left(-\frac{\cos nx}{n^3}\right)$

Bernoulli's Theorem

$$\int_0^{2\pi} x^2 \sin nx = uv_1 - u'v_2 + u''v_3$$

$$= \left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \left[\left(-\frac{4\pi^2}{n}\cos n(2\pi) + \frac{4\pi}{n^2}\sin n(2\pi) + \frac{2}{n^3}\cos n(2\pi)\right) - \left(0 + 0 + \frac{2}{n^3}\cos 0\right)\right]$$

$$= \left[\left(-\frac{4\pi^2}{n} + 0 + \frac{2}{n^3}\right) - \frac{2}{n^3}\right]$$

$$= -\frac{4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3}$$

$$= -\frac{4\pi^2}{n} \; //$$

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9.34 Evaluate $\int_{-1}^{1} e^{-\lambda x}(1-x^2)\,dx$

$u = 1-x^2$ $\quad v = e^{-\lambda x}$.

$u' = -2x$ $\quad v_1 = \frac{e^{-\lambda x}}{-\lambda}$

$u'' = -2$ $\quad v_2 = \frac{e^{-\lambda x}}{\lambda^2}$

$v_3 = \frac{e^{-\lambda x}}{-\lambda^3}$

Bernoulli's Th

$\int uv\,dx = uv_1 - u'v_2 + u''v_3$

$\therefore \int_{-1}^{1} e^{-\lambda x}(1-x^2)\,dx$

$= \left[(1-x^2)\frac{e^{-\lambda x}}{-\lambda} - (-2x)\frac{e^{-\lambda x}}{\lambda^2} + (-2)\frac{e^{-\lambda x}}{-\lambda^3}\right]_{-1}^{1}$

$= \left[\frac{e^{-\lambda x}}{\lambda^3}\left(-\lambda^2(1-x^2) + 2x\lambda + 2\right)\right]_{-1}^{1}$

$= \frac{e^{-\lambda}}{\lambda^3}\left(-\lambda^2(0) + 2(0)\lambda + 2\right) - \frac{e^{+\lambda}}{\lambda^3}\left(-\lambda^2(1-1) + 2(-1)\lambda + 2\right)$

$= \frac{e^{-\lambda}}{\lambda^3}(2\lambda+2) - \frac{e^{\lambda}}{\lambda^3}(-2\lambda+2)$

$= \frac{2e^{-\lambda}}{\lambda^2} + \frac{2e^{-\lambda}}{\lambda^3} + \frac{2e^{\lambda}}{\lambda^2} - \frac{2e^{\lambda}}{\lambda^3}$

$= \frac{2}{\lambda^2}(e^{-\lambda}+e^{\lambda}) + \frac{2}{\lambda^3}(e^{-\lambda}-e^{\lambda})$

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EXERCISE 9.4

1) Evaluate $\int_0^1 x^3 e^{-2x}\,dx$

Let $u = x^3$ | $v = e^{-2x}$

$u' = 3x^2$ | $v_1 = \frac{e^{-2x}}{-2}$

$u'' = 6x$ | $v_2 = \frac{e^{-2x}}{4}$

$u''' = 6$ | $v_3 = \frac{e^{-2x}}{-8}$

| $v_4 = \frac{e^{-2x}}{16}$

Bernoulli's formula

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4$

$\therefore \int_0^1 x^3 e^{-2x}\,dx$

$= \left[x^3\left(\frac{e^{-2x}}{-2}\right) - 3x^2\left(\frac{e^{-2x}}{4}\right) + 6x\left(\frac{e^{-2x}}{-8}\right) - 6\left(\frac{e^{-2x}}{16}\right)\right]_0^1$

$= \left[\frac{e^{-2x}}{16}\left(-8x^3 - 12x^2 + 12x - 6\right)\right]_0^1$

$= \left[\left(\frac{e^{-2}}{16}\left(-8-12-12-6\right) - \frac{e^{-0}}{16}\left(0+0+0-6\right)\right]$

$= -\frac{38}{16e^2} + \frac{6}{16} = \frac{3}{8} - \frac{19}{8e^2}$ //

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2). $\int_0 \frac{\sin(3\tan^{-1}x)\tan^{-1}x}{1+x^2}\,dx$

put $\tan^{-1}x = t$

Diff w.r to x

$\frac{1}{1+x^2}\,dx = dt$

$x=0$, $\tan^{-1}(0) = t \Rightarrow t = 0$

$x=1$, $\tan^{-1}(1) = t \Rightarrow t = \pi/4$

$\therefore \int_0^{\pi/4} \sin(3t)\cdot t\,dt$

Let $u = t$ $\quad v = \sin 3t$

$u' = 1$ $\quad v_1 = -\frac{\cos 3t}{3}$

$v_2 = -\frac{\sin 3t}{9}$

Bernoulli's formula

$\int uv\,dx = uv_1 - u'v_2$

$\therefore \int_0^1 \frac{\sin(3\tan^{-1}x)\tan^{-1}x}{1+x^2}\,dx = \int_0^{\pi/4} t\cdot\sin 3t\,dt$

$= \left[t\left(-\frac{\cos 3t}{3}\right) - \left(-\frac{\sin 3t}{9}\right)\right]_0^{\pi/4}$

$= \left[\left(-\frac{\cos 3\pi/4}{3}(\pi/4) + \frac{\sin 3\pi/4}{9}\right) - \left(0 + \frac{\sin 0}{9}\right)\right]$

$= -\frac{\pi}{4}\frac{\cos 135}{3} + \frac{\sin 135}{9} - 0$

$= -\frac{\pi}{4}\left(\frac{\cos(90+45)}{3}\right) + \frac{\sin(90+45)}{9}$

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$$-\frac{\pi}{12}\left(-\sin 45\right) + \frac{\cos 45}{9}$$

$$\frac{\pi}{12}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{9\sqrt{2}} = \frac{\pi}{12\sqrt{2}} + \frac{1}{9\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\pi}{12} + \frac{1}{9}\right)$$ //

---

3) $\int_0^{\frac{1}{\sqrt{2}}} \frac{e^{a\sin^{-1}x} \cdot \sin^{-1}x}{\sqrt{1-x^2}}\,dx$

put $t = \sin^{-1}x$

$dt = \frac{1}{\sqrt{1-x^2}}\,dx$

$x = 0,\ t = \sin^{-1}0 = 0$

$x = \frac{1}{\sqrt{2}},\ t = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \pi/4$

$\therefore \int_0^{\pi/4} e^{at} \cdot t\,dt$

Let $u = t$, $v = e^{at}$

$u' = 1$ $v_1 = \frac{e^{at}}{a}$

$v_2 = \frac{e^{at}}{a^2}$

Bernoulli's formula,

$\int uv\,dx = uv_1 - u'v_2$

$\therefore \int_0^{\pi/4} e^{at} \cdot t\,dt = \left[t\left(\frac{e^{at}}{a}\right) - \left(\frac{e^{at}}{a^2}\right)\right]_0^{\pi/4}$

$= \left[\left(\pi/4\ \frac{e^{\frac{\pi}{4}a}}{a} - \frac{e^{a\frac{\pi}{4}}}{a^2}\right) - \left(0 - \frac{e^0}{a^2}\right)\right]$

$\frac{e^{\frac{\pi}{4}a}}{a^2}\left(a\frac{\pi}{4} - 1\right) - \frac{1}{a^2}$ //.

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3) $\int_0^{1/\sqrt{2}} \frac{e^{\sin^{-1}x}\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Let $t = \sin^{-1}x$

$dt = \frac{1}{\sqrt{1-x^2}}\,dx$

| $x=0,\ t=\sin^{-1}0=0$

| $x=\frac{1}{\sqrt{2}},\ t=\sin^{-1}\frac{1}{\sqrt{2}} = \pi/4$

$\therefore \int_0^{\pi/4} e^t\, t.\, dt$

Let $u = t$ $\quad v = e^t$

$u' = 1$ $\quad v_1 = e^t$

$t'' = 0$ $\quad v_2 = e^t$

$v_3 = e^t$

$\therefore$

Applying Bernoulli's formula

$\int_0^{\pi/4} e^t.t\,dt = uv_1 - u'v_2 + u''v_3 - \cdots$

$= \left[te^t - e^t\right]_0^{\pi/4} = \left[e^t(t-1)\right]_0^{\pi/4}$

$= e^{\pi/4}(\pi/4 - 1) - e^0(0-1)$

$= e^{\pi/4}(\pi/4 - 1) + 1$

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Eg 9.35 Evaluate $\int_b^\infty \frac{1}{a^2+x^2}\,dx$

$$\int_b^\infty \frac{1}{a^2+x^2}\,dx = \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_b^\infty$$

$$= \frac{1}{a}\tan^{-1}\frac{\infty}{a} - \frac{1}{a}\tan^{-1}\frac{b}{a}$$

$$= \frac{1}{a}\tan^{-1}(\infty) - \frac{1}{a}\tan^{-1}\left(\frac{b}{a}\right)$$

$$= \frac{1}{a}\,\pi/2 - \frac{1}{a}\tan^{-1}\left(\frac{b}{a}\right)$$

$$= \frac{1}{a}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{b}{a}\right)\right]$$

Note:

1) $$\int_0^\infty \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\left[\frac{\pi}{2} - \tan^{-1}(0)\right]$$

$$= \frac{1}{a}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{2a}$$

2) $$\int_a^\infty \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{a}{a}\right)\right]$$

$$= \frac{1}{a}\left[\frac{\pi}{2} - \tan^{-1}(1)\right]$$

$$= \frac{1}{a}\left[\frac{\pi}{2} - \frac{\pi}{4}\right]$$

$$= \frac{1}{a}\frac{\pi}{4} = \frac{\pi}{4a}$$

Improper Integrals

$$\int_a^\infty f(x)dx, \quad \int_{-\infty}^a f(x)dx, \quad \int_{-\infty}^\infty f(x)dx$$

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(iii) $\int_{-\infty}^{\infty} \frac{1}{a^2+x^2}\,dx = 2\int_{0}^{\infty} \frac{dx}{a^2+x^2}$

$= 2\left(\frac{\pi}{2a}\right)$

$= \boxed{\pi/a}$

Eg 9.36 Evaluate $\int_0^{\pi/2} \frac{dx}{4\sin^2 x + 5\cos^2 x}$

Let $I = \int_0^{\pi/2} \frac{dx}{4\sin^2 x + 5\cos^2 x}$

$= \int_0^{\pi/2} \frac{dx}{\cos^2 x\left(4\frac{\sin^2 x}{\cos^2 x} + 5\right)}$

$= \int_0^{\pi/2} \frac{\sec^2 x\,dx}{4\tan^2 x + 5}$

Let $u = \tan x$, $du = \sec x\,dx$ | $x = 0,\ u = \tan 0 = 0$; $x = \pi/2,\ u = \tan \pi/2 = \infty$

$\therefore I = \int_0^{\infty} \frac{du}{4u^2+5}$ (This is improper Integral)

$= \frac{1}{4}\int_0^{\infty} \frac{du}{u^2 + \frac{5}{4}}$

$= \frac{1}{4}\int_0^{\infty} \frac{du}{u^2 + \left(\frac{\sqrt{5}}{2}\right)^2}$

$= \frac{1}{4} \times \frac{1}{\frac{\sqrt{5}}{2}}\left[\tan^{-1}\frac{u}{\frac{\sqrt{5}}{2}}\right]_0^{\infty}$

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$$\frac{1}{4} \times \frac{2}{\sqrt{5}} \left[\tan^{-1} \frac{2t}{\sqrt{5}} \right]_0^{\infty}$$

$$= \frac{1}{2\sqrt{5}} \left[\left(\tan^{-1}(\infty) - \tan^{-1}(0) \right) \right]$$

$$= \frac{1}{2\sqrt{5}} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{4\sqrt{5}} \text{ //}$$

EXERCISE 9.5

1) Evaluate $\int_0^{\pi/2} \frac{dx}{1+5\cos^2 x}$

Let $I = \int_0^{\pi/2} \frac{dx}{1+5\cos^2 x}$

$$= \int_0^{\pi/2} \frac{dx}{\cos^2 x \left(\frac{1}{\cos^2 x} + 5 \right)}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x \, dx}{\sec^2 x + 5}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x \, dx}{1 + \tan^2 x + 5} = \int_0^{\pi/2} \frac{\sec^2 x \, dx}{\tan^2 x + 6}$$

Consider $t = \tan x$ | $x = 0,\ t = \tan 0 = 0$

$dt = \sec^2 x \, dx$ | $x = \pi/2,\ t = \tan \pi/2 = \infty$

$$\therefore \int_0^{\infty} \frac{dt}{t^2+6} = \int_0^{\infty} \frac{dt}{t^2 + (\sqrt{6})^2}$$

$$= \left[\frac{1}{\sqrt{6}} \tan^{-1} \left(\frac{t}{\sqrt{6}} \right) \right]_0^{\infty}$$

$$= \frac{1}{\sqrt{6}} \tan^{-1}(\infty) - 0$$

$$= \frac{1}{\sqrt{6}} \pi/2 = \frac{\pi}{2\sqrt{6}} \text{ //}$$

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(11) $\int_0^{\pi/2} \frac{dx}{5+4\sin^2 x}$

Let $I = \int_0^{\pi/2} \frac{dx}{5+4\sin^2 x}$

Dividing both NR and DR by $\cos^2 x$

$$I = \int_0^{\pi/2} \frac{\frac{dx}{\cos^2 x}}{\frac{5}{\cos^2 x} + \frac{4\sin^2 x}{\cos^2 x}}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\, dx}{5\sec^2 x + 4\tan^2 x}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\, dx}{5(1+\tan^2 x) + 4\tan^2 x}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\, dx}{5+5\tan^2 x + 4\tan^2 x}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\, dx}{5+9\tan^2 x}$$

$$= \frac{1}{9}\int_0^{\pi/2} \frac{\sec^2 x\, dx}{\frac{5}{9}+\tan^2 x} \quad \text{---(1)}$$

put $t = \tan x$, $dt = \sec^2 x\, dx$

$x = 0,\ t = \tan 0 = 0$

$x = \pi/2,\ t = \tan \pi/2 = \infty$

$\therefore$ (1) $\Rightarrow \frac{1}{9}\int_0^{\infty} \frac{dt}{\frac{5}{9}+t^2}$

$$= \frac{1}{9}\int_0^{\infty} \frac{dt}{t^2+\left(\frac{\sqrt{5}}{3}\right)^2}$$

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XII/ADC/9.5/PS

$$= \frac{1}{9}\left[\frac{1}{\frac{\sqrt{5}}{3}}\tan^{-1}\left(\frac{t}{\frac{\sqrt{5}}{3}}\right)\right]_0^{\infty}$$

$$= \frac{1}{9}\left[\frac{3}{\sqrt{5}}\tan^{-1}\frac{3t}{\sqrt{5}}\right]_0^{\infty}$$

$$= \frac{1}{9}\left[\frac{3}{\sqrt{5}}\tan^{-1}(\infty) - 0\right]$$

$$= \frac{1}{9}\times\frac{3}{\sqrt{5}}\times\frac{\pi}{2} = \frac{3\pi}{18\sqrt{5}} = \frac{\pi}{6\sqrt{5}}$$

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Reduction formula :-

1) If n is positive integer, Then

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$

$$= \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3}, & \text{when } n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & \text{when } n \text{ is even} \end{cases}$$

2) $$\int_0^{\pi/2} \sin^n x \cos^m x\,dx$$

$$\begin{cases} \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{2}{3+n} \cdot \frac{1}{1+n}; & \text{if } m \text{ is odd and } n \text{ may be even or odd} \\ \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{1}{2+n} \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} & \end{cases}$$

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if m is even and n is odd

$$\frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{1}{2+n} \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

if m is even and n is even.

These formula can be expressed as a single formula

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x\,dx$$

$$= \frac{[(m-1)(m-3)\cdots][(n-1)(n-3)\cdots]}{(m+n)(m+n-2)\cdots}$$

to be multiplied by $\frac{\pi}{2}$ when m and n are both even integers.

4) $$\int_0^1 x^m (1-x)^n\,dx = \frac{m! \times n!}{(m+n+1)!}$$

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EXERCISE 9.6

i) Evaluate $\int_0^{\pi/2} \sin^{10} x \, dx$

$$I_n = \int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} I_{n-2}$$

Let $I_{10} = \int_0^{\pi/2} \sin^{10} x \, dx$

$$= \frac{9}{10} \times \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{9}{\cancel{10}_{2}} \times \frac{7}{8} \times \frac{\cancel{5}}{\cancel{6}_{2}} \times \frac{\cancel{3}}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{63\pi}{512}$$

(ii) $\int_0^{\pi/2} \cos^7 x \, dx$

$$\int_0^{\pi/2} \cos^7 x \, dx = \frac{6}{7} \times \frac{\cancel{4}}{5} \times \frac{2}{3} \times 1 = \frac{16}{35}$$

(iii) $\int_0^{\pi/4} \sin^6 2x \, dx$

Let $t = 2x$ | $x = 0, \ t = 0$

$dt = 2dx$ | $x = \pi/4, \ t = 2 \times \pi/4 = \pi/2$

$$\therefore \int_0^{\pi/2} \sin^6 t \, \frac{dt}{2} = \frac{1}{2} \int_0^{\pi/2} \sin^6 t \, dt$$

$$\frac{1}{2} \int_0^{\pi/2} \sin^6 t \, dt = \frac{1}{\cancel{2}} \times \frac{\cancel{5}}{\cancel{6}} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{5\pi}{64} //$$

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(IV) $\int_0^{\pi/6} \sin^5 3x\,dx$

put $t = 3x$ | $x = 0,\ t = 0$

$dt = 3\,dx$ | $x = \pi/6,\ t = 3 \times \frac{\pi}{6} = \pi/2$

$dx = \frac{dt}{3}$

$\therefore \int_0^{\pi/2} \sin^5 t \frac{dt}{3} = \frac{1}{3} \int_0^{\pi/2} \sin^5 t\,dt$

$= \frac{1}{3} \times \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{8}{45}$

(V) $\int_0^{\pi/2} \sin^2 x \cos^4 x\,dx$

$\frac{(2-1)(4-1)(4-3)}{6(6-2)(6-4)} \times \frac{\pi}{2}$

$\frac{1 \times 3 \times 1}{6 \times 4 \times 2} \times \frac{\pi}{2} = \pi/32$

$= \int_0^{\pi/2} (1 - \cos^2 x) \cos^4 x\,dx$

$= \int_0^{\pi/2} (\cos^4 x - \cos^6 x)\,dx$

$= \int_0^{\pi/2} \cos^4 x\,dx - \int_0^{\pi/2} \cos^6 x\,dx$

$= \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} - \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \pi/2$

$= \frac{3\pi}{16} - \frac{15\pi}{96}$

$= \frac{18\pi - 15\pi}{96} = \frac{3\pi}{96} = \frac{\pi}{32}$

(VI) $\int_0^{2\pi} \sin^7 \frac{x}{4}\,dx$

Let $t = \frac{x}{4}$ | $x = 0,\ t = 0$

$dt = \frac{dx}{4}$ | $x = 2\pi,\ t = \frac{2\pi}{4} = \pi/2$

$\therefore dx = 4\,dt$

$\therefore \int_0^{\pi/2} \sin^7 t\ 4\,dt = 4 \int_0^{\pi/2} \sin^7 t\,dt$

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$$\therefore = 4 \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{64}{35}$$

VII) $\int_0^{\pi/2} \sin^3\alpha \cos^5\alpha \, d\alpha$

$\int_0^{\pi/2} \sin^m x \cos^n x \, dx$

WKT $= \frac{[(m-1)(m-3)\cdots 2\cdot 1][(n-1)(n-3)\cdots 2\cdot 1]}{(m+n)(m+n-2)(m+n-4)\cdots}$

$= \frac{2}{8} \times \frac{4}{6} \times \frac{2}{4} \cdot \frac{1}{2} = \frac{1}{24}$

(VIII) $\int_0^1 x^2(1-x)^3 \, dx$

WKT $\int_0^1 x^m (1-x)^n dx = \frac{m! \times n!}{(m+n)!}$

$$\therefore I = \frac{2! \times 3!}{6!} = \frac{2 \times 1 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 2 \times 1 \times 3}$$

$$= \frac{1}{5 \times 4 \times 3} = \frac{1}{60} //$$

Eg 9.37 $\int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^4 x) \, dx$

Let $I = \int_0^{\pi/2} (\sin^2 x + \cos^4 x) \, dx$

$= \int_0^{\frac{\pi}{2}} \sin^2 x \, dx + \int_0^{\pi/2} \cos^4 x \, dx$

$= \frac{1}{2} \times \frac{\pi}{2} + \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$

$= \frac{\pi}{4} + \frac{3\pi}{16} = \frac{4\pi + 3\pi}{16} = \frac{7\pi}{16}$

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9.38 Evaluate $\int_0^{\pi/2} \begin{vmatrix} \cos^4 x & 7 \\ \sin^5 x & 3 \end{vmatrix} dx$

$$I = \int_0^{\pi/2} (3\cos^4 x - 7\sin^5 x)\, dx$$

$$= 3\int_0^{\pi/2} \cos^4 x - 7\int_0^{\pi/2} \sin^5 x\, dx$$

$$= 3 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} - 7 \times \frac{4}{5} \times \frac{2}{3}$$

$$= \frac{9\pi}{16} - \frac{56}{15} \text{ //}$$

9.39 Find the value of

(i) $\int_0^{\pi/2} \sin^5 x \cos^4 x$

WKT $\int_0^{\pi/2} \sin^m x \cos^n x$

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XII/AI9.6/P5

$$= \frac{[(m-1)(m-3)\cdots 2.1][(n-1)(n-3)\cdots 2.1]}{(m+n)(m+n-2)(m+n-4)\cdots}$$

$$\therefore \int_0^{\pi/2} \sin^5 x \cos^4 x\, dx = \frac{4 \times 2 \times 3 \times 1}{9 \times 7 \times 5 \times 3}$$

$$= \frac{8}{315}$$

9.39 (ii) $\int_0^{\pi/2} \sin^4 x \cos^6 x\, dx$

$$\int_0^{\pi/2} \sin^4 x \cos^6 x\, dx = \frac{3 \times 1 \times 5 \times 3 \times 1}{2 \times 8 \times 6 \times 4 \times 2} \times \frac{\pi}{2}$$

$$= \frac{3\pi}{512} = \frac{3\pi}{512} \text{ //}$$

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9.40 Evaluate $\int_0^{2a} x^2 \sqrt{2ax - x^2}\, dx$

put $x = 2a\cos^2\theta$

$dx = -4a\cos\theta \cdot \sin\theta$

when $x = 0$, $2a\cos^2\theta = 0 \Rightarrow \theta = \pi/2$

when $x = 2a$, $2a\cos^2\theta = 2a$

$\Rightarrow \cos^2\theta = 1$

$\cos\theta = 1$

$\theta = 0$

$\therefore I = \int_{\pi/2}^{0} 4a^2\cos^4\theta \sqrt{4a^2\cos^2\theta - 4a^2\cos^4\theta} \cdot (-4a\cos\theta \cdot \sin\theta)$

$= \int_0^{\pi/2} 4a^2\cos^4\theta \sqrt{4a^2\cos^2\theta(1 - \cos^2\theta)} \cdot 4a\cos\theta \cdot \sin\theta$

$= \int_0^{\pi/2} 4a^2\cos^4\theta \sqrt{4a^2\cos^2\theta \cdot \sin^2\theta} \cdot 4a\cos\theta \cdot \sin\theta$

$= \int_0^{\pi/2} 4a^2\cos^4\theta \cdot 2a\cos\theta \cdot \sin\theta \cdot 4a\cos\theta \cdot \sin\theta \, d\theta$

$= 32 \int_0^{\pi/2} a^4 \cos^6\theta \sin^2\theta \, d\theta$

$= 32a^4 \frac{5 \times 3 \times 1 \times 1}{8 \times 6 \times 4 \times 2} \times \frac{\pi}{2}$

$= \frac{5a^4\pi}{8}$

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9.42 Evaluate $\int_0^1 x^3(1-x)^4\,dx$

WKT $\int_0^1 x^m(1-x)^n\,dx = \frac{m!\,n!}{(m+n+1)!}$

$\therefore \int_0^1 x^3(1-x)^4\,dx = \frac{3!\times 4!}{8!}$

$= \frac{1\times 2\times 3\times 1\times 2\times 3\times 4}{1\times 2\times 3\times 4\times 5\times 6\times 8\times 7}$

$= \frac{1}{280}$ //

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Eg 9.43 prove that $\int_0^\infty e^{-x} x^n dx = n!$

$\int_0^\infty e^{-x} x^n dx$

Applying Integration by parts

$u = x^n \quad dv = e^{-x}, \quad v = e^{-x}(-1) = -e^{-x}$

$\int u\,dv = uv - \int v\,du$

$\int_0^\infty e^{-x} x^n = \left[x^n(-e^{-x})\right]_0^\infty - \left[\int -e^{-x}(nx^{n-1})\,dx\right]$

$= \left[-\infty^n e^{-\infty} - (-0)\right] + \int_0^\infty n e^{-x} x^{n-1} dx$

Let $I_n = \int_0^\infty e^{-x} x^n dx$ Then

$I_n = n I_{n-1}$

$= n(n-1) I_{n-2}$

$= n(n-1)(n-2) I_{n-3}$

proceeding like this

$I_n = n(n-1)(n-2) \cdots 2.1$

$= n!$

Result

$\int_0^\infty e^{-x} x^n dx = n!$

9.44 Evaluate $\int_0^\infty e^{-ax} x^n dx$

$t = ax$ | $x = 0, \quad t = a(0) = 0$

$dt = a\,dx$ | $x = \infty, \quad t = a(\infty) = \infty$

$\therefore dx = \frac{dt}{a}$

$\therefore I = \int_0^\infty e^{-t} \left(\frac{t}{a}\right)^n \frac{dt}{a}$

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$$I = \frac{1}{a^{n+1}} \int_0^\infty e^{-t} t^n dt$$

$$= \frac{1}{a^{n+1}} \cdot n! = \frac{n!}{a^{n+1}} //.$$

9.45 Show that $\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$

put $x = \sqrt{u}$

$dx = \frac{1}{2\sqrt{u}} du$

when $x = 0$, $u = 0$

when $x = \infty$, $u = \infty$

$$\therefore 2\int_0^\infty e^{-x^2} x^{2n-1} dx$$

$$= 2\int_0^\infty e^{-u} (\sqrt{u})^{2n-1} \frac{1}{2\sqrt{u}} du$$

XII/A2/9.7/P2

$$= \int_0^\infty e^{-u} (\sqrt{u})^{2n} \times (\sqrt{u})^{-1} \frac{1}{\sqrt{u}} du$$

$$= \int_0^\infty e^{-u} u^n \frac{1}{u} du$$

$$= \int_0^\infty e^{-u} u^{n-1} du = \Gamma n .$$

9.46 Evaluate $\int_0^\infty \frac{x^n}{n^x} dx$.

Using the formula, $n = e^{\log_e n}$

$$\therefore I = \int_0^\infty \frac{x^n}{n^x} dx = \int_0^\infty n^{-x} x^n dx$$

$$= \int_0^\infty \left(e^{\log_e n}\right)^{-x} x^n dx$$

$$= \int_0^\infty e^{-x \log n} x^n dx$$

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$$= \int_0^\infty e^{-x\log n} x^n dx$$

put $u = x\log n \Rightarrow x = \frac{u}{\log n}$

$du = x(0) + \log n\, dx$.

$\therefore dx = \frac{du}{\log n}$

when $x = 0$, $u = 0$

when $x = \infty$, $u = \infty$

$\therefore I = \int_0^\infty e^{-x\log n} x^n dx$

$$= \int_0^\infty e^{-u} \left(\frac{u}{\log n}\right)^n \cdot \frac{du}{\log n}$$

$$= \frac{1}{(\log n)^{n+1}} \int_0^\infty e^{-u} u^n du = \frac{n!}{(\log n)^{n+1}}$$

Exercise 9.7

1) (i) Evaluate $\int_0^\infty x^5 e^{-3x} dx$

WKT $\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$

$\therefore \int_0^\infty x^5 e^{-3x} dx = \frac{5!}{3^{5+1}} = \frac{5!}{3^6}$ //

(ii) $\int_0^{\pi/2} \frac{e^{-\tan x}}{\cos^6 x} dx$

$= \int_0^{\pi/2} e^{-\tan x} \sec^6 x\, dx$

put $t = \tan x$

$dt = \sec^2 x\, dx$.

when $x = 0$, $t = 0$.

when $x = \pi/2$, $t = \infty$

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$$I = \int_0^{\pi/2} e^{-\tan x} \sec^2 x \cdot \sec^4 x \, dx$$

$$= \int_0^{\infty} e^{-t} (1+\tan^2 x)^2 \sec^2 x \, dx$$

$$= \int_0^{\infty} e^{-t} (1+t^2)^2 \, dt$$

$$= \int_0^{\infty} e^{-t} (1+t^4+2t^2) \, dt$$

$$= \int_0^{\infty} e^{-t} dt + \int_0^{\infty} e^{-t} t^4 dt + 2\int_0^{\infty} e^{-t} t^2 dt$$

$$= \left[e^{-t}\right]_0^{\infty} + 4! + 2 \times 2!$$

$$= 0 + 1 + 4! + 2(2!)$$

$$= 1 + 1 \times 2 \times 3 \times 4 + 4$$

$$= 5 + 24 = 29$$

2) If $\int_0^{\infty} e^{-\alpha x^2} x^3 \, dx = 32$ find α — XII/AI/9-7/p4

put $t = x^2$

$dt = 2x\,dx$.

$x\,dx = \frac{dt}{2}$

when $x = 0$, $t = 0$

$x = \infty$, $t = \infty$

$$\therefore I = \int_0^{\infty} e^{-\alpha x^2} \cdot x^2 \cdot x \, dx = 32$$

$$= \int_0^{\infty} e^{-\alpha t} \, t \, \frac{dt}{2} = 32$$

$$= \int_0^{\infty} e^{-\alpha t} \, t \, dt = 64 \quad (n=1, a=\alpha)$$

$$= \frac{1!}{\alpha^{1+1}} = 64$$

$$\frac{1}{\alpha^2} = 64 \Rightarrow \alpha^2 = \frac{1}{64}$$

$$\alpha = \frac{1}{8} \text{ //.}$$

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Area of the region bounded by a curve and lines :-

1) The area of the region bounded by a curve, above x-axis and the line $x = a$ and $x = b$ is

$$A = \int_a^b y\,dx$$

2) The area of the region bounded by a curve, below x-axis and the lines $x = a$, $x = b$ is

$$A = -\int_a^b y\,dx = \left| \int_a^b y\,dx \right|$$

3) Thus area of the region bounded by the curve to the right of y-axis, the lines $y = c$ and $y = d$ is $A = \int_c^d x\,dy$

4) The area of the region bounded by the curve to the left of y-axis, the lines $y = c$ and $y = d$ is

$$A = -\int_c^d x\,dy = \left| \int_c^d x\,dy \right|$$

Volume of the solid of revolution :-

a) The volume of the solid of revolution about x-axis is $V = \pi \int_a^b y^2\,dx$

b) The volume of the solid of revolution about y-axis is

$$V = \pi \int_c^d x^2\,dy .$$

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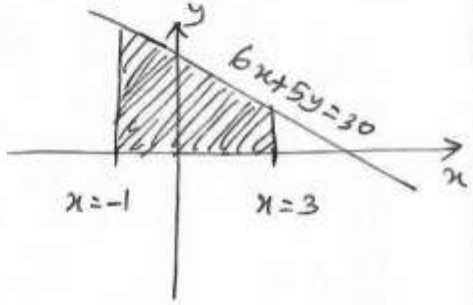
Eg 9.47 Find the area of the region bounded by the line $6x+5y=30$, x-axis and the lines $x=-1$ and $x=3$.



$6x+5y=30$

$5y = 30-6x$

$y = \frac{30-6x}{5}$

∴ The required area $A = \int_{-1}^{3} y\,dx$

$= \int_{-1}^{3} \left(\frac{30-6x}{5}\right) dx$

$= \frac{1}{5}\int_{-1}^{3} (30-6x)\,dx$

$= \frac{1}{5}\left[30x - 6\frac{x^2}{2}\right]_{-1}^{3}$

$= \frac{1}{5}\left[30x - 3x^2\right]_{-1}^{3}$

$= \frac{1}{5}\left[\left(30(3)-3(3)^2\right) - \left(30(-1)-3(-1)^2\right)\right]$

$= \frac{1}{5}\left[(90-27) - (-30-3)\right]$

$= \frac{1}{5}\left[63-(-33)\right] = \frac{1}{5}\left[63+33\right]$

$= \frac{1}{5}(96)$ sq.unit

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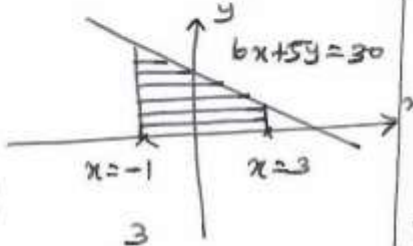
9.47 Find the area of the region bounded by the line $6x+5y=30$, x-axis and the line $x=-1$, $x=3$.

$6x+5y=30$

$5y=30-6x$

$y=\frac{1}{5}(30-6x)$



The required Area $A=\int_{-1}^{3} y\,dx$

$=\int_{-1}^{3}\frac{1}{5}(30-6x)$

$=\frac{1}{5}\left[30x-\frac{6x^2}{2}\right]_{-1}^{2}$

$=\frac{1}{5}\left[\left(30(3)-3(3)^2\right)-\left(30(-1)-3(-1)^2\right)\right]$

$=\frac{1}{5}\left[(90-27)-(-30-3)\right]$

$=\frac{1}{5}\left[63-(-33)\right]=\frac{1}{5}\left[63+33\right]$

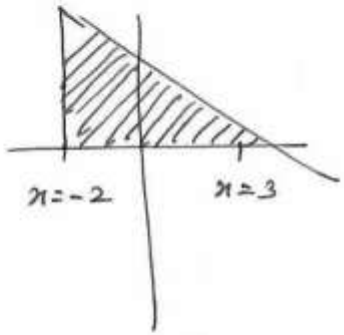
$=\frac{1}{5}(96)$ Sq. unit

9.48 Find the area of the region bounded by the line $7x-5y=35$, x-axis and the line $x=-2$, $x=3$

$7x-5y=35$

$7x-35=5y$

$y=\frac{1}{5}(7x-35)$



The required area

$A=\int_{-2}^{3} y\,dx$

$A=\int_{-2}^{3}\frac{1}{5}(7x-35)$

$=\frac{1}{5}\int_{-2}^{3}(7x-35)$

$=\frac{1}{5}\left[7\frac{x^2}{2}-35x\right]_{-2}^{3}$

$=\frac{1}{5}\left[\left(\frac{7(3)^2}{2}-35(3)\right)-\left(\frac{7(-2)^2}{2}-35(-2)\right)\right]$

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$$= \frac{1}{5}\left[\left(\frac{63}{2} - 105\right) - (14+70)\right]$$

$$= \frac{1}{5}\left[\frac{63-210}{2} - 84\right]$$

$$= \frac{1}{5}\left[\frac{-147}{2} - 84\right] = \frac{1}{5}\left[\frac{-147-168}{2}\right]$$

$$= \frac{1}{5}\left[\frac{-315}{2}\right]$$

$$= -\frac{63}{2}$$

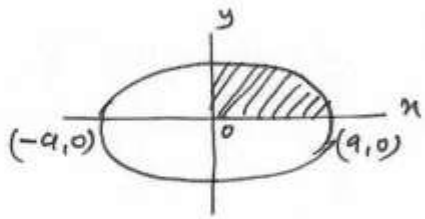
Area can't be negative

∴ Required Area = $\frac{63}{2}$ sq.unit.

9.49 Find the area of the Region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The ellipse is symmetric about both major and minor axes.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{a^2-x^2}{a^2}$$

$$y^2 = \frac{b^2(a^2-x^2)}{a^2} \Rightarrow y = \frac{b}{a}\sqrt{a^2-x^2}$$

∴ The required area = 4 x Area in the 1st Q.

$$= 4\int_0^a y\,dx$$

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{4b}{a}\left[\frac{a}{2}\sqrt{a^2-a^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right) + 0\right]$$

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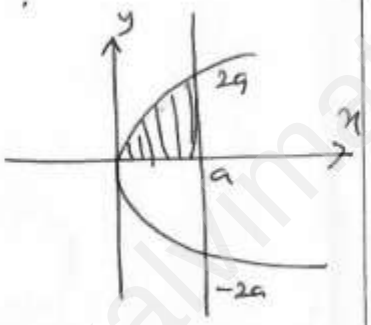
$$\frac{4b}{a}\left[\frac{a}{2}\sqrt{0} + \frac{a^2}{2}\sin^{-1}(1)\right]$$

$$\frac{4b}{a}\left[\frac{a^2}{2} \times \frac{\pi}{2}\right] = \pi ab \text{ sq. units.}$$

9.50 Find the area of region bounded between the parabola $y^2 = 4ax$ and its latus rectum.



The parabola is symmetric about x-axis

$$y^2 = 4ax$$

$$y = 2\sqrt{a}\sqrt{x}.$$

$\therefore$ The required area $A = 2\int_0^a y\,dx$

$$= 2\int_0^a 2\sqrt{a}\sqrt{x}\,dx$$

$$= 4\sqrt{a}\int_0^a \sqrt{x}\,dx$$

$$= 4\sqrt{a}\int_0^a x^{\frac{1}{2}}\,dx = 4\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^a$$

$$= 4\sqrt{a} \times \frac{2}{3}\left[a^{3/2}\right]$$

$$= \frac{8}{3}a^{\frac{3}{2}+\frac{1}{2}}$$

$$= \frac{8}{3}a^2 \text{ sqnits.}$$

9.51 Find the area of the region bounded by y axis and the parabola $x = 5 - 4y - y^2$

$$x = 5 - 4y - y^2$$

$$= -(y^2 + 4y - 5)$$

$$x = -\left[y^2 + 4y + 4 - 4 - 5\right]$$

$$x = -\left[(y+2)^2 - 9\right]$$

$$x = -(y+2)^2 + 9$$

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$x - 9 = -(y+2)^2$

$(y+2)^2 = -(x-9)$

Now $x = 5 - 4y - y^2$

The parabola meets the y axis, put $x = 0$ $\therefore$ $-(y^2+4y-5) = 0$

$y^2 + 4y - 5 = 0$

$(y+5)(y-1) = 0$

$y = -5, \; y = 1$

$\therefore$ The parabola meets the y-axis at $(0,1)$ and $(0,-5)$.

$\therefore$ The required Area $A = \int_{-5}^{1} x\,dy$

$= \int_{-5}^{1} (5 - 4y - y^2)\,dy$

$= \int_{-5}^{1} (5 - 4y - y^2)\,dy = \left[5y - \frac{4y^2}{2} - \frac{y^3}{3} \right]_{-5}^{1}$

$= \left[5y - 2y^2 - \frac{y^3}{3} \right]_{-5}^{1}$

$= \left[\left(5 - 2 - \frac{1}{3}\right) - \left(-25 - 50 + \frac{125}{3}\right) \right]$

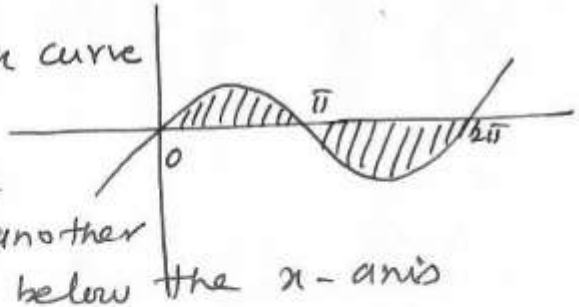
$= \left[\left(3 - \frac{1}{3}\right) - \left(-75 + \frac{125}{3}\right) \right]$

$= \left[\frac{8}{3} - \left(\frac{125 - 225}{3}\right) \right]$

$= \frac{8}{3} + \frac{100}{3} = \frac{108}{3} = 36$ sq.units.

9.52 Find the area of the region bounded by x-axis, the sine curve $y = \sin x$, the lines $x = 0$ and $x = 2\pi$

From the $\sin x$ curve one portion lies above the x-axis and another portion lies below the x-axis



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$\therefore$ The required area A

$$= \int_0^{\pi} y\,dx + \left| \int_{\pi}^{2\pi} y\,dx \right|$$

$$= \int_0^{\pi} \sin x\,dx + \left| \int_{\pi}^{2\pi} \sin x\,dx \right|$$

$$= \left[-\cos x\right]_0^{\pi} + \left|\left[-\cos x\right]_{\pi}^{2\pi}\right|$$

$$= \left[-\cos\pi - (-\cos 0)\right] + \left|\left[-\cos 2\pi - (-\cos\pi)\right]\right|$$

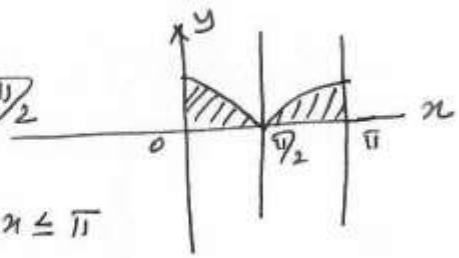
$$= \left[-(-1)+1\right] + \left|\left[-(1)+(-1)\right]\right|$$

$$= 2 + |-2| = 2+2 = 4 \text{ squnb.}$$

Ex 9.53 find the area of the region bounded by x-axis, the curve $y = |\cos x|$, the lines $x = 0$ and $x = \pi$

The given curve is

$$y = \begin{cases} \cos x, & 0 \le x \le \pi/2 \\ -\cos x, & \pi/2 \le x \le \pi \end{cases}$$



The required Area is

$$A = \int_0^{\pi} y\,dx = \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} (-\cos x)\,dx$$

$$= \left[\sin x\right]_0^{\pi/2} - \left[\sin x\right]_{\pi/2}^{\pi}$$

$$= (\sin\pi/2 - \sin 0) - (\sin\pi - \sin\pi/2)$$

$$= 1 - 0 - (0-1)$$

$$= 1+1 = 2$$

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EXERCISE 9.8

1. Find the area of the region by $3x-2y+6=0$, $x=-3$, $x=1$ and x-axis.

The given eqn is $3x-2y+6=0$

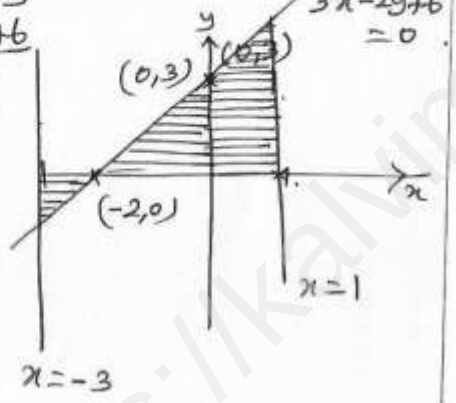
put $x=0$, $2y=6$, $y=3$, $(0,3)$

put $y=0$, $3x=-6$, $x=-2$, $(-2,0)$

$3x+6=2y$

$y=\frac{3x+6}{2}$

$3x-2y+6=0$



The line intersect the x-axis at $x=2$ and y-axis at $y=3$

limits:- $x=-3$ and $x=-2$ (below)

$x=-2$ and $x=1$ (above)

∴ The required Area $A=\int_{-3}^{1} y\,dx$

$$A=\int_{-3}^{-2} y\,dx+\int_{-2}^{1} y\,dx.$$

$$=\int_{-3}^{-2}-\left(\frac{3x+6}{2}\right)dx+\int_{-2}^{1}\left(\frac{3x+6}{2}\right)dx$$

$$=-\frac{1}{2}\int_{-3}^{-2}(3x+6)\,dx+\frac{1}{2}\int_{-2}^{1}(3x+6)\,dx$$

$$=-\frac{1}{2}\left[\left(\frac{3x^2}{2}+6x\right)\right]_{-3}^{-2}+\frac{1}{2}\left[\frac{3x^2}{2}+6x\right]_{-2}^{1}$$

$$=-\frac{1}{2}\left[\left(\frac{12}{2}-12\right)-\left(\frac{27}{2}-18\right)\right]+\frac{1}{2}\left[\left(\frac{3}{2}+6\right)-\left(\frac{12}{2}-12\right)\right]$$

$$=-\frac{1}{2}\left[(6-12)-\left(\frac{27-36}{2}\right)\right]+\frac{1}{2}\left[\frac{15}{2}-(6-12)\right]$$

$$=-\frac{1}{2}\left[-6-\left(-\frac{9}{2}\right)\right]+\frac{1}{2}\left[\frac{15}{2}+6\right]$$

$$=\frac{1}{2}\left[6-\frac{9}{2}+\frac{15}{2}+6\right]=\frac{1}{2}\left[12+\frac{6}{2}\right]$$

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$= \frac{1}{2}[12+3] = \frac{15}{2} = 7.5$ sq units.

Q2 Find the area of the region bounded by $2x-y+1=0$, $y=-1$, $y=3$ and y-axis.

Given line is $2x-y+1=0$

put $x=0$, $y=1$

put $y=0$, $2x=-1$

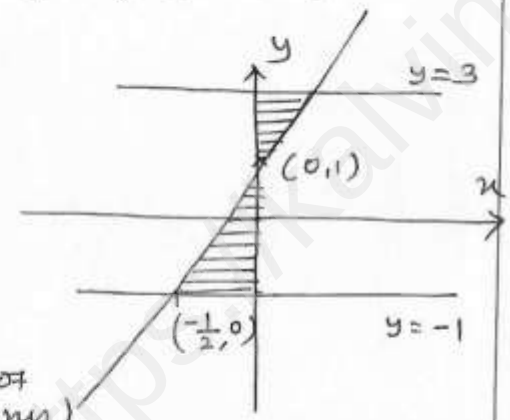
$x=-\frac{1}{2}$

∴ The pts are $(0,1)$, $(-\frac{1}{2},0)$.

$2x-y+1=0$

$2x=y-1$

$x=\frac{y-1}{2}$



limits

$y=-1$ and $y=1$ (Left of y axis)

$y=1$, and $y=3$ (Right of y-axis).

∴ The required Area $A = \int_{-1}^{3} x\,dy$.

$$A = \int_{-1}^{1} x\,dy + \int_{1}^{3} x\,dy$$

$$= \int_{-1}^{1} -\left(\frac{y-1}{2}\right)dy + \int_{1}^{3} \left(\frac{y-1}{2}\right)dy$$

$$= -\frac{1}{2}\int_{-1}^{1} (y-1)\,dy + \frac{1}{2}\int_{1}^{3} (y-1)\,dy$$

$$= -\frac{1}{2}\left[\frac{y^2}{2}-y\right]_{-1}^{1} + \frac{1}{2}\left[\frac{y^2}{2}-y\right]_{1}^{3}$$

$$= -\frac{1}{2}\left[\left(\frac{1}{2}-1\right)-\left(\frac{1}{2}+1\right)\right] + \frac{1}{2}\left[\left(\frac{9}{2}-3\right)-\left(\frac{1}{2}-1\right)\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{2}-\frac{3}{2}\right] + \frac{1}{2}\left[\frac{3}{2}+\frac{1}{2}\right]$$

$$= -\frac{1}{2}\left[-\frac{4}{2}\right] + \frac{1}{2}\left[\frac{4}{2}\right]$$

$$= \frac{1}{2}\times\frac{4}{2} + \frac{1}{2}\times\frac{4}{2}$$

$$= 1+1 = 2 \text{ sq units} //$$

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3) The area of the region bounded by the curve $2+x-x^2+y=0$, x-axis, $x=-3$, and $x=3$.

Given eqn is $2+x-x^2+y=0$

put $y=0 \Rightarrow -x^2+x+2=0$

$-(x^2-x-2)=0$

$x^2-x-2=0$

$(x-2)(x+1)=0$

$x=2, \quad x=-1$

put $x=0, \quad y=-2$

$\therefore$ The points are $(2,0), (-1,0)$ $(0,-2)$.

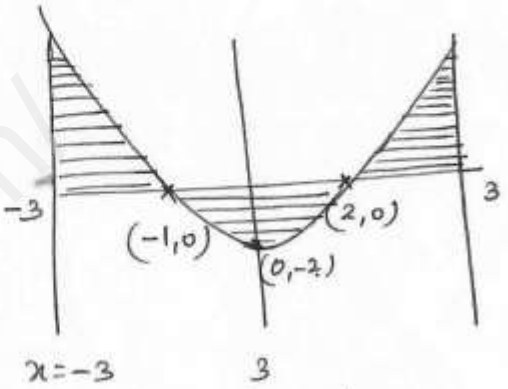
$2+x-x^2+y=0$

$y = x^2-x-2$

limits:- $x=-3$ and $x=-1$ (above)

$x=-1$, and $x=2$ (below)

$x=2$ and $x=3$ (above)



$x=-3 \qquad 3$

$\therefore$ Required Area $A = \int_{-3}^{3} y\,dx$.

$$A = \int_{-3}^{-1} y\,dx + \int_{-1}^{2} -y\,dx + \int_{2}^{3} y\,dx.$$

$$= \int_{-3}^{-1} (x^2-x-2)\,dx - \int_{-1}^{2} (x^2-x-2)\,dx + \int_{2}^{3} (x^2-x-2)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-3}^{-1} - \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{2} + \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{2}^{3}$$

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$$= \left[\left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{-27}{3} - \frac{9}{2} + 6\right)\right] -$$

$$\left[\left(\frac{8}{3} - \frac{4}{2} - 4\right) - \left(-\frac{1}{3} - \frac{1}{2} + 2\right)\right] +$$

$$\left[\left(\frac{27}{3} - \frac{9}{2} - 6\right) - \left(\frac{8}{3} - \frac{4}{2} - 4\right)\right]$$

$$= \left[\left(\frac{-2-3+12}{6}\right) - \left(\frac{-54-27+36}{6}\right)\right] -$$

$$\left[\left(\frac{16-12-24}{6}\right) - \left(\frac{-2-3+12}{6}\right)\right] +$$

$$\left[\left(\frac{54-27-36}{6}\right) - \left(\frac{16-12-24}{6}\right)\right]$$

$$= \left[\frac{7}{6} - \left(-\frac{45}{6}\right)\right] - \left[-\frac{20}{6} - \left(\frac{7}{6}\right)\right] +$$

$$\left[-\frac{9}{6} + \frac{20}{6}\right]$$

$$= \left(\frac{52}{6}\right) - \left(\frac{-27}{6}\right) + \left(\frac{11}{6}\right)$$

$$= \frac{63}{6} + \frac{27}{6} = \frac{36}{6} \quad \frac{90}{6} = 15$$

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4) Find the area of the region bounded by the line $y = 2x+5$ and the parabola $y = x^2-2x$

Eqn of parabola is $y = x^2-2x$ —①

Eqn of line is $y = 2x+5$ —②

To get the point of interseection solve ① and ②

$$x^2-2x = 2x+5$$
$$x^2-4x-5 = 0$$
$$(x-5)(x+1) = 0$$
$$x = -1 \text{ or } x = 5$$

put $x=-1$ in ① | put $x=5$ in ②

$y = 1+2 = 3$ | $y = 2(5)+5 = 15$

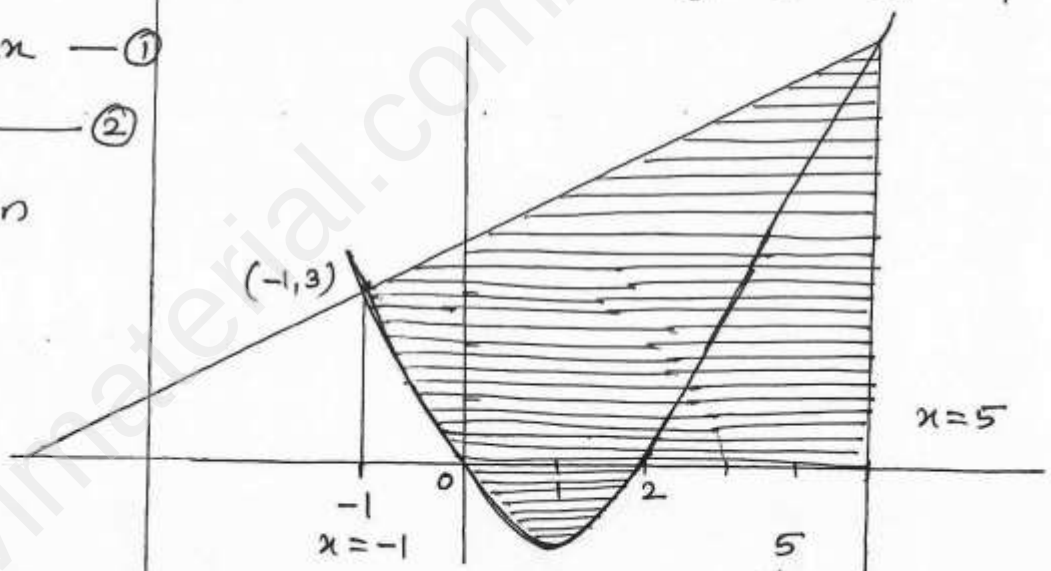
∴ The point of interseections are $(-1,3)$ and $(5,15)$

For parabola

x	0	1	2
y	0	-1	0

For line

x	0	-1	-2
y	5	3	1



∴ The required area $A = \int_{-1}^{5} (y_1-y_2)\,dx$

$$= \int_{-1}^{5} \left[(2x+5)-(x^2-2x)\right] dx$$

$$= \int_{-1}^{5} \left[4x+5-x^2\right] dx$$

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$$= \left[\frac{4x^2}{2} + 5x - \frac{x^3}{3} \right]_{-1}^{5}$$

$$= \left(\frac{100}{2} + 25 - \frac{125}{3} \right) - \left(\frac{4}{2} - 5 + \frac{1}{3} \right)$$

$$= \left(50 + 25 - \frac{125}{3} \right) - \left(2 - 5 + \frac{1}{3} \right)$$

$$= \left(\frac{225}{3} - \frac{125}{3} \right) - \left(-3 + \frac{1}{3} \right)$$

$$= \frac{100}{3} - \left(-\frac{8}{3} \right)$$

$$= \frac{108}{3} = \frac{108}{3} \text{ squnit.}$$

Q5 Find the area of the region bounded betn the curves $y = \sin x$ and $y = \cos x$ and the lines $x = 0$ and $x = \pi$

Given that $y = \sin x$ —— (1)

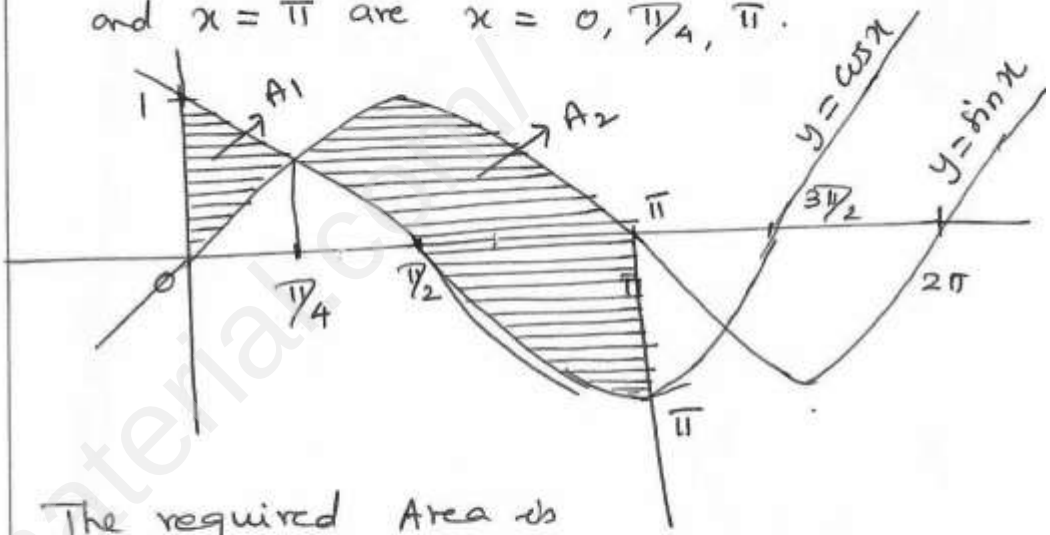
$y = \cos x$ —— (2)

from (1) and (2) $\sin x = \cos x$

$\Rightarrow x = \pi/4$

XII | A1 | 9.8 | P12

$\therefore$ The value of x betn $x = 0$ and $x = \pi$ are $x = 0, \pi/4, \pi$.



The required Area is combination of two pieces A_1 and A_2.

$\therefore$ Required Area $= A_1 + A_2$

$$= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi} (\sin x - \cos x)\, dx$$

$$= \left[\sin x + \cos x \right]_0^{\pi/4} + \left[-\cos x - \sin x \right]_{\pi/4}^{\pi}$$

$$= \left(\sin \pi/4 + \cos \pi/4 \right) - \left(\sin 0 + \cos 0 \right) + \left(-\cos \pi - \sin \pi \right) - \left(-\cos \pi/4 - \sin \pi/4 \right)$$

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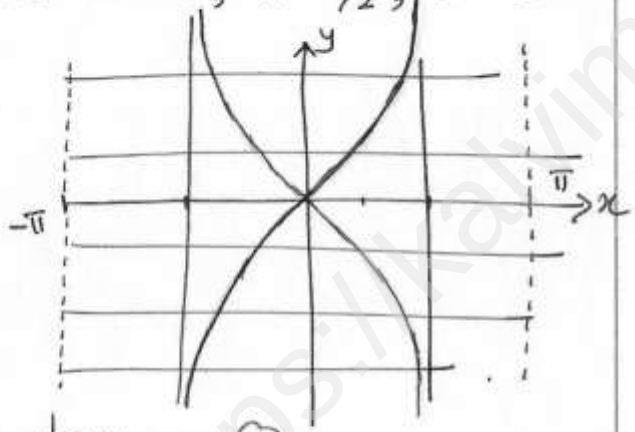
$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1) + (1-0) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{2}{\sqrt{2}} - 1 + 1 + \frac{2}{\sqrt{2}}$$

$$= \frac{4}{\sqrt{2}} = \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$ squnt.

---

6) Find the area of the region bounded by $y = \tan x$, $y = \cot x$ and the lines $x = 0$, $x = \pi/2$, $y = 0$



Given that

$y = \tan x$ — (1)

$y = \cot x$ — (2)

To get point of interseation Solve (1) and (2)

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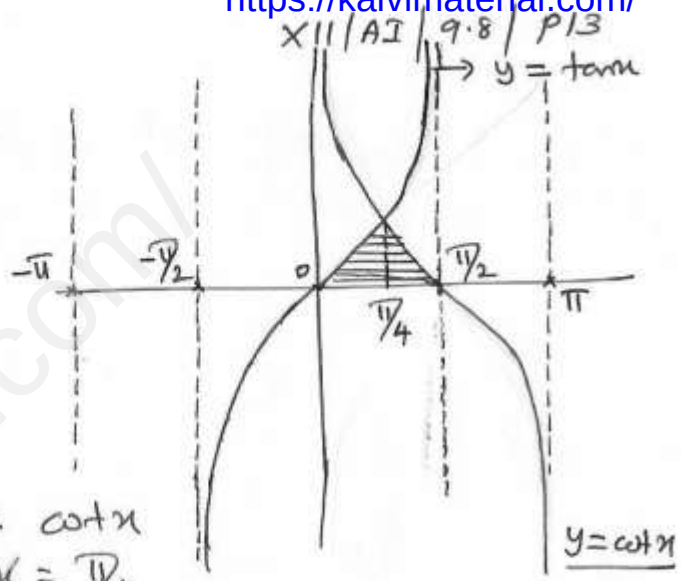

$\tan x = \cot x$

$\Rightarrow x = \pi/4$

∴ The value of x lies betn $x = 0$ and $x = \pi/2$ ~~are~~ is $\pi/4$.

limits: $x = 0$ and $x = \pi/4$

$x = \pi/4$ and $x = \pi/2$

$$\therefore \text{Required Area} = \int_0^{\pi/2} y\,dx$$

$$= \int_0^{\pi/4} \tan x\,dx + \int_{\pi/4}^{\pi/2} \cot x\,dx$$

$$= \left[\log \sec x\right]_0^{\pi/4} + \left[\log \sin x\right]_{\pi/4}^{\pi/2}$$

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$$\left[\log \sec \tfrac{\pi}{4} - \log \sec 0\right] + \left[\log \sin \tfrac{\pi}{2} - \log \sin \tfrac{\pi}{4}\right]$$

$$= \log \sqrt{2} - 0 + \log 1 - \log \frac{1}{\sqrt{2}}$$

$$= \log \sqrt{2} - \left(\log \frac{1}{\sqrt{2}}\right)$$

$$= \log \sqrt{2} - \left[\log 1 - \log \sqrt{2}\right]$$

$$= \log \sqrt{2} + \log \sqrt{2} = \log \sqrt{2} \times \sqrt{2}$$

$$= \log 2$$

7) Find the area of the region bounded by the parabola $y^2 = x$ and the line $y = x - 2$.

Eqn of parabola is $y^2 = x$ — ①

Eqn of line is $y = x - 2$ — ②

$x = y^2$ — ①

$x = y + 2$ — ②

$y^2 = y + 2$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

$y = -1$ and $y = 2$

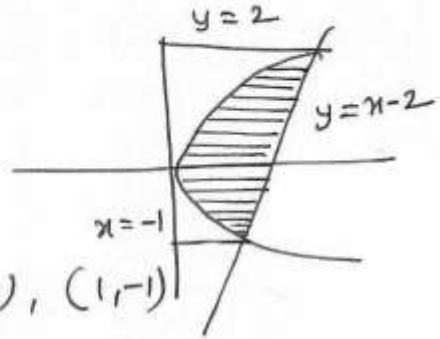
$y = -1$ in ①

$x = 1$

$y = 2$ in ②

$x = 4$

∴ The pts are $(4,2)$, $(1,-1)$



line $x = y + 2$

x	0	-1
y	-2	-3

$$\text{Required Area} = \int_{-1}^{2} (y_1 - y_2)\, dx$$

$$= \int_{-1}^{2} \left[y^2 - (y+2)\right] dx.$$

$$= \int_{-1}^{2} (y^2 - y - 2)\, dx$$

$$= \left[\frac{y^3}{3} - \frac{y^2}{2} - 2y\right]_{-1}^{2}$$

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$$= \left(\frac{8}{3} - \frac{4}{2} - 4\right) - \left(-\frac{1}{3} - \frac{1}{2} + 2\right)$$

$$= \left(\frac{16-12-24}{6}\right) - \left(\frac{-2-3+12}{6}\right)$$

$$= -\frac{20}{6} - \left(\frac{7}{6}\right)$$

$$= -\frac{27}{6} = -\frac{9}{2}$$

Area can't be negative

$\therefore$ Area $= \frac{9}{2}$ squnt.

---

8) Father of a family wishes to divide his square field bounded by $x=0$, $x=4$, $y=4$ and $y=0$ along the curve $y^2 = 4x$ and $x^2 = 4y$ into three equal parts for his wife, daughter and son. Is it possible to divide? If so, find the area tobe divided among Them.

$y^2 = 4x$ — ①

$x^2 = 4y$ — ②

$①^2 \Rightarrow y^4 = 16x^2$

$y^4 = 16(4y)$

$y^4 = 64y$

$y^4 - 64y = 0 \Rightarrow y(y^3 - 64) = 0$

$\Rightarrow y = 0$ or $y^3 - 64 = 0$

$y^3 = 64$

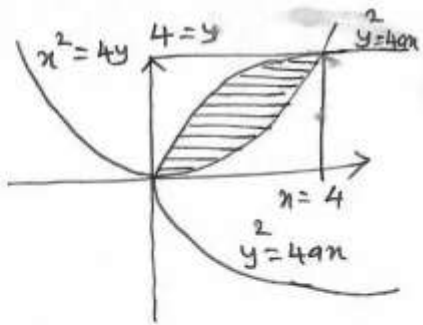
$y = \pm 4.$

$y = 0$ in ① $x = 0$

$y = 4$ in ① $16 = 4x \Rightarrow x = 4$

$y = -4$ in ① $16 = 4x \Rightarrow x = 4.$

$\therefore$ The point of intersections are $(0,0)$, $(4,4)$



$\therefore$ Required Area

$$A = \int_0^4 (y_1 - y_2)\,dx$$

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$$

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$$= \left[2\frac{x^{3/2}}{3/2} - \frac{x^3}{12} \right]_0^4$$

$$= \left[\frac{4}{3} x^{\frac{3}{2}} - \frac{x^3}{12} \right]_0^4$$

$$= \left(\frac{4}{3} \times 4^{3/2} - \frac{64}{12} \right)$$

$$= \frac{4}{3} \times 4 \times 2 - \frac{16}{3}$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3}$$ squnt// three equal parts.

Yes it is possible to divide the square into

---

9) The curve $y = (x-2)^2 + 1$ has a minimum point at P. A point Q on the curve is such that the slope of PQ is 2. Find the area bounded by the curve and the chord PQ.

$$y = (x-2)^2 + 1$$

$$\frac{dy}{dx} = 2(x-2)$$

$$\frac{d^2y}{dx^2} = 2$$

put $\frac{dy}{dx} = 0 \Rightarrow 2(x-2) = 0 \Rightarrow x = 2$

At $x = 2$, $\frac{d^2y}{dx^2} = 2 > 0$.

$\therefore$ $x = 2$ is minimum point.

$\therefore$ The ~~point is (2,1)~~

when $x = 2$, $y = (2-2)^2 + 1 = 1$

$\therefore$ The point is (2,1).

But slope PQ is 2.

$\therefore$ Eqn of chord is $y - y_1 = m(x - x_1)$

i.e. $y - 1 = 2(x-2)$

$y - 1 = 2x - 4$

$y = 2x - 3$

$y = (x-2)^2 + 1$ —— (1)

$y = 2x - 3$ —— (2)

solving (1) and (2)

$(x-2)^2 + 1 = 2x - 3$

$x^2 + 4 - 4x + 1 - 2x + 3 = 0$

$x^2 - 6x + 8 = 0$

$(x-2)(x-4) = 0$

$x = 2$ or $x = 4$.

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when $x = 2$, $y = 2(2) - 3$

$y = 1$

when $x = 4$, $y = 2(4) - 3$

$= 8 - 3$

$= 5$

$\therefore$ The pt of intersection are $(2,1), (4,5)$

$\therefore$ The required Area $= \int_2^4 (y_1 - y_2)\, dx$

$= \int_2^4 \left((2x-3) - [(x-2)^2 + 1] \right) dx$

$= \int_2^4 \left((2x-3) - [x^2 + 4 - 4x + 1] \right) dx$

$= \int_2^4 (2x - 3 - x^2 + 4x - 5)\, dx$

$= \int_2^1 (-x^2 + 6x - 8)\, dx$

$= \left[-\frac{x^3}{3} + \frac{6x^2}{2} - 8x \right]_2^1$

$= \left[\left(-\frac{1}{3} + 3 - 8\right) - \left(-\frac{8}{3} + 12 - 16\right) \right]$

$= \left[\left(-\frac{1}{3} - 5\right) - \left(-\frac{8}{3} - 4\right) \right]$

$= -\frac{16}{3} - \left(-\frac{20}{3}\right)$

$= \frac{20}{3} - \frac{16}{3} = \frac{4}{3}$ squnits.

---

10) Find the area of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$.

Eqn of circle is $x^2 + y^2 = 16$ — ①

Eqn of parabola is $y^2 = 6x$ — ②

Solving ① and ②

Sub ② in ①

$x^2 + 6x - 16 = 0$

$(x+8)(x-2) = 0$

$x = 2, \; x = -8$

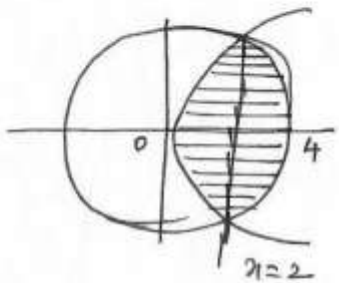
$x = -8$ is not possible.

$\therefore x = 2$, in ②

$y^2 = 6(2)$

$y^2 = 12$

$y = \pm 2\sqrt{3}$



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$\therefore$ pt of intersections are

$(2, 2\sqrt{3})$, $(2, -2\sqrt{3})$

Due to symmetry

$$\text{Area} = 2\int_0^2 y\,dx + 2\int_2^4 y\,dx$$

$$= 2\int_0^2 \sqrt{6x}\,dx + 2\int_2^4 \sqrt{16-x^2}\,dx$$

$$= 2\sqrt{6}\left[\frac{x^{3/2}}{3/2}\right]_0^2 + 2\left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_2^4$$

$$= \frac{4\sqrt{6}}{3}\left[x^{3/2}\right]_0^2 + 2\left[\left(2\sqrt{16-16} + 8\sin^{-1}(1)\right) - \left(\frac{2}{2}\sqrt{16-4} + 8\sin^{-1}\left(\frac{1}{2}\right)\right)\right]$$

$$= \frac{4\sqrt{6}}{3}(2)^{3/2} + 2\left[\left(0 + 8\frac{\pi}{2}\right) - \left(\sqrt{12} + 8\sin^{-1}\left(\frac{1}{2}\right)\right)\right]$$

$$= \frac{4\sqrt{6}}{3}\cdot 2\cdot\sqrt{2} + 2\left[8\frac{\pi}{2} - \sqrt{12} - 8\frac{\pi}{6}\right]$$

$$= \frac{8}{3}\sqrt{12} + 2\left[\frac{24\pi - 6\sqrt{12} - 8\pi}{6}\right]$$

$$= \frac{8}{3}\sqrt{12} + 2\left[\frac{16\pi - 6\sqrt{12}}{6}\right]$$

$$= \frac{8}{3}\sqrt{12} + \frac{16\pi - 6\sqrt{12}}{3}$$

$$= \frac{8\sqrt{12} + 16\pi - 6\sqrt{12}}{3} = \frac{2\sqrt{12} + 16\pi}{3}$$

$$= \frac{2\sqrt{4\times 3} + 16\pi}{3}$$

$$= \frac{2\times 2\sqrt{3} + 16\pi}{3}$$

$$= \frac{4}{3}\left(\sqrt{3} + 4\pi\right) \text{ squnt}$$

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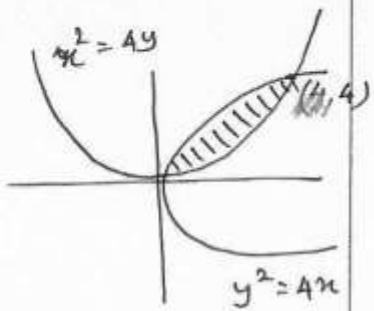
9.54 Find the area of the region bounded betn the parabolas $y^2 = 4x$ and $x^2 = 4y$



$y^2 = 4x$ —— ①

$x^2 = 4y$ —— ②

① $^2 \Rightarrow$ $y^4 = 16x^2$

$y^4 = 16(4y)$

$y^4 = 64y$

$y^3 = 64 \Rightarrow y = 4$

put $y = 4$ in ① $16 = 4x$

$\therefore x = \frac{16}{4} = 4$

put $x = 0 \Rightarrow y = 0$

$\therefore$ The pt of intersections are $(0,0)$, $(4,4)$

① $\Rightarrow y^2 = 4x$ | ② $\Rightarrow y = \frac{x^2}{4}$

$y = 2\sqrt{x}$

$\therefore$ Required Area $A = \int_0^4 (y_1 - y_2)\, dx$

$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$

$= \left[2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{12} \right]_0^4$

$= \left[\frac{4}{3} x^{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4$

$= \left(\frac{4}{3} \cdot 4 \cdot 2 - \frac{64}{12}\right) - (0 - 0)$

$= \frac{32}{3} - \frac{64}{12} \Rightarrow \frac{32}{3} - \frac{16}{3}$

$= \frac{16}{3}$ squnt.

9.55 Find the area of the region bounded betn the parabola $x^2 = y$ and the curve $y = |x|$.

The curve $y = |x|$ is

$$y = \begin{cases} -x & \text{if } x \le 0 \\ x & \text{if } x \ge 0 \end{cases}$$

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$x^2 = y$ — ①

$y = |x|$ — ②

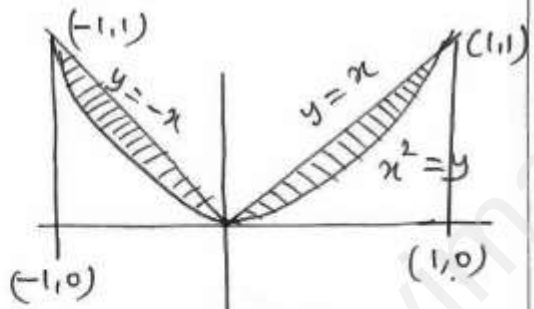
Solving ① and ②

$x^2 = |x|$.

put $x = -1$, $y = 1$

$x = 1$, $y = 1$

∴ The point of intersections are $(-1,1)$, $(1,1)$



By symmetric property

The required Area is twice the area in the first quadrant.

$$\therefore \text{Area} = 2\int_0^1 (y_1 - y_2)\,dx$$

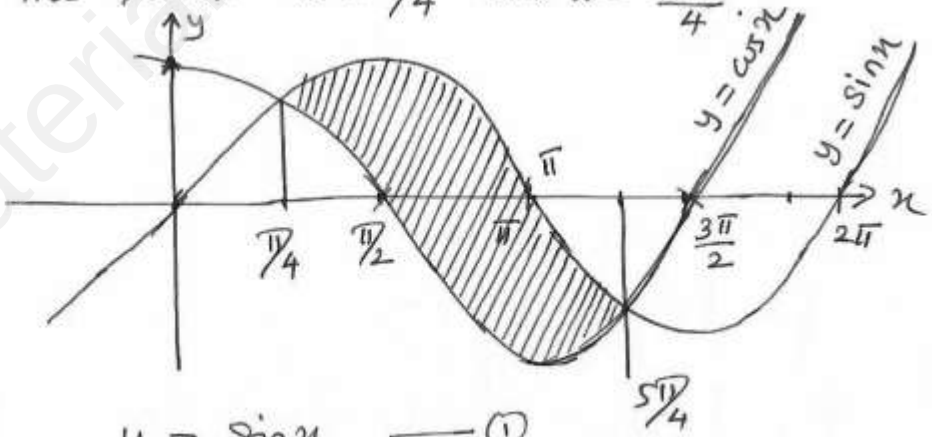
$$= 2\int_0^1 [x - x^2]\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\left(\frac{1}{2} - \frac{1}{3}\right) - 0\right]$$

$$= 2\left[\frac{3-2}{6}\right] = 2\left(\frac{1}{6}\right) = \frac{1}{3} \text{ Squnits}.$$

9.56 Find the area of the region bounded by $y = \cos x$, $y = \sin x$, the lines $x = \pi/4$ and $x = \frac{5\pi}{4}$.



$y = \sin x$ — ①

$y = \cos x$ — ②

The required Area lies betn $x = \pi/4$ and $x = 5\pi/4$.

$$\therefore \text{The required Area } A = \int_{\pi/4}^{5\pi/4} (y_1 - y_2)\,dx$$

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$$A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx$$

$$= \left[-\cos x - \sin x \right]_{\pi/4}^{5\pi/4}$$

$$= \left[\left(-\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4} \right) - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4} \right) \right]$$

$$= \left[-\cos\left(\pi + \pi/4\right) - \sin\left(\pi + \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right]$$

$$= + \cos \pi/4 + \sin \pi/4 + \frac{2}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}}$$

$$= \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2}$$

$$= 2\sqrt{2} \text{ sq units}$$

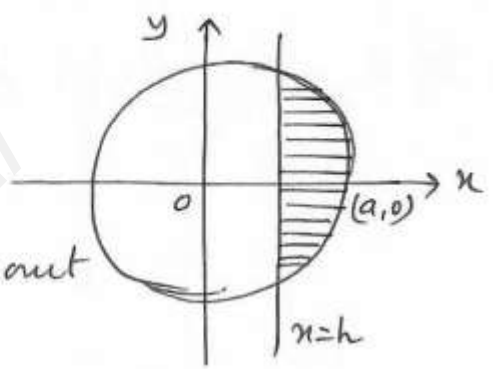
9.57 The region enclosed by the circle $x^2 + y^2 = a^2$ is divided into two segments by the line $x = h$. Find the area of the smaller segment.

XII | AI | 9.8 | P21

$$x^2 + y^2 = a^2$$

$$y^2 = a^2 - x^2$$

$$y = \sqrt{a^2 - x^2}$$



By symmetry about the x-axis,

The required Area

$$A = 2\int_h^a y\, dx = 2\int_h^a \sqrt{a^2 - x^2}\, dx$$

$$= 2\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \right]_h^a$$

$$= 2\left[\left(0 + \frac{a^2}{2}\sin^{-1}(1) \right) - \left(\frac{h}{2}\sqrt{a^2-h^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{h}{a}\right) \right) \right]$$

$$= 2 \times \frac{a^2}{2}\frac{\pi}{2} - \frac{h}{2}\sqrt{a^2-h^2} - \frac{a^2}{2}\sin^{-1}\left(\frac{h}{a}\right)$$

$$= \frac{\pi a^2}{2} - \frac{h}{2}\sqrt{a^2-h^2} - \frac{a^2}{2}\sin^{-1}\left(\frac{h}{a}\right)$$

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$= \frac{a^2}{2}\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{h}{a}\right)\right) - \frac{h}{2}\sqrt{a^2-h^2}$ squnt.

9.58 Find the area of the region in the first quadrant bounded by the parabola $y^2 = 4x$, the line $x+y=3$ and y-axis.

$$y^2 = 4x \quad \text{——} (1)$$
$$x+y = 3 \quad \text{——} (2)$$

To get the point of interseection. solve (1) and (2)

(2) $\Rightarrow$ $y = 3-x$

$\therefore$ (1) $\Rightarrow$ $(3-x)^2 = 4x$

$9+x^2-6x-4x = 0$

$x^2-10x+9 = 0$

$(x-1)(x-9) = 0$

$x = 1, \quad x = 9$

$x=1$ in (2) $y = 2$

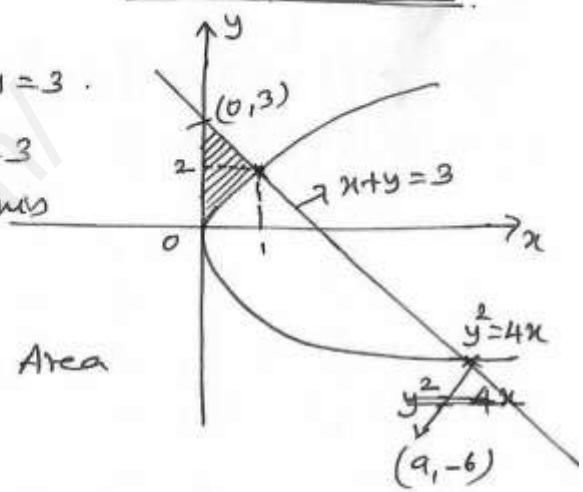
$x=9$ in (2) $y = -6$

$\therefore$ The point of intersections are $(1,2), (9,-6)$

put $x=0$ in (1)

$0+y=3 \Rightarrow y=3$.

$\therefore$ The line $x+y=3$ meets the y-axis at $y=3$



$\therefore$ The required Area

$$A = \int_0^3 x\,dy$$

$$= \int_0^2 x\,dy + \int_2^3 x\,dy$$

$$= \int_0^2 \frac{y^2}{4}\,dy + \int_2^3 (3-y)\,dy$$

$$= \left[\frac{y^3}{12}\right]_0^2 + \left[3y - \frac{y^2}{2}\right]_2^3$$

$$= \frac{8}{12} + \left[\left(9-\frac{9}{2}\right) - (6-2)\right]$$

$$= \frac{8}{12} + \left[\frac{9}{2} - 4\right]$$

$$= \frac{8}{12} + \frac{1}{2} = \frac{8+6}{12} = \frac{14}{12} = \frac{7}{6}$$

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9.59 Find by integration, the area of the region bounded by the lines $5x - 2y = 15$, $x + y + 4 = 0$ and x-axis.

$$5x - 2y = 15 \quad \text{—— } ①$$
$$x + y = -4 \quad \text{—— } ②$$

② × 2 ⇒ $2x + 2y = -8$ —— ③

$5x - 2y = 15$ —— ①

③ + ① ⇒ $7x = 7 \Rightarrow x = 1$.

Sub $x = 1$ in ① $\quad 5 - 2y = 15$

$5 - 15 = 2y$

$-10 = 2y$

$y = -5$

∴ point of intersection of two lines is $(1, -5)$.

The line $5x - 2y = 15$ meets x-axis,

put $y = 0$, $\quad 5x = 15$

$x = 3$

The line $x + y = -4$ meets x-axis,

put $y = 0$ $\quad \therefore x = -4$

∴ The pts are $(3, 0)$ and $(-4, 0)$

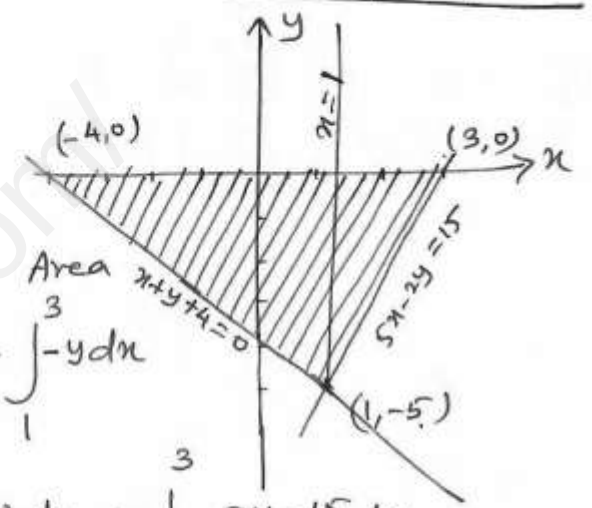
$x + y + 4 = 0$

$y = -x - 4$

$5x - 2y = 15$

$2y = 5x - 15$

$y = \frac{5x - 15}{2}$



The required Area

$$A = \int_{-4}^{1} -y\,dx + \int_{1}^{3} -y\,dx$$

$$= \int_{-4}^{1} (+x+4)\,dx + -\int_{1}^{3} \frac{5x-15}{2}\,dx.$$

$$= \left[+\frac{x^2}{2} + 4x\right]_{-4}^{1} - \frac{1}{2}\left[\frac{5x^2}{2} - 15x\right]_{1}^{3}$$

$$= \left(+\frac{1}{2} + 4\right) - \left(+\frac{16}{2} - 16\right) - \frac{1}{2}\left[\left(\frac{45}{2} - 45\right) - \left(\frac{5}{2} - 15\right)\right]$$

$$= +\frac{9}{2} - \left(-\frac{16}{2}\right) - \frac{1}{2}\left[\left(-\frac{45}{2}\right) - \left(-\frac{25}{2}\right)\right]$$

$$= \left(+\frac{9}{2} + 8\right) - \frac{1}{2}\left[-\frac{45}{2} + \frac{25}{2}\right]$$

$$= +\frac{25}{2} - \frac{1}{2}\left[-\frac{20}{2}\right]$$

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$= +\frac{25}{2} + 5 = +\frac{35}{2}$

$\therefore$ Area $= \frac{35}{2}$ sq units.

— x — x —

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Volume of the solid revolution

a) The volume of the solid of revolution about x-axis is

$$V = \pi \int_a^b y^2 dx$$

b) The volume of the solid of revolution about y-axis is

$$V = \pi \int_a^b x^2 dy$$

EXERCISE 9.9

1) Find by integration, the volume of the solid generated by revolving about the x-axis, the region enclosed by $y = 2x^2$, $y = 0$ and $x = 1$

$y = 2x^2$, $y = 0$ and $x = 1$ is revolved about x-axis

Given curve is $y = 2x^2$ ——— (1)

Required volume $= \pi \int_a^b y^2 dx$

$$= \pi \int_0^1 (2x^2)^2 dx = \pi \int_0^1 4x^4 dx$$

$$= 4\pi \int_0^1 x^4 dx$$

$$= 4\pi \left[\frac{x^5}{5} \right]_0^1$$

$$= \frac{4\pi}{5} \text{ cubic units}$$

2) Find, by integration, the volume of the solid generated by revolving about the x-axis, the region enclosed by $y = e^{-2x}$, $y = 0$, $x = 0$, $x = 1$

$y = e^{-2x}$, $y = 0$, $x = 0$, $x = 1$ is revolved about x-axis.

∴ The required volume $V = \pi \int_a^b y^2 dx$

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$$= \int_0^1 \pi (e^{-2x})^2 dx$$

$$= \pi \int_0^1 e^{-4x} dx = \pi \left[\frac{e^{-4x}}{-4} \right]_0^1$$

$$= \frac{\pi}{-4} \left[e^{-4x} \right]_0^1$$

$$= \frac{\pi}{-4} \left[e^{-4} - e^{0} \right]$$

$$= \frac{\pi}{-4} \left[\frac{1}{e^4} - 1 \right]$$

$$= \pi/4 \left[1 - \frac{1}{e^4} \right] \text{ cubic unit.}$$

XII|AI|9.9|P2

Eqn of the given curve is $x^2 = 1+y$ — ①

put $x=0$ in ① $y+1=0 \Rightarrow y=-1$

∴ The required volume $= \pi \int_a^b x^2 dy$

$$V = \pi \int_{-1}^{3} (1+y) dy$$

$$= \pi \left[y + \frac{y^2}{2} \right]_{-1}^{3}$$

$$= \pi \left[\left(3 + \frac{9}{2}\right) - \left(-1 + \frac{1}{2}\right) \right]$$

$$= \pi \left[\frac{15}{2} - \left(-\frac{1}{2}\right) \right]$$

$$= \pi \left[\frac{15}{2} + \frac{1}{2} \right] = 8\pi \text{ cubic units}$$

3) Find by integration, the volume of the solid generated by revolving about the y-axis the region enclosed by $x^2 = 1+y$ and $y = 3$

4) The region enclosed between the graph $y = x$ and $y = x^2$ is denoted by R, Find the volume generated when R is rotated through 360° about x-axis.

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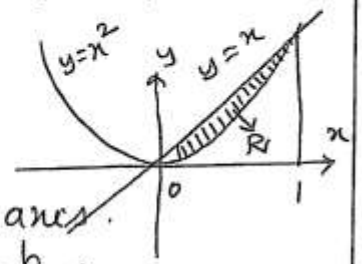
Eqn of parabola is $y = x^2$

Eqn of the line is $y = x$.



The required volume is the line is rotated through 360° about x-axes.

$\therefore$ Volume $V = \pi \int_a^b y^2 dx$

$y = x$
$y^2 = x^2$
$y = x^2$
$y^2 = x^4$.

$$= \pi \int_0^1 (x^2 - x^4)\, dx.$$

$$= \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1$$

$$= \pi \left[\frac{1}{3} - \frac{1}{5} \right]$$

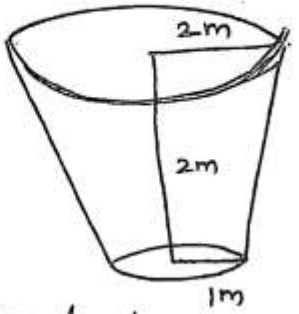
$$= \pi \left[\frac{5-3}{15} \right] = 2\pi/15$$

cubic units.

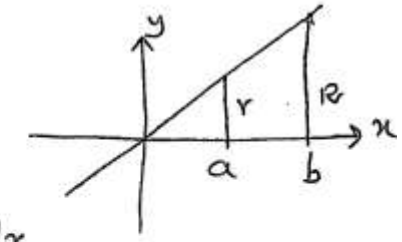
---

5) Find by integration, the volume of the container which is in the shape of a right circular conical frustum as shown in the fig.

XII | AI | 9-9 | P3



Volume of the right circular conical frustum is obtained by revolving the line $y = x$ betn $x = a$ and $x = b$ around the x-axis.



$\therefore$ Height of the frustum $h = b - a$

$$\therefore \text{Volume} = \pi \int_a^b x^2 dx$$

$$= \pi \left(\frac{x^3}{3} \right)_a^b$$

$$= \frac{\pi}{3} \left[b^3 - a^3 \right]$$

$$= \frac{\pi}{3} \left[(b-a)(b^2 + ba + a^2) \right]$$

Substitute $b - a = h$

$b = R$

$a = r$

we get volume of frustum

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$\frac{\pi}{3}\left[h\left(R^2+Rr+r^2\right)\right]$ cu. unit.

Given $R = 2m$, $h = 2m$, $r = 1m$.

$\therefore$ Required volume $V = \frac{\pi}{3}\left[2\left(4+2+1\right)\right]$

$V = \frac{\pi}{3}(14)$

$= \frac{14\pi}{3}$ cubic units.

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6) A watermelon has an ellipsoid shape which can be obtained by revolving an ellipse with major-axis 20 cm and minor axis 10cm about its major axis. find its volume using integration.

Given

major axis $= 2a = 20cm \Rightarrow a = 10cm$

minor axis $= 2b = 10cm \Rightarrow b = 5cm$.

$\therefore$ Eqn of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

i $\frac{x^2}{100} + \frac{y^2}{25} = 1$

$\Rightarrow \frac{y^2}{25} = 1 - \frac{x^2}{100} = \frac{100 - x^2}{100}$

$y^2 = \frac{100 - x^2}{100} \times 25 = \frac{100 - x^2}{4}$

$\therefore$ Required volume $2\pi \int_0^{10} y^2 dx$

$= 2\pi \int_0^{10} \frac{100 - x^2}{4} dx$

$= \frac{2\pi}{4} \int_0^{10} (100 - x^2) dx$

$= \frac{\pi}{2} \left[100x - \frac{x^3}{3} \right]_0^{10}$

$= \frac{\pi}{2} \left[\left(1000 - \frac{1000}{3}\right) - 0 \right]$

$= \frac{\pi}{2} \left[\frac{2000}{3} \right]$

$= \frac{1000\pi}{3}$ cu units.

Eg 9.62 Find the volume of a sphere of radius 'a'.

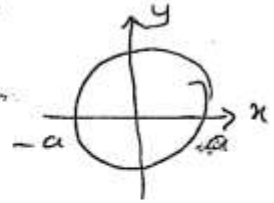
$x^2 + y^2 = a^2$

$y^2 = a^2 - x^2$

Required volume $= \pi \int_{-a}^{a} y^2 dx$

$= 2\pi \int_0^a (a^2 - x^2) dx$

$\because a^2 - x^2$ is an even fun



$\therefore V = 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a$

$= 2\pi \left[a^3 - \frac{a^3}{3} \right] = 2\pi \left(\frac{2a^3}{3} \right)$

$= \frac{4\pi a^3}{3}$ cubic units.

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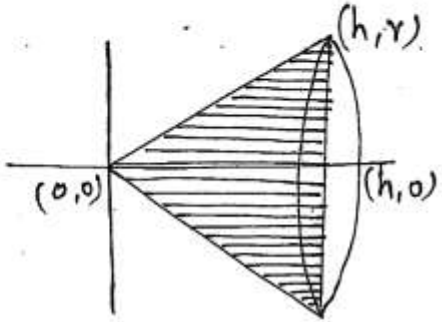
9.63 Find the volume of a right circular cone of base radius r and height h.



The volume of the right circular cone with base radius 'r' and height h is obtained by revolving the right angled triangle having vertices (0,0), (h,0), and (h,r)

Eqn of line joining the points (0,0) and (h,r) is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

i.e. $\frac{y-0}{r-0} = \frac{x-0}{h-0} \Rightarrow \frac{y}{r} = \frac{x}{h}$

$$\Rightarrow y = \frac{rx}{h}$$

∴ The required volume of the cone is obtained by revolving the area bounded by $y = \frac{r}{h}x$, $x=0$, $x=h$ and x-axis about x-axis is given by

$$V = \int_0^h \pi y^2 dx$$

$$= \int_0^h \pi \frac{r^2x^2}{h^2} dx$$

$$= \frac{\pi r^2}{h^2} \int_0^h x^2 dx$$

$$= \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h$$

$$= \frac{\pi r^2}{h^2} \left[\frac{h^3}{3}\right]$$

$$= \frac{\pi r^2 h}{3} \text{ cu. units.}$$

x

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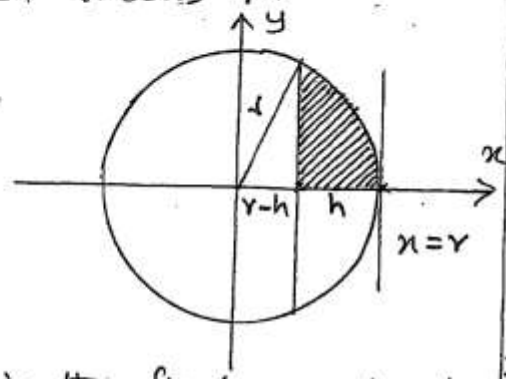
9.64 Find the volume of spherical cap of height h cut off from a sphere of radius r.



The region in the first quadrant bounded by the circle $x^2+y^2=r^2$, the x-axis, the line $x=r-h$ and $x=r$ is revolved about x-axis.

Hence the required volume

$$V = \pi \int_a^b y\,dx$$

$$= \pi \int_{r-h}^{r} (r^2 - x^2)\,dx$$

$$= \pi \left[r^2 x - \frac{x^3}{3} \right]_{r-h}^{r}$$

$$= \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(r^2(r-h) - \frac{(r-h)^3}{3} \right) \right]$$

$$= \pi \left[\frac{2r^3}{3} - \left(r^3 - r^2 h - \left(\frac{r^3 - h^3 - 3r^2 h + 3rh^2}{3} \right) \right) \right]$$

$$= \pi \left[\frac{2r^3}{3} - \left(\frac{3r^3 - 3r^2 h - r^3 + h^3 + 3r^2 h - 3rh^2}{3} \right) \right]$$

$$= \pi \left[\frac{2r^3}{3} - \left(\frac{2r^3 + h^3 - 3rh^2}{3} \right) \right]$$

cont.

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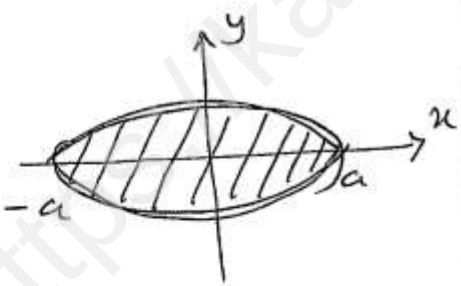
$$= \pi\left[\frac{2r^3 - 2r^3 - h^3 + 3rh^2}{3}\right]$$

$$= \frac{\pi(3rh^2 - h^3)}{3}$$

$$= \frac{\pi h^2}{3}(3r - h) \text{ //.}$$

9.65 Find the volume of a solid formed by revolving the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, about the major axis.

Eqn of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Here the major axis is x-axis

put $y = 0$ then $\frac{x^2}{a^2} = 1 \Rightarrow x^2 = a^2$

$x = \pm a$.

∴ The ellipse meets the x-axis at $x = -a$ and $x = a$.

∴ Required volume $V = \pi \int_{-a}^{a} y^2 dx$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

∴ Volume $V = \pi \int_{-a}^{a} \frac{b^2}{a^2}(a^2 - x^2)\,dx$

$$= 2\pi \int_{0}^{a} \frac{b^2}{a^2}(a^2 - x^2)\,dx$$

$$= \frac{2\pi b^2}{a^2}\left[a^2 x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{2\pi b^2}{a^2}\left[a^3 - \frac{a^3}{3}\right]$$

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$= \frac{2\pi b^2}{a^2}\left[\frac{2a^3}{3}\right]$

$= \frac{4\pi a b^2}{3}$ cu units.

9.67 Find, by integration, the volume of the solid generated by revolving about y-axis the region bounded between the parabola $x = y^2+1$, the y-axis, and the lines $y=1$ and $y=-1$

Given curve $x = 1+y^2$

Required volume $V = \int_a^b \pi x^2 dy$

$\therefore V = \int_{-1}^{1} \pi (1+y^2)^2 dy$

$= 2\pi \int_0^1 (1+y^2)^2 dy$

$= 2\pi \int_0^1 [1+y^4+2y^2]\, dy$ XII/AI/9.9/P9

$= 2\pi \left[y + \frac{y^5}{5} + \frac{2y^3}{3}\right]_0^1$

$= 2\pi \left[\left(1+\frac{1}{5}+\frac{2}{3}\right)-0\right] = 2\pi\left(\frac{15+3+10}{15}\right)$

$= \frac{56\pi}{15}$ cu.units.

Eg 9.68 Find, by integration, the volume of the solid generated by revolving about y-axis the region bounded betn the curve $y = \frac{3}{4}\sqrt{x^2-16}$, $x \geq 4$, the y-axis and the lines $y=1$ and $y=6$.

Given curve $y = \frac{3}{4}\sqrt{x^2-16}$

$4y = 3\sqrt{x^2-16}$

squaring $16y^2 = 9(x^2-16)$

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$16y^2 = 9x^2 - 144$

$9x^2 - 16y^2 = 144$

$\frac{9x^2}{144} - \frac{16y^2}{144} = 1$

$\frac{x^2}{16} - \frac{y^2}{9} = 1$ ——— (1)

∴ Required volume $V = \pi \int_a^b x^2 dy$

From (1) $\frac{x^2}{16} = 1 + \frac{y^2}{9}$

$\frac{x^2}{16} = \frac{9+y^2}{9}$

$x^2 = \frac{16}{9}(9+y^2)$

∴ $V = \pi \int_1^b \frac{16}{9}(9+y^2)\,dy$

$= \frac{16\pi}{9}\left[9y + \frac{y^3}{3}\right]_1^b$

$= \frac{16\pi}{9}\left[(54+72) - (9+\frac{1}{3})\right]$

$= \frac{16\pi}{9}\left[126 - \frac{28}{3}\right]$

$= \frac{16\pi}{9}\left[\frac{378-28}{3}\right] = \frac{16\pi}{9} \times \frac{350}{3}$

$= \frac{5600}{9}\pi$ cu.unit.

×

9.69 Find, by integration, the volume of the solid generated by revolving about y-axis the region bounded by the curves $y = \log x$, $y = 0$, $x = 0$, $y = 2$

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The given curve is $y = \log x$

The curve $y = \log x$, $y = 0$, $x = 0$, $y = 2$ revolved about x-axes

∴ The required volume

$$V = \pi \int_0^2 x^2 dy$$

Given $y = \log x$

Taking exponential on both sides

$$e^y = x$$

$$\therefore \quad x^2 = (e^y)^2 = e^{2y}$$

$$\therefore \quad V = \pi \int_0^2 e^{2y} dy$$

$$= \pi \left[\frac{e^{2y}}{2} \right]_0^2$$

$$= \frac{\pi}{2} \left[e^4 - e^0 \right]$$

$$= \frac{\pi}{2} \left[e^4 - 1 \right] \text{ cu. units.}$$

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Chapter 10 – Ordinary Differential Equations

* Exercise are hyperlink with page number. To get into particular exercise, please click exercise number

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10. Ordinary Differential Equations.

Order of differential equation

The order of differential eqn is the highest order derivative present in the differential eqn

Degree of a differential equation

The integral power of the highest order derivative is called the degree of the differential eqn.

Eg 10.1 Determine the order and degree.

(I) $\frac{dy}{dx} = x+y+5$

order = 1, degree = 1

(II) $\left(\frac{d^4y}{dx^4}\right)^3 + 4\left(\frac{dy}{dx}\right)^7 + 6y = 5\cos 3x$

order = 4, degree = 3

(III) $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right) = x^2 \log\left(\frac{d^2y}{dx^2}\right)$

order = 2

Degree is not defined ∵ The given differential eqn is not a polynomial eqn.

(IV) $3\left(\frac{d^2y}{dx^2}\right) = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$

Squaring on both sides

$$9\left(\frac{d^2y}{dx^2}\right)^2 = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^3$$

order = 2, Degree 2

(V) $dy + (xy - \cos x)\,dx = 0$

$dy = -(xy - \cos x)\,dx$

$\frac{dy}{dx} = -xy + \cos x$

$\frac{dy}{dx} + xy - \cos x$

order - 1 degree - 1

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EXERCISE 10.1

Determine order and degree.

(i) $\frac{dy}{dx} + xy = \cot x$

$\frac{dy}{dx} + xy - \cot x = 0$

order – 1 Degree – 1

(ii) $\left(\frac{d^3y}{dx^3}\right)^{\frac{2}{3}} - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4 = 0$

$\left(\frac{d^3y}{dx^3}\right)^{\frac{2}{3}} = 3\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 4$

raising the power 3

$\left(\frac{d^3y}{dx^3}\right)^2 = \left(3\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 4\right)^3$

order 3, degree – 2

(iii) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x \sin\left(\frac{d^2y}{dx^2}\right)$

Highest derivative is $\frac{d^2y}{dx^2}$.

∴ Its order – 2.

The given differential eqn is not a polynomial eqn. ∴ its degree is not defined. (OR) $\frac{d^2y}{dx^2}$ cannot be Separated.

(iv) $\sqrt{\frac{dy}{dx}} - 4\frac{dy}{dx} - 7x = 0$

$\sqrt{\frac{dy}{dx}} = 4\frac{dy}{dx} + 7x$

squaring, $\frac{dy}{dx} = \left(4\frac{dy}{dx} + 7x\right)^2$

∴ order 1 degree – 2.

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v) $y\left(\frac{dy}{dx}\right) = \frac{x}{\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3}$

$y\left(\frac{dy}{dx}\right)\left[\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3\right] = x$

$y\left(\frac{dy}{dx}\right)^2 + y\left(\frac{dy}{dx}\right)^4 = x$

order - 1 Degree - 4

(vi) $x^2 \frac{d^2y}{dx^2} + \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} = 0$

$x^2 \frac{d^2y}{dx^2} = -\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}}$

squaring

$x^4 \left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$

∴ order - 2, Degree - 2

(x) $x = e^{xy\left(\frac{dy}{dx}\right)}$

$\log x = xy\left(\frac{dy}{dx}\right)$

order - 1

Degree does not exist

(vii) $\left(\frac{d^2y}{dx^2}\right)^3 = \sqrt{1 + \left(\frac{dy}{dx}\right)}$

Squaring $\left(\frac{d^2y}{dx^2}\right)^6 = 1 + \frac{dy}{dx}$.

∴ order 2, Degree - 6.

(viii) $\frac{d^2y}{dx^2} = xy + \cos\left(\frac{dy}{dx}\right)$

order - 2, Degree does not exists. Since $\frac{dy}{dx}$ cannot be seperated from $\cos\left(\frac{dy}{dx}\right)$.

---

ix $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + \int y\,dx = x^3$

Diff w.r.to x

$\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} + d\left(\int y\,dx\right) = 3x^2$

order 3 Degree 1

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EXERCISE 10.2

(iii) For a certain substance the rate of change of vapour pressure p w.r to the temperature T is proportional to the vapour pressure and inversely proportional to the square of the temp.

Rate of change of p w.r to t is $\frac{dp}{dt}$.

Given that $\frac{dp}{dt} \propto \frac{p}{T^2}$

$\therefore \frac{dp}{dt} = K \frac{p}{T^2}$ ($\because$ K is constant).

(iv) Assume that a spherical rain drop evaporates at the rate proportional to its surface area. form a differential eqn involving the rate of change of the radius of the rain drop.

Volume of the sphere $V = \frac{4}{3}\pi r^3$

surface area $= 4\pi r^2$.

Given that $\frac{dV}{dt} \propto$ S.A.

i.e $\frac{4}{3} 3\pi r^2 \frac{dr}{dt} \propto 4\pi r^2$

$4\pi r^2 \frac{dr}{dt} = -K\, 4\pi r^2$

$\frac{dr}{dt} = -K$ or $\frac{dr}{dt} + K = 0$.

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Eg 10.3 Form the differential eqn by eliminating the orbitrary constant A and B from

$y = A\cos x + B\sin x$

$y = A\cos x + B\sin x$.

$\frac{dy}{dx} = -A\sin x + B\cos x$.

$\frac{d^2y}{dx^2} = -A\cos x - B\sin x$

$= -(A\cos x + B\sin x)$

$= -y$

$\frac{d^2y}{dx^2} + y = 0$. is the required Differential eqn .

Eg 10.4 Find the differential eqn of the family of circles passing through the points $(a,0)$ and $(-a,0)$.

The circle passing through $(a,0)$ and $(-a,0)$ has its centre on y-axis.

∴ let $(0,b)$ be the centre of the circle .

∴ radius $r = \sqrt{(-a+0)^2 + (0-b)^2}$

$= \sqrt{a^2+b^2}$.

∴ The eqn of family of circle passing through $(-a,0)$ and $(a,0)$ is

$(x-0)^2 + (y-b)^2 = a^2+b^2$ ——— (1)

$x^2 + y^2 + b^2 - 2yb = a^2+b^2$

$x^2 + y^2 - 2yb = a^2$

Differentiate. w.r.to x

$2x + 2y\frac{dy}{dx} - 2b\frac{dy}{dx} = 0$.

$2\frac{dy}{dx}(y-b) + 2x = 0$. ——— (2)

$2\frac{dy}{dx}(y-b) = -2x$

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$$y - b = \frac{-2x}{2\frac{dy}{dx}} = -\frac{x}{\frac{dy}{dx}}$$

$$b = y + \frac{x}{\frac{dy}{dx}}$$

sub b in ①.

$$x^2 + \left[y - y - \frac{x}{\frac{dy}{dx}} \right]^2 = a^2 + \left(y + \frac{x}{\frac{dy}{dx}} \right)^2$$

$$x^2 + \frac{x^2}{\left(\frac{dy}{dx}\right)^2} = a^2 + y^2 + \frac{x^2}{\left(\frac{dy}{dx}\right)^2} + \frac{2xy}{\frac{dy}{dx}}$$

$$x^2 = a^2 + y^2 + \frac{2xy}{\frac{dy}{dx}}$$

$$x^2 = \frac{a^2\frac{dy}{dx} + y^2\frac{dy}{dx} + 2xy}{dy/dx}.$$

$$x^2\frac{dy}{dx} = a^2\frac{dy}{dx} + y^2\frac{dy}{dx} + 2xy.$$

$$\frac{dy}{dx}(x^2 - a^2 - y^2) = 2xy \quad \text{or}$$

$$\frac{dy}{dx}(x^2 - a^2 - y^2) - 2xy = 0$$

is the required differential eqn.

10.5 Find the differential eqn of the family of parabolas $y^2 = 4ax$, where 'a' is arbitrary constant.

The eqn of parabola is $y^2 = 4ax$ —— ①

Diff w.r.to x

$$2y\frac{dy}{dx} = 4a \quad \therefore a = \frac{y}{2}\frac{dy}{dx}.$$

$$y\frac{dy}{dx} = 2a$$

sub a value in ①

$$y^2 = 2\cancel{4}\left(\frac{y}{\cancel{2}}\frac{dy}{dx}\right)x$$

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$$y = 2x\frac{dy}{dx}$$

$\therefore \frac{dy}{dx} = \frac{y}{2x}$ is the required Differential eqn.

10.6 Find the differential eqn of the family of ellipses having foci on the x-axis and centre at the origin.

The eqn of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ——— (1)

Diff w.r to x

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0. \quad \text{——— (2)}$$

Again Diff w.r to x.

$$\frac{2}{a^2}(1) + \frac{2}{b^2}\left[y\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{dy}{dx}\right] = 0.$$

$$\frac{2}{a^2} = -\frac{2}{b^2}\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]$$

$$\frac{1}{a^2} = -\frac{1}{b^2}\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]$$

Sub $\frac{1}{a^2}$ value in (2).

$$-\frac{2x}{b^2}\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$-\frac{2}{b^2}\left[xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx}\right] = 0$$

$$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0.$$

is the required differential eqn.

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EXERCISE 10.3

1. Find the differential eqn of the family

(i) all non-vertical lines in a plane.

Eqn of all non vertical lines is

$y = mx + c$, $m \neq 0$.

Diff w.r.to x

$\frac{dy}{dx} = m$, Again, $\frac{d^2y}{dx^2} = 0$.

(ii) all non-horizontal lines in a plane.

Eqn of family non-horizontal line is

$y = mx + c$, $m \neq \infty$

$\frac{dy}{dx} = m$ ~~Again $\frac{d^2y}{dx^2} =$~~

$\frac{dx}{dy} = \frac{1}{m}$

$\frac{d^2x}{dy^2} = 0$.

2) Form the differential eqn of all st lines touching the circle $x^2 + y^2 = r^2$

The line $y = mx + c$ is tobe tgt to the circle $x^2 + y^2 = r^2$.

∴ The eqn of the tgt is

$y = mx \pm r\sqrt{1+m^2}$. ——— (1)

Diff w.r.to x

$\frac{dy}{dx} = m$

Sub in (1) $y = \frac{dy}{dx}x \pm r\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$y - \frac{dy}{dx}x = \pm r\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Squaring,

$\left(y - \frac{dy}{dx}x\right)^2 = r^2\left(1+\left(\frac{dy}{dx}\right)^2\right)$

$\left(\frac{dy}{dx}x - y\right)^2 = r^2\left[1+\left(\frac{dy}{dx}\right)^2\right]$

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3) Find the differential eqn of the family of circle passing through the origin and having their centres on the $x-$axis.

Since the centre is on the $x-$axis, let it be $(a,0)$. and its radius is 'a'.

$\therefore$ Eqn of the circle is

$$(x-a)^2+(y-0)^2 = a^2 \quad \text{---} (1)$$

$$x^2+a^2-2xa+y^2 = a^2$$

$$x^2-2xa+y^2 = 0$$

Diff w.r.to x

$$2x-2a+2y\frac{dy}{dx} = 0.$$

$$2\left(x+y\frac{dy}{dx}\right) = 2a$$

$$\therefore a = x+y\frac{dy}{dx}.$$

Sub 'a' value in (1)

$$\left(x-x-y\frac{dy}{dx}\right)^2+y^2 = \left(x+y\frac{dy}{dx}\right)^2$$

$$y^2\left(\frac{dy}{dx}\right)^2+y^2 = x^2+y^2\left(\frac{dy}{dx}\right)^2+2xy\frac{dy}{dx}.$$

$$x^2+2xy\frac{dy}{dx}-y^2 = 0.$$

---

4) Find the differential eqn of the family of all the parabolas with latus rectum $4a$ and whose axes are parallel to $x-$axis.

Eqn of parabola whose axis is parallel to $x-$axis is of the form

$$(y-k)^2 = 4a(x-h) \quad \text{---} (1)$$

Diff w.r.to x —(2)

$$2(y-k)\frac{dy}{dx} = 4a \Rightarrow (y-k)\frac{dy}{dx} = 2a$$

$$y-k = \frac{2a}{dy/dx} \quad \text{---} (2)$$

Diff. w.r.to x

$$(y-k)\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(\frac{dy}{dx}\right) = 0$$

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(i) $\frac{2a}{dy/dx}\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0.$

$$\frac{2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3}{dy/dx} = 0$$

$$\Rightarrow 2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0.$$

---

5) Find the differential eqn of the family of parabolas with vertex at (0,-1) and having axis along the y-axis.

Given vertex is (0,-1) and axis is along y-axis.

Eqn of parabola is

$$(x+1)^2 = -4ay \quad \text{——— (1)}$$

Diff w.r.to x.

$$2(x+1) = -4ay'$$

$$\frac{2(x+1)}{y'} = -4a$$

sub in (1).

$$(x+1)^2 = \frac{2(x+1)}{y'}\cdot y$$

$$x+1 = \frac{2y}{y'}$$

$$y'(x+1) - 2y = 0.$$

---

6) Find the differential eqns of the family of all the ellipses having foci on the y-axis and centre at the origin.

Eqn of ellipse foci on the y-axis and centre at the origin is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad \text{——— (1)}.$$

Diff w.r.to x

$$\frac{2x}{b^2} + \frac{2y\,dy/dx}{a^2} = 0$$

$$\frac{x}{b^2} + \frac{y}{a^2}\frac{dy}{dx} = 0 \quad \text{——— (2)}$$

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Diff w.r.to x

$$\frac{1}{b^2} + \frac{1}{a^2}\left\{ y\frac{d^2y}{dx^2}\cdot\frac{dy}{dx} + \frac{dy}{dx}\right\}.$$

$$\frac{1}{b^2} + \frac{1}{a^2}\left\{ y\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{dy}{dx}\right\} = 0$$

$$\frac{1}{b^2} = -\frac{1}{a^2}\left\{ yy'' + (y')^2\right\} \quad *$$

Sub in (2)

$$-\frac{x}{a^2}\left\{ yy'' + (y')^2\right\} + \frac{y}{a^2}y' = 0$$

$$-xyy'' - xy'^2 + yy' = 0.$$

$$xy'^2 + xyy'' - yy' = 0.$$

---

7) Find the differential eqn corresponding to the family of curves represented by the eqn $y = Ae^{8x} + Be^{-8x}$ where A, B orbitrary constents.

$$y = Ae^{8x} + Be^{-8x}$$

$$\frac{dy}{dx} = Ae^{8x}\cdot 8 + Be^{-8x}(-8)$$

Again Diff w.r.to x.

$$\frac{d^2y}{dx^2} = 8Ae^{8x}\cdot 8 - 8Be^{-8x}(-8)$$

$$= 64Ae^{8x} + 64Be^{-8x}$$

$$= 64\left(Ae^{8x} + Be^{-8x}\right)$$

$$= 64y$$

$\frac{d^2y}{dx^2} - 64y = 0$ is the required Differential eqn.

---

8) Find the differential eqn of the curve represented by

$$xy = ae^x + be^{-x} + x^2$$

$$xy = ae^x + be^{-x} + x^2$$

Diff w.r.to x.

$$x\frac{dy}{dx} + y = ae^x - be^{-x} + 2x$$

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$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}(1) + \frac{dy}{dx} =$$

$$ae^x + be^{-x} + 2$$

$$x\,y'' + 2y' - 2 = xy - x^2$$

$$xy'' + 2y' - xy + x^2 - 2 = 0.$$

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10.7 Show that $x^2+y^2=r^2$, where r is a constant, is a soln of the differential eqn $\frac{dy}{dx} = -\frac{x}{y}$.

Given $x^2+y^2=r^2$

Diff w.r.to x

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Hence $x^2+y^2=r^2$ satisfies the differential eqn $\frac{dy}{dx} = -\frac{x}{y}$

$\therefore$ $x^2+y^2=r^2$ is a soln of the differential eqn $\frac{dy}{dx} = -\frac{x}{y}$.

10.8 Show that $y = mx + \frac{7}{m}$, $m \neq 0$ is a soln of the differential eqn

$$xy' + 7\frac{1}{y'} - y = 0.$$

Given eqn is $y = mx + \frac{7}{m}$ — ①

Diff w.r.to x

$$y' = m \quad — ②$$

Sub ② in ①

$$y = y'x + \frac{7}{y'} \Rightarrow y = \frac{y'^2x+7}{y'}$$

$$\Rightarrow yy' = y'^2x + 7.$$

$$\Rightarrow y'x - y + \frac{7}{y'} = 0.$$

Hence the given fun is soln of the differential eqn $xy' - y + \frac{7}{y'} = 0$.

10.9 Show that $y = 2(x^2-1) + ce^{-x^2}$ is a soln of the differential eqn

$$\frac{dy}{dx} + 2xy - 4x^3 = 0$$

$$y = 2(x^2-1) + ce^{-x^2} \quad — ①$$

Diff w.r.to x

$$y' = 4x + ce^{-x^2}(-2x)$$

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Now $\frac{dy}{dx} + 2xy - 4x^3$

$= 4x - 2xce^{-x^2} + 2xy - 4x^3$

$= 2x\left[2 - ce^{-x^2} + y - 2x^2\right]$

$= 2x\left[-2(x^2-1) - ce^{-x^2} + y\right]$

$= 2x\left[-y + y\right] = 2x(0) = 0$.

Hence the given fun is soln of the differential eqn $\frac{dy}{dx} + 2xy - 4x^3 = 0$.

10.10. Show that $y = a\cos(\log x) + b\sin(\log x)$ is a soln of the differential eqn $x^2y'' + xy' + y = 0$.

$y = a\cos(\log x) + b\sin(\log x)$

Diff w.r to x

$y' = a(-\sin(\log x)) \cdot \frac{1}{x} + b\cos\log x \cdot \frac{1}{x}$

$xy' = -a\sin\log x + b\cos\log x$.

Diff w.r to x.

$xy'' + y' = -a\left[\cos\log x \cdot \frac{1}{x}\right] + b\left[-\sin\log x \cdot \frac{1}{x}\right]$

$xy'' + y' = -\frac{1}{x}\left[a\cos\log x + b\sin\log x\right]$

$x^2y'' + xy' = -y$

$\therefore x^2y'' + xy' + y = 0$.

EXERCISE 10.4

1) Show that each of the following expressions is a soln of the corresponding given differential eqn

(i) $y = 2x^2$, $xy' = 2y$

$y = 2x^2$

Diff w.r to x

$y' = 4x \Rightarrow x = \frac{y'}{4}$

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$\therefore \quad y = 2x^2$

$y = 2x(x)$

$y = 2x\left(\frac{y'}{4}\right)$

$y = \frac{xy'}{2} \Rightarrow 2y = xy'$

$\therefore$ $y = 2x^2$ is the soln of the given diff eqn.

(ii) $y = ae^x + be^{-x}$, $y'' - y = 0$

$y = ae^x + be^{-x}$

Diff w.r.to x

$y' = ae^x - be^{-x}$

Again Diff. w.r.to x

$y'' = ae^x + be^{-x}$

$y'' = y \Rightarrow y'' - y = 0$

$\therefore$ The given eqn is the soln of the given differential eqn.

---

2) Find the value of m so that the function $y = e^{mx}$ is a soln of the given differential eqn.

(i) $y' + 2y = 0$ ——— (1)

Given eqn is $y = e^{mx}$

Diff w.r.to x $\quad y' = e^{mx} \cdot m$.

i.e. $y' = my$

i.e. $y' - my = 0$ ——— (2)

comparing (1) and (2)

$y' = 2x = y' - my \Rightarrow \boxed{m = -2}$

(ii) $y'' - 5y' + 6y = 0$ ——— (1)

$y = e^{mx}$ ——— (2)

Diff w.r.to x

$y' = e^{mx} \cdot m$ ——— (3)

$y'' = e^{mx} \cdot m^2$ ——— (4)

(1) $\Rightarrow m^2 e^{mx} - 5me^{mx} + 6e^{mx} = 0$

$=$

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$\div e^{mx} \Rightarrow m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

$m = 2$ or $m = 3$.

3) The slope of the tgt to the curve is the reciprocal of four times the ordinate at that point. The curve passes through (2,5) find the eqn of the curve.

Given the tgt is the reciprocal of four times of the ordinate

i $\frac{dy}{dx} = \frac{1}{4y}$

$4\frac{dy}{dx} = \frac{1}{y}$

$\int 4y\,dy = \int dx$

$4\frac{y^2}{2} = x + c$

$2y^2 = x + c$

It passes through (2,5)

$\therefore 2(5)^2 - 2 = c$

$50 - 2 = c \Rightarrow c = 48$

$\therefore$ The eqn of the curve is

$2y^2 = x + 48$

4) Show that $y = e^{-x} + mx + n$ is a soln of the differential eqn

$e^x\left(\frac{d^2y}{dx^2}\right) - 1 = 0$

$y = e^{-x} + mx + n$

Diff w.r.to x

$y' = e^{-x}(-1) + m$

Again diff w.r.to x

$y'' = e^{-x} = \frac{1}{e^x}$

$e^x y'' = 1 \Rightarrow e^x y'' - 1 = 0.$

Hence proved.

5) Show that $y = ax + \frac{b}{x}$ is a soln of the differential eqn $x^2y'' + xy' - y = 0$

$y = ax + \frac{b}{x}$

$y = \frac{ax^2 + b}{x} \Rightarrow yx = ax^2 + b$ —— (1)

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Diff w.r.to x

$$y + xy' = 2ax \quad \text{——} (2)$$

Again Diff w.r.to x

$$y' + xy'' + y' = 2a \quad \text{——} (3)$$

Sub (3) in (2)

$$y + xy' = (y' + xy'' + y')x$$

$$y + xy' = xy' + x^2y'' + xy'$$

$$x^2y'' + xy' - y = 0.$$

Hence proved.

6) Show that $y = ae^{-3x} + b$ is a soln of the diff eqn $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 0$

$$y = ae^{-3x} + b$$

Diff w.r.to x

$$y' = ae^{-3x}(-3)$$

$$y'' = -3ae^{-3x}(-3)$$

i.e. $y'' = -3y'$

$$y'' + 3y' = 0$$

i.e. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 0$

7) Show that the differential eqn representing the family of curves $y^2 = 2a(x + a^{2/3})$ is $\left(y^2 - 2xy\frac{dy}{dx}\right)^3 = 8\left(y\frac{dy}{dx}\right)^5$

$$y^2 = 2a(x + a^{2/3})$$

$$y^2 = 2ax + 2a^{5/3} \quad \text{——} (1)$$

Diff w.r.to x

$$2yy' = 2a \Rightarrow yy' = a \quad \text{——} (2)$$

Sub (2) in (1)

$$y^2 = 2yy'x + 2(yy')^{5/3}$$

$$y^2 - 2yy'x = 2(yy')^{5/3}$$

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Raising the power 3 on both sides.

$$(y^2 - 2yy'x)^3 = 8(yy')^5$$

ii $$\left(\frac{d^2y}{dx^2} - 2xy\frac{dy}{dx}\right)^3 = 8\left(y\frac{dy}{dx}\right)^5$$

Hence proved.

8) Show that $y = a\cos bx$ is a soln of the differential eqn

$$\frac{d^2y}{dx^2} + b^2y = 0.$$

$$y = a\cos bx \quad \text{——} \quad (1)$$

$$\frac{dy}{dx} = -a\sin bx\,(b)$$

$$: \quad -ab\sin bx.$$

Again diff. w.r.to x

$$\frac{d^2y}{dx^2} = -ab\cos bx \cdot b$$

$$: \quad \cancel{-b^2(a\cos bx)}$$

$$: \quad \cancel{b^2(-a\cos bx)}$$

$$= -b^2(a\cos bx) = -b^2y$$

$$\frac{d^2y}{dx^2} + b^2y = 0$$

Hence proved.

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1) Variable separable method.

10.11 Solve $(1+x^2)\frac{dy}{dx} = 1+y^2$

Given $(1+x^2)\frac{dy}{dx} = 1+y^2$

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

Integrating, $\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$

$\tan^{-1}(y) = \tan^{-1}(x) + c$

$\tan^{-1}(y) - \tan^{-1}(x) = c$

$\Rightarrow \tan^{-1}\left(\frac{y-x}{1+xy}\right) = c$

$\therefore \frac{y-x}{1+xy} = \tan c = a$ (say)

$y-x = a(1+xy)$ is the required soln..

10.12 find the particular soln of $(1+x^3)dy - x^2y\,dx = 0$ satisfying the condition $y(1) = 2$

Given that $(1+x^3)dy - x^2y\,dx = 0$

$(1+x^3)dy = x^2y\,dx$.

$\frac{dy}{y} = \frac{x^2}{1+x^3}dx$

Integrating $\int \frac{dy}{y} = \int \frac{x^2}{1+x^3}dx$.

$\log y = \frac{1}{3}\log(1+x^3) + c$

$3\log y = \log(1+x^3) + c$

$\log y^3 = \log(1+x^3) + \log C$

$= \log C(1+x^3)$

$\Rightarrow y^3 = C(1+x^3)$ is the general soln of the given eqn.

put $x=1, y=2$

Then $8 = C(1+1) \Rightarrow 2C = 8 \Rightarrow C = 4$

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Hence the particular soln is

$$y^3 = 4(1+x^3)$$

II Substitution Method.

10.13 Solve $y' = \sin^2(x-y+1)$

Given $y' = \sin^2(x-y+1)$

put $z = x-y+1$

$$\frac{dz}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx} = 1 - \frac{dz}{dx}$$

i $\quad \sin^2(x-y+1) = 1 - \frac{dz}{dx}$

$$\sin^2 z = 1 - \frac{dz}{dx}$$

$$\frac{dz}{dx} = 1 - \sin^2 z = \cos^2 z$$

$$\frac{dz}{\cos^2 z} = dx$$

$$\sec^2 z\, dz = dx.$$

Integrating $\int \sec^2 z\, dz = \int dx$

$$\tan z = x + c.$$

i $\quad \tan(x-y+1) = x+c.$

10.14 Solve $\frac{dy}{dx} = \sqrt{4x+2y-1}$

put $z = 4x+2y-1$

$$\frac{dz}{dx} = 4 + 2\frac{dy}{dx} = 4 + 2y'$$

$$\therefore = 4 + 2\sqrt{z}.$$

$$\frac{dz}{4+2\sqrt{z}} = dx.$$

Integrating $\int \frac{dz}{4+2\sqrt{z}} = \int dx.$

$$\therefore \int \frac{dz}{4+2\sqrt{z}} = \int dx \Rightarrow \int \frac{dz}{4+2\sqrt{z}} = x + c.$$

putting $z = u^2$

$$dz = 2u\,du.$$

$$\therefore \int \frac{dz}{4+2\sqrt{z}} = x+c \Rightarrow \int \frac{2u\,dy}{4+2u} = x+c$$

$$\int \frac{u\,dy}{2+u} = x+c.$$

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$$u - 2\ln|u+2| + c = x + c$$

(or) $$\sqrt{z} - 2\ln\left(\sqrt{4x+2y-1}\right) = x + c.$$

$$\sqrt{4x+2y-1} - 2\ln\left(\sqrt{4x+2y-1}\right) = x + c.$$

Eg 10.15 Solve $\frac{dy}{dx} = \frac{x-y+5}{2(x-y)+7}$

Given $\frac{dy}{dx} = \frac{x-y+5}{2(x-y)+7}$ — ①

put $z = x-y$

$$\frac{dz}{dx} = 1 - \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 1 - \frac{dz}{dx} \quad — ②$$

From ① and ②

$$1 - \frac{dz}{dx} = \frac{z+5}{2z+7}$$

$$\frac{dz}{dx} = 1 - \frac{z+5}{2z+7}$$

$$\frac{dz}{dx} = \frac{2z+7-z-5}{2z+7}$$

$$\frac{dz}{dx} = \frac{z+2}{2z+7}$$

$$\left(\frac{2z+7}{z+2}\right)dz = dx.$$

$$\left(\frac{2z+4+3}{z+2}\right)dz = dx \Rightarrow \left(\frac{2(z+2)+3}{z+2}\right)dz = dx$$

$$\Rightarrow \frac{2(z+2)}{z+2}dz + \frac{3}{z+2}dz = dx$$

$$2\,dz + \frac{3}{z+2}dz = dx.$$

Integrating $2\int dz + 3\int \frac{dz}{z+2} = \int dx.$

$$= 2z + 3\log(z+2) = x + c$$

i.e. $$2(x-y) + 3\log(x-y+2) = x + C$$

Eg 10.16 Solve $\frac{dy}{dx} = (3x+y+4)^2$ — ①

put $z = 3x+y+4$

$$\frac{dz}{dx} = 3 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 3 \quad — ②$$

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from ① and ②

$$\frac{dz}{dx} - 3 = (3x+y+4)^2$$

$$\frac{dz}{dx} - 3 = z^2$$

$$\frac{dz}{dx} = 3 + z^2$$

$$\frac{dz}{3+z^2} = dx \Rightarrow \frac{dz}{(\sqrt{3})^2 + z^2} = dx .$$

Integrating,

$$\int \frac{dz}{(\sqrt{3})^2 + z^2} = \int dx$$

$$\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{z}{\sqrt{3}}\right) = x + c$$

$$\frac{1}{\sqrt{3}} \tan^{-1}\left[\frac{3x+y+4}{\sqrt{3}}\right] = x + c$$

EXERCISE 10.5

1. If F is the constant force generated by the motor of an automobile of Mass M, its velocity V is given by $M\frac{dV}{dt} = F - KV$, where K is constant Express V interms of t given that V = 0 when t = 0.

Given eqn is $M\frac{dV}{dt} = F - KV$

$$M\frac{dV}{dt} = K\left(\frac{F}{K} - V\right)$$

$$\int \frac{dV}{\frac{F}{K} - V} = \int \frac{K}{M}\, dt .$$

$$-\log\left(\frac{F}{K} - V\right) = \frac{K}{M} t + c \quad \text{——} ①$$

Given V = 0 and t = 0.

$$-\log\left(\frac{F}{K}\right) = 0 + c \Rightarrow c = -\log\left(\frac{F}{K}\right)$$

$$\therefore ① \Rightarrow -\log\left(\frac{F}{K} - V\right) = \frac{K}{M} t - \log\left(\frac{F}{K}\right)$$

$$\log\left(\frac{F}{K}\right) - \log\left(\frac{F}{K} - V\right) = \frac{Kt}{M}$$

$$\log\left[\frac{F/K}{\frac{F}{K} - V}\right] = \frac{Kt}{M}$$

$$\frac{F/K}{\frac{F}{K} - V} = e^{\frac{Kt}{M}}$$

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~~Variable Separable.~~

Given eqn is $m\frac{dv}{dt} = f - kv$

On separating the variables, we get

$$\frac{dv}{f-kv} = \frac{1}{m}dt$$

$$\int \frac{dv}{f-kv} = \int \frac{dt}{m}$$

$$\frac{\log(f-kv)}{-k} = t/m + \log c$$

$$\log(f-kv) = -\frac{kt}{m} + \log c$$

$$\log(f-kv) - \log c = -\frac{kt}{m}$$

$$\log\left(\frac{f-kv}{c}\right) = -\frac{kt}{m}$$

$$\frac{f-kv}{c} = e^{-kt/m}$$

$$\frac{f-kv}{c} = \frac{1}{e^{kt/m}}$$

$$c = (f-kv)\,e^{kt/m} \quad \text{—— (1)}$$

when $v=0, t=0$

then $c = (f-0)e^0$

$c = fe^0 \Rightarrow c = f \quad \because e^0 = 1$

$\therefore$ eqn (1) becomes

$$f = (f-kv)\,e^{kt/m}$$

———— x ———— x ————

2) The velocity v of a parachute falling vertically satisfies the eqn $v\frac{dv}{dx} = g\left(1 - \frac{v^2}{k^2}\right)$ where g and k are constants. If v and x are both initially zero, find v interms of x.

Given $v\frac{dv}{dx} = g\left(1 - \frac{v^2}{k^2}\right)$

$$v\frac{dv}{dx} = g\left(\frac{k^2-v^2}{k^2}\right)$$

$$\frac{v\,dv}{k^2-v^2} = \frac{g}{k^2}dx$$

Integrating, $\int \frac{v\,dv}{k^2-v^2} = \frac{g}{k^2}\int dx$

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$$-\frac{1}{2}\log(k^2-v^2) = \frac{g}{k^2}x + c \quad \text{— ①}$$

Given $v=0,\ x=0$

$$\therefore \quad -\frac{1}{2}\log k^2 = c$$

$$-\frac{1}{2}\,2\log k = c$$

$$-\log k = c$$

$$\log 1 - \log k = c$$

$$\log\frac{1}{k} = c$$

$\therefore$ From ①

$$-\frac{1}{2}\log(k^2-v^2) = \frac{g}{k^2}x + \log\left(\frac{1}{k}\right)$$

$$-\log\sqrt{k^2-v^2} = \frac{g}{k^2}x + \log\left(\frac{1}{k}\right)$$

$$\log\left(\frac{1}{\sqrt{k^2-v^2}}\right) - \log\left(\frac{1}{k}\right) = \frac{g}{k^2}x$$

$$\log\left[\frac{\frac{1}{\sqrt{k^2-v^2}}}{\frac{1}{k}}\right] = \frac{g}{k^2}x$$

$$\log\left[\frac{k}{\sqrt{k^2-v^2}}\right] = \frac{g}{k^2}x$$

$$\frac{k}{\sqrt{k^2-v^2}} = e^{\frac{gx}{k^2}}$$

$$\frac{k}{e^{\frac{gx}{k^2}}} = \sqrt{k^2-v^2}$$

$$k\,e^{-\frac{gx}{k^2}} = \sqrt{k^2-v^2}$$

Squaring on both sides

$$k^2 e^{-\frac{2gx}{k^2}} = k^2 - v^2$$

$$v^2 = k^2 - k^2 e^{-\frac{2gx}{k^2}}$$

$$v^2 = k^2\left(1 - e^{-\frac{2gx}{k^2}}\right)$$

//

3) Find the eqn of the curve whose slope is $\frac{y-1}{x^2+x}$ and which passes through the point $(1,0)$

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Given that slope $= \frac{y-1}{x^2+x}$

i.e. $\frac{dy}{dx} = \frac{y-1}{x^2+x}$

$\Rightarrow \frac{dy}{y-1} = \frac{dx}{x^2+x}$

$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

$A(x+1) + Bx = 1$

put $x=0$, $A=1$,

put $x=-1$, $-B=1 \Rightarrow B=-1$

$\therefore \frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1}$

$\therefore \frac{dy}{y-1} = \frac{dx}{x} - \frac{dx}{x+1}$

Integrating,

$\int \frac{dy}{y-1} = \int \frac{dx}{x} - \int \frac{dx}{x+1}$

$\log(y-1) = \log x - \log(x+1) + \log C$

$\log(y-1) = \log\left(\frac{x}{x+1}\right) + \log C$

$\log(y-1) = \log \frac{xC}{x+1} \Rightarrow y-1 = \frac{xC}{x+1}$

Given that the curve passes through $(1,0)$

$\therefore -1 = \frac{C}{2} \Rightarrow C = -2$

$\therefore y-1 = \frac{-2x}{x+1}$

$y = 1 - \frac{2x}{x+1} \Rightarrow y = \frac{x+1-2x}{x+1}$

$\therefore y = \frac{1-x}{x+1}$ //

---

4) solve

(i) $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{\sqrt{1-x^2}}$

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Integrating,

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{\sqrt{1-x^2}}$$

$$\sin^{-1} y = \sin^{-1} x + c$$

(ii) $y\,dx + (1+x^2)\tan^{-1} x\,dy = 0$

$$y\,dx = -(1+x^2)\tan^{-1} x\,dy$$

$$\frac{dx}{(1+x^2)\tan^{-1} x} = -\frac{dy}{y}$$

Integrating

$$\int \frac{dx}{(1+x^2)\tan^{-1} x} = -\int \frac{dy}{y}$$

$$\log \tan^{-1} x = -\log y + \log c$$

$$\log \tan^{-1} x + \log y = \log c$$

$$\log y \tan^{-1} x = \log c$$

$$y \tan^{-1} x = c \quad //$$

(iii) $\sin \frac{dy}{dx} = a, \quad y(0) = 1$

$$\sin\left(\frac{dy}{dx}\right) = a$$

$$\Rightarrow \frac{dy}{dx} = \sin^{-1}(a)$$

$$dy = \sin^{-1}(a)\,dx$$

$$\int dy = \sin^{-1}(a) \int dx$$

$$y = \sin^{-1}(a)\,x + c \quad \text{——— } ①$$

Since $y(0) = 1$ we get-

$$1 = \sin^{-1}(a)(0) + c \Rightarrow c = 1$$

$\therefore$ ① $\Rightarrow$ $y = \sin^{-1}(a)\,x + 1$

$$y - 1 = \sin^{-1}(a)\,x$$

$$\frac{y-1}{x} = \sin^{-1}(a)$$

$$\sin\left(\frac{y-1}{x}\right) = a \quad //$$

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(iv) $\frac{dy}{dx} = e^{x+y} + x^3 e^y$

$\frac{dy}{dx} = e^x \cdot e^y + x^3 e^y$

$= e^y (e^x + x^3)$

$\frac{dy}{e^y} = (e^x + x^3)\,dx$

$e^{-y}\,dy = (e^x + x^3)\,dx$

Integrating

$\int e^{-y}\,dy = \int e^x\,dx + \int x^3\,dx$

$-e^{-y} = e^x + \frac{x^4}{4} + c$

$e^x + e^{-y} + \frac{x^4}{4} = -c = c$

(v) $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$

$(e^y + 1)\cos x\,dx = -e^y \sin x\,dy$

$\frac{\cos x}{\sin x}\,dx = -\frac{e^y}{e^y + 1}\,dy$ ——— (1)

put $t = e^y + 1$

$dt = e^y\,dy$

$\therefore$ (1) $\Rightarrow \cot x\,dx = \frac{-dt}{t}$

$\int \cot x\,dx = -\int \frac{dt}{t}$

$\log \sin x = -\log t$

$\log \sin x = -\log (e^y + 1) + \log c$

$\log \sin x + \log (e^y + 1) = \log c$

$\log \sin x \cdot (e^y + 1) = \log c$

$\sin x\,(e^y + 1) = c$ //

(vi) $(y\,dx - x\,dy)\cot\left(\frac{x}{y}\right) = ny^2\,dx$

$\frac{y\,dx - x\,dy}{y^2} \cot\left(\frac{x}{y}\right) = n\,dx$ ——— (1)

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put $\frac{x}{y} = t$

$$\frac{y\,dx - x\,dy}{y^2} = dt$$

$\therefore$ (1) $\Rightarrow$ $dt \cot t = n\,dx$

$$\cot t\,dt = n\,dx$$

$$\int \cot t\,dt = \int n\,dx$$

$$\log \sin t = nx + c$$

$$\sin t = e^{nx+c}$$

$$\sin\left(\frac{x}{y}\right) = e^{nx+c}$$

(VII) $\frac{dy}{dx} - x\sqrt{25-x^2} = 0$

$$\frac{dy}{dx} = x\sqrt{25-x^2} \quad \text{---------- (1)}$$

put $25 - x^2 = t$

Diff

$$-2x\,dx = dt$$

$$x\,dx = -\frac{dt}{2}$$

(1) $\Rightarrow$ $dy = x\,dx\sqrt{25-x^2}$

$$dy = -\frac{dt}{2}\sqrt{t}$$

Integrating $\int dy = -\frac{1}{2}\int \sqrt{t}\,dt$

$$y = -\frac{1}{2}\frac{t^{3/2}}{3/2} + c$$

$$y = -\frac{1}{2} \times \frac{2}{3} t^{3/2} + c$$

$$y = -\frac{1}{3} t^{3/2} + c$$

$$3y = -(25-x^2)^{3/2} + 3c$$

(VIII) $x \cos y\,dy = e^x (x \log x + 1)\,dx$

$$\cos y\,dy = \frac{e^x (x\log x + 1)\,dx}{x}$$

$$\cos y\,dy = e^x \left(\log x + \frac{1}{x}\right) dx$$

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Integrating,

$$\int \cos y \, dy = \int e^x \left(\log x + \frac{1}{x}\right) dx$$

$$\sin y = \int e^x \left(f(x) + f'(x)\right) dx .$$

where $f(x) = \log x$

$$\sin y = e^x f(x) + c$$

$$\sin y = e^x \log x + c$$

(X) $\tan y \frac{dy}{dx} = \cos(x+y) + \cos(x-y)$

$$\tan y \frac{dy}{dx} = \cos x \cos y - \sin x \sin y + \cos x \cos y + \sin y \sin y$$

$$\tan y \frac{dy}{dx} = 2 \cos x \cos y$$

$$\frac{\tan y}{\cos y} dy = 2 \cos x \, dx$$

$$\tan y \sec y \, dy = 2 \cos x \, dx$$

Integrating

$$\int \tan y \sec y \, dy = 2 \int \cos x \, dx$$

$$\sec y = 2 \sin x + c .$$

(X) $\frac{dy}{dx} = \tan^2(x+y)$ ——— (1)

put $x+y = t$

$$1 + \frac{dy}{dx} = dt/dx$$

$$\therefore \frac{dy}{dx} = \frac{dt}{dx} - 1$$

$$\therefore (1) \Rightarrow \frac{dt}{dx} - 1 = \tan^2 t$$

$$\frac{dt}{dx} = \tan^2 t + 1$$

$$\frac{dt}{dx} = \sec^2 t$$

$$\frac{dt}{\sec^2 t} = dx \Rightarrow \cos^2 t \, dt = dx$$

$$\left(\frac{1 + \cos 2t}{2}\right) dt = dx$$

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$\because \left[\cos 2x = 2\cos^2 x - 1 \right.$

$2\cos^2 x = 1 + \cos 2x$

$\left. \cos^2 x = \frac{1+\cos 2x}{2} \right]$

Integrating

$$\frac{1}{2}\int (1+\cos 2t)\,dt = \int dx$$

$$\frac{1}{2}\left[t + \frac{\sin 2t}{2} \right] = x + c$$

$$\frac{1}{2}\left[t + \frac{2\sin t \cos t}{2} \right] = x + c$$

$$\frac{1}{2}\left[t + \sin t \cos t \right] = x + C$$

ii) $\frac{1}{2}\left[x+y + \sin(x+y)\cos(x+y) \right] = x + C$



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Homogeneous form or Homogeneous Differential eqn.

A function $f(x,y)$ is said to be a homogeneous function of degree n in the variables x and y if, $f(tx, ty) = t^n f(x,y)$ for some $n \in R$ for all suitable restricted x, y and t.

Homogeneous Differential eqn:-

An ordinary differential eqn is said to be in homogeneous form, if the differential eqn is written as $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$.

If $M(x,y)\,dx + N(x,y)\,dy = 0$ is a homogeneous eqns, then the change of variable $y = vx$, transforms in to a separable eqn in the variable v and x.

Ex 10.17 Solve $(x^2 - 3y^2)\,dx + 2xy\,dy = 0$

$$(x^2 - 3y^2)\,dx + 2xy\,dy = 0$$

$$2xy\,dy = -(x^2 - 3y^2)\,dx$$

$$\frac{dy}{dx} = -\frac{(x^2 - 3y^2)}{2xy}$$

$$= -\frac{x^2}{2xy} + \frac{3y^2}{2xy}$$

$$\frac{dy}{dx} = -\frac{x}{2y} + \frac{3y}{2x} \quad \text{——— (1)}$$

This is a homogeneous differential eqn.

put $y = vx$

$$\frac{dy}{dx} = v \cdot 1 + x\frac{dv}{dx}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{——— (2)}$$

Sub (2) in (1)

$$v + x\frac{dv}{dx} = \frac{3vx}{2x} - \frac{x}{2vx}$$

$$= \frac{3v}{2} - \frac{1}{2v}$$

$$\therefore x\frac{dv}{dx} = \frac{3v}{2} - \frac{1}{2v} - v$$

$$= \frac{3v^2 - 1 - 2v^2}{2v}$$

$$= \frac{v^2 - 1}{2v}.$$

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$$\frac{2v}{v^2-1}\,dv = \frac{dx}{x}$$

Integrating, $\int \frac{2v}{v^2-1}\,dv = \int \frac{1}{x}\,dx$

$$\log(v^2-1) = \log x + \log c$$

$$\log(v^2-1) - \log x = \log c$$

$$\log \frac{v^2-1}{x} = \log c$$

$$\frac{\frac{y^2}{x^2}-1}{x} = c \Rightarrow \frac{y^2-x^2}{x^3} = \pm c$$

$$\therefore\ y^2-x^2 = \pm c x^3$$

Hence $y^2-x^2 = Kx^3$ is the general soln.

Example 10.18

Solve $(y+\sqrt{x^2+y^2})\,dx - x\,dy = 0$, $y(1)=0$

$$(y+\sqrt{x^2+y^2})\,dx = x\,dy$$

$$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} \quad \text{——— (1)}$$

which is homogeneous differential eqn.

put $y = vx$

$$\therefore \frac{dy}{dx} = v(1) + x\frac{dv}{dx}$$

$$= v + x\frac{dv}{dx} \quad \text{——— (2)}$$

Sub (2) in (1)

$$v + x\frac{dv}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$$

$$= \frac{vx+\sqrt{x^2+v^2x^2}}{x}$$

$$= \frac{vx+\sqrt{x^2(1+v^2)}}{x}$$

$$= \frac{vx+x\sqrt{1+v^2}}{x}$$

$$= \frac{x\left(v+\sqrt{1+v^2}\right)}{x}$$

$$= v+\sqrt{1+v^2}$$

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$$\therefore \quad x\frac{dv}{dx} = v + \sqrt{1+v^2} - v$$

$$x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integrating, $\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{1}{x}\,dx$

WKT $\int \frac{dx}{\sqrt{a^2+x^2}} = \log\left|x+\sqrt{a^2+x^2}\right| + c$

$$\therefore \int \frac{dv}{\sqrt{1+v^2}} = \log|x| + \log c$$

$$= \log\left|\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right| = \log|x| + \log c$$

$$= \log\left|\frac{y}{x} + \sqrt{\frac{x^2+y^2}{x^2}}\right| = \log|x| + \log c$$

$$= \log\left|\frac{y+\sqrt{x^2+y^2}}{x}\right| = \log|x| + \log c$$

$$\log\left|\frac{y+\sqrt{x^2+y^2}}{x}\right| - \log|x| = \log c$$

$$\log\left|\frac{y+\sqrt{x^2+y^2}}{x^2}\right| = \log c$$

$$\frac{y+\sqrt{x^2+y^2}}{x^2} = c \Rightarrow y+\sqrt{x^2+y^2} = cx^2 \quad \hookrightarrow (3).$$

Given $y(1) = 0$

i.e. when $x=1$, $y=0$

Sub $x=1$ and $y=0$ in (3)

$$0+\sqrt{1} = c(1)$$

$$\therefore \boxed{c=1}$$

$$\therefore \quad y+\sqrt{x^2+y^2} = x^2$$

10.19 Solve

$$(2x+3y)\,dx + (y-x)\,dy = 0$$

$$(y-x)\,dy = -(2x+3y)\,dx$$

$$\frac{dy}{dx} = -\frac{(2x+3y)}{y-x}$$

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$$\frac{dy}{dx} = \frac{-(2x+3y)}{-(x-y)} = \frac{2x+3y}{x-y} \quad \text{—— (1)}$$

put $y = vx$

$$\therefore \frac{dy}{dx} = v(1) + x\frac{dv}{dx}$$

$$= v + x\frac{dv}{dx} \quad \text{—— (2)}$$

Sub (2) in (1)

$$v + x\frac{dv}{dx} = \frac{2x+3y}{x-y}$$

$$v + x\frac{dv}{dx} = \frac{2x+3(vx)}{x-vx}$$

$$= \frac{x(2+3v)}{x(1-v)}$$

$$= \frac{2+3v}{1-v}$$

Now $x\frac{dv}{dx} = \frac{2+3v}{1-v} - v$

$$= \frac{2+3v-v(1-v)}{1-v}$$

$$x\frac{dv}{dx} = \frac{2+3v-v+v^2}{1-v}$$

$$x\frac{dv}{dx} = \frac{v^2+2v+2}{1-v}$$

$$\frac{1-v}{v^2+2v+2}\,dv = \frac{dx}{x} \quad \text{—— (3)}$$

put $1-v = A\frac{d}{dx}(v^2+2v+2) + B$

$1-v = A(2v+2) + B$

$1-v = 2Av + 2A + B$

compare the coefficients

$2A = -1$ | $2A + B = 1$

$\boxed{A = -\frac{1}{2}}$ | $2(-\frac{1}{2}) + B = 1$

$-1 + B = 1 \Rightarrow \boxed{B = 2}$

$\therefore 1-v = -\frac{1}{2}(2v+2) + 2$

Sub in (3)

$$\frac{-\frac{1}{2}(2v+2)+2}{v^2+2v+2}\,dv = \frac{dx}{x}$$

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Integrating,

$$\int \frac{-\frac{1}{2}(2v+2)+2}{v^2+2v+2}dv = \int \frac{dx}{x}$$

$$-\frac{1}{2}\int \frac{2v+2}{v^2+2v+2}dv + 2\int \frac{dv}{v^2+2v+2} = \int \frac{dx}{x}$$

$$-\frac{1}{2}\int \frac{2v+2}{v^2+2v+2}dv + 2\int \frac{dv}{v^2+2v+1+1} = \int \frac{dx}{x}$$

$$-\frac{1}{2}\log|v^2+2v+2| + 2\int \frac{dv}{(v+1)^2+1^2} = \int \frac{dx}{x}$$

$$-\frac{1}{2}\log|v^2+2v+2| + 2\tan^{-1}\left(\frac{v+1}{1}\right) = \log x + C$$

$$2\tan^{-1}(v+1) = \frac{1}{2}\log|v^2+2v+2| + \log x + C$$

$$2\tan^{-1}\left(\frac{y}{x}+1\right) = \frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{2y}{x}+2\right| + \log x + C$$

$$2\tan^{-1}\left(\frac{y+x}{x}\right) = \frac{1}{2}\log\left|\frac{y^2+2xy+2x^2}{x^2}\right| + \log x + C$$

$$2\tan^{-1}\left(\frac{x+y}{x}\right) = \frac{1}{2}\log|2x^2+2xy+y^2| - \frac{1}{2}\log x^2 + \log x + C$$

$$2\tan^{-1}\left(\frac{x+y}{x}\right) = \frac{1}{2}\log|2x^2+2xy+y^2| - \log\sqrt{x^2} + \log x + C$$

$$2\tan^{-1}\left(\frac{x+y}{x}\right) = \frac{1}{2}\log|2x^2+2xy+y^2| - \log x + \log x + C$$

$$4\tan^{-1}\left(\frac{x+y}{x}\right) = \log|2x^2+2xy+y^2| + 2C$$

$$\log|2x^2+2xy+y^2| - 4\tan^{-1}\left(\frac{x+y}{x}\right) = -2C$$

$$\therefore \log|2x^2+2xy+y^2| - 4\tan^{-1}\left(\frac{x+y}{x}\right) = k$$

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10.20 Solve $y^2 + x^2 \frac{dy}{dx} = xy \frac{dy}{dx}$

$$y^2 + x^2 \frac{dy}{dx} = xy \frac{dy}{dx}$$

$$y^2 = xy \frac{dy}{dx} - x^2 \frac{dy}{dx}$$

$$= (xy - x^2) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y^2}{xy - x^2} \quad \text{——— (1)}$$

put $y = vx$

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad \text{——— (2)}$$

Sub (2) in (1)

$$v + x \frac{dv}{dx} = \frac{y^2}{xy - x^2}$$

$$v + x \frac{dv}{dx} = \frac{v^2 x^2}{x \cdot vx - x^2}$$

$$= \frac{v^2 x^2}{x^2 (v-1)} = \frac{v^2}{v-1}$$

$$v + x \frac{dv}{dx} = \frac{v^2}{v-1}$$

$$x \frac{dv}{dx} = \frac{v^2}{v-1} - v = \frac{v^2 - v(v-1)}{v-1}$$

$$= \frac{v^2 - v^2 + v}{v-1} = \frac{v}{v-1}$$

$$\therefore \frac{v-1}{v} dv = \frac{dx}{x}$$

Integrating,

$$\int \frac{v-1}{v} dv = \int \frac{dx}{x}$$

$$\int \left(\frac{v}{v} - \frac{1}{v}\right) dv = \int \frac{dx}{x}$$

$$\int \left(1 - \frac{1}{v}\right) dv = \int \frac{dx}{x}$$

$$\int dv - \int \frac{1}{v} dv = \int \frac{dx}{x}$$

$$v - \log|v| = \log|x| + \log|c|$$

$$v = \log|v| + \log|x| + \log|c|$$

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$V = \log Vxc$

$\frac{y}{x} = \log \frac{y}{x} \cdot x \cdot c$

$\frac{y}{x} = \log yc$

$e^{y/x} = yc$

$y = \frac{1}{c} e^{y/x} = k e^{\frac{y}{x}}$

$\therefore \boxed{y = k e^{y/x}}$

10.21 Solve $\left(1 + 2e^{\frac{x}{y}}\right) dx + 2e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right) dy = 0$

$\left(1 + 2e^{\frac{x}{y}}\right) dx = -2e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right) dy$

[Given eqn is in the power of x/y so we find $\frac{dx}{dy}$]

$$\frac{dx}{dy} = \frac{-2e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right)}{1 + 2e^{x/y}} \quad \text{——} (1)$$

put $V = \frac{x}{y}$, then $x = Vy$

$\therefore \frac{dx}{dy} = V + y \frac{dv}{dy}$ —(2)

sub (2) in (1)

$$V + y \frac{dv}{dy} = \frac{-2e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right)}{1 + 2e^{x/y}}$$

$$y \frac{dv}{dy} = \frac{-2e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right)}{1 + 2e^{x/y}} - V$$

$$= \frac{-2e^{x/y} \left(1 - \frac{x}{y}\right) - V\left(1 + 2e^{\frac{x}{y}}\right)}{1 + 2e^{x/y}}$$

$$= \frac{-2e^{V} + 2e^{V} \cdot V - V - 2e^{V} \cdot V}{1 + 2e^{V}}$$

$$= \frac{-(V + 2e^{V})}{1 + 2e^{V}}$$

$$\frac{1 + 2e^{V}}{V + 2e^{V}} dv = -\frac{dy}{y}$$

Integrating,

$$\int \frac{1 + 2e^{V}}{V + 2e^{V}} dv = -\int \frac{dy}{y}$$

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$$\log|(Cv + 2e^{v})| = -\log|y| + \log c$$

$$\log\left|\frac{x}{y} + 2e^{\frac{x}{y}}\right| = -\log|y| + \log c$$

$$\log\left|\frac{x + 2ye^{x/y}}{y}\right| = -\log|y| + \log c$$

$$\log|x + 2ye^{x/y}| - \log|y| = -\log|y| + \log c$$

$$\therefore \log|x + 2ye^{x/y}| = \log c$$

$$\therefore x + 2ye^{x/y} = \pm c = k.$$

$$\therefore x + 2ye^{x/y} = k.$$

---

1) **EXERCISE 10.6**

Solve: $\left[x + y\cos\frac{y}{x}\right]dx = x\cos\frac{y}{x}\,dy$

$$\frac{dy}{dx} = \frac{x + y\cos\frac{y}{x}}{x\cos\frac{y}{x}} \quad \text{——— (1)}$$

This is homogeneous differential eqn.

put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{——— (2)}$$

sub (2) in (1).

$$v + x\frac{dv}{dx} = \frac{x + y\cos\frac{y}{x}}{x\cos\frac{y}{x}}$$

$$v + x\frac{dv}{dx} = \frac{x + vx\cos v}{x\cos v} \qquad \left|\ y = vx,\ \frac{y}{x} = v\right.$$

$$= \frac{x(1 + v\cos v)}{x\cos v}$$

$$= \frac{1 + v\cos v}{\cos v}$$

$$\therefore x\frac{dv}{dx} = \frac{1 + v\cos v}{\cos v} - v$$

$$= \frac{1 + v\cos v - v\cos v}{\cos v}$$

$$= \frac{1}{\cos v}$$

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$$\cos v\, dv = \frac{dx}{x}$$

Integrating,

$$\int \cos v\, dv = \int \frac{dx}{x}$$

$$\sin v = \log x + \log c$$

$$\sin\left(\frac{y}{x}\right) = \log|cx|.$$

---

2) $(x^3+y^3)\,dy - x^2y\,dx = 0.$

$$(x^3+y^3)\,dy = x^2y\,dx$$

$$\frac{dy}{dx} = \frac{x^2y}{x^3+y^3} \quad \text{——— (1)}$$

This is homogeneous differential eqn.

put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{— (2)}$$

Sub (2) in (1)

$$v + x\frac{dv}{dx} = \frac{x^2y}{x^3+y^3}$$

$$v + x\frac{dv}{dx} = \frac{x^2vx}{x^3+v^3x^3}$$

$$= \frac{x^3v}{x^3(1+v^3)} = \frac{v}{1+v^3}$$

$$\therefore x\frac{dv}{dx} = \frac{v}{1+v^3} - v$$

$$= \frac{v - v(1+v^3)}{1+v^3}$$

$$= \frac{v - v - v^4}{1+v^3} = \frac{-v^4}{1+v^3}$$

$$\frac{1+v^3}{v^4}\,dv = -\frac{dx}{x}$$

Integrating,

$$\int \frac{1+v^3}{v^4}\,dv = -\int \frac{dx}{x}$$

$$\int\left(\frac{1}{v^4} + \frac{v^3}{v^4}\right)dv = -\int \frac{dx}{x}$$

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$$\int v^{-4}\,dv + \int \frac{1}{v}\,dv = -\int \frac{dx}{x}$$

$$\frac{v^{-4+1}}{-4+1} + \log|v| = -\log|x| + \log|c|$$

$$\frac{v^{-3}}{-3} + \log\left|\frac{y}{x}\right| = -\log|x| + \log|c|$$

$$-\frac{1}{3v^3} + \log|y| - \log|x| = -\log|x| + \log|c|$$

$$-\frac{1}{3\left(\frac{y}{x}\right)^3} + \log|y| = \log|c|$$

$$-\frac{x^3}{3y^3} + \log|y| = \log|c|$$

$$\log|y| - \log|c| = \frac{x^3}{3y^3}$$

$$\log\left|\frac{y}{c}\right| = \frac{x^3}{3y^3}$$

$$\frac{y}{c} = e^{x^3/3y^3}$$

$$y = c\,e^{\frac{x^3}{3y^3}}$$ //

3) $y e^{\frac{x}{y}}\,dx = \left(x e^{\frac{x}{y}} + y\right)dy$

$$y e^{\frac{x}{y}}\,dx = \left(x e^{\frac{x}{y}} + y\right)dy$$

$$\frac{dx}{dy} = \frac{x e^{x/y} + y}{y e^{x/y}} \quad \text{---} \quad (1).$$

put $x = vy$ or $\frac{x}{y} = v$

$$\frac{dx}{dy} = v + y\frac{dv}{dy} \quad \text{---} \quad (2)$$

Sub (2) in (1)

$$v + y\frac{dv}{dy} = \frac{x e^{v} + \frac{x}{v}}{\frac{x}{v} e^{v}}$$

$$= \frac{x\left(e^{v} + \frac{1}{v}\right)}{x\,\frac{e^{v}}{v}}$$

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$$v + y\frac{dv}{dy} = \frac{\frac{ve^v+1}{v}}{\frac{e^v}{v}} = \frac{ve^v+1}{e^v}$$

$$y\frac{dv}{dy} = \frac{ve^v+1}{e^v} - v = \frac{ve^v+1-ve^v}{e^v}$$

$$= \frac{1}{e^v}$$

$$\therefore \quad e^v dv = \frac{dy}{y}$$

Integrating, $\int e^v dv = \int \frac{dy}{y}$

$$e^v = \log y + \log c$$

$$e^{\frac{x}{y}} = \log |yc|$$

---

4) $2xy\,dx + (x^2+2y^2)\,dy = 0$

$$2xy\,dx = -(x^2+2y^2)\,dy$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2+2y^2} \quad \text{——— (1)}$$

This is homogeneous differential eqn.

put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{——— (2)}$$

Sub (2) in (1)

$$v + x\frac{dv}{dx} = \frac{-2xy}{x^2+2y^2}$$

$$v + x\frac{dv}{dx} = \frac{-2x(vx)}{x^2+2v^2x^2}$$

$$= \frac{-2x^2v}{x^2+2x^2v^2}$$

$$= \frac{-2x^2v}{x^2(1+2v^2)}$$

$$= \frac{-2v}{1+2v^2}$$

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$$x\frac{dv}{dx} = -\frac{2v}{1+2v^2} - v$$

$$= \frac{-2v - v(1+2v^2)}{1+2v^2}$$

$$= \frac{-2v - v - 2v^3}{1+2v^2}$$

$$= -\frac{(3v+2v^3)}{1+2v^2}$$

$$\frac{1+2v^2}{3v+2v^3}\,dv = -\frac{dx}{x}$$

Integrating,

$$\int \frac{1+2v^2}{2v^3+3v}\,dv = -\int \frac{dx}{x} \quad \text{—— (3)}$$

put $2v^3+3v = t$

Diff, $(6v^2+3)dv = dt$

$3(2v^2+1)dv = dt$

$$(2v^2+1)\,dv = \frac{dt}{3}$$

∴ from (3)

$$\int \frac{1}{3}\frac{dt}{t} = -\int \frac{dx}{x}$$

$$\frac{1}{3}\log|t| = -\log|x| + \log|c|$$

$$\frac{1}{3}\log|2v^3+3v| = -\log|x| + \log|c|$$

$$\frac{1}{3}\log\left|2\frac{y^3}{x^3}+3\frac{y}{x}\right| = -\log|x| + \log|c|$$

$$\frac{1}{3}\log\left|\frac{2y^3+3x^2y}{x^3}\right| = -\log|x| + \log|c|$$

$$\frac{1}{3}\left[\log|2y^3+3x^2y| - \log|x^3|\right]$$

$$= -\log|x| + \log|c|$$

$$\frac{1}{3}\log|2y^3+3x^2y| - \frac{1}{3}\log|x^3|$$

$$= -\log|x| + \log|c|$$

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$$\frac{1}{3}\log|2y^3+3x^2y| - \log|x| = -\log|x| + \log|c|$$

$$\frac{1}{3}\log|2y^3+3x^2y| = \log|c|$$

$$2y^3+3x^2y = 3|c| = k$$

$$\therefore \quad 2y^3+3x^2y = k$$

---

5) $(y^2-2xy)\,dx = (x^2-2xy)\,dy$

$$\frac{dy}{dx} = \frac{y^2-2xy}{x^2-2xy} \quad \text{——— (1)}$$

This is homogeneous differential eqn.

put $y = vx$, Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ ——(2)

Sub (2) in (1)

$$v + x\frac{dv}{dx} = \frac{y^2-2xy}{x^2-2xy}$$

$$= \frac{v^2x^2-2xvx}{x^2-2x(vx)}$$

i.e.,

$$v + x\frac{dv}{dx} = \frac{x^2(v^2-2v)}{x^2(1-2v)}$$

$$v + x\frac{dv}{dx} = \frac{v^2-2v}{1-2v}$$

$$x\frac{dv}{dx} = \frac{v^2-2v}{1-2v} - v$$

$$= \frac{v^2-2v-v(1-2v)}{1-2v}$$

$$= \frac{v^2-2v-v+2v^2}{1-2v}$$

$$= \frac{3v^2-3v}{1-2v} = \frac{3(v^2-v)}{1-2v}$$

$$\frac{1-2v}{3(v^2-v)}dv = \frac{dx}{x}$$

Integrating,

$$\frac{1}{3}\int \frac{1-2v}{v^2-v}\,dv = \int \frac{dx}{x}$$

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consider $\frac{1-2v}{v^2-v} = \frac{1-2v}{v(v-1)}$

$= \frac{A}{v} + \frac{B}{v-1}$

$= \frac{A(v-1)+B(v)}{v(v-1)}$

$\therefore A(v-1) + Bv = 1-2v$

put $v=0$, $-A = 1 \Rightarrow \boxed{A=-1}$

put $v=1$, $B = -1 \Rightarrow \boxed{B=-1}$

$\therefore \frac{1}{3}\int \frac{1-2v}{v^2-v}\,dv = \int \frac{dx}{x}$

$\Rightarrow \frac{1}{3}\int\left[\frac{-1}{v} + \frac{-1}{v-1}\right]dv = \int \frac{dx}{x}$

$\Rightarrow -\frac{1}{3}\int \frac{1}{v}dv - \frac{1}{3}\int \frac{1}{v-1}dv = \int \frac{dx}{x}$

$-\frac{1}{3}\log v - \frac{1}{3}\log|v-1| = \log|x| + \log c$

$-\frac{1}{3}\log\left|\frac{y}{x}\right| - \frac{1}{3}\log\left|\frac{y}{x}-1\right| = \log|x| + \log|c|$

$-\frac{1}{3}\left[\log|y| - \log|x|\right] - \frac{1}{3}\log\left|\frac{y-x}{x}\right|$

$= \log|x| + \log|c|$

$-\frac{1}{3}\log|y| + \frac{1}{3}\log|x| - \frac{1}{3}\left[\log|y-x| - \log|x|\right]$

$= \log|x| + \log|c|$

$-\frac{1}{3}\log|y| + \frac{1}{3}\log|x| - \frac{1}{3}\log|y-x| +$

$\frac{1}{3}\log|x| = \log|x| + \log|c|$

$-\frac{1}{3}\left[\log|y| + \log|y-x| - 2\log|x|\right]$

$= \log|x| + \log|c|$

$\log|y| + \log|y-x| - 2\log|x|$

$= -3\log|x| - 3\log|c|$

$\log|y| + \log|y-x| - 2\log|x| + 3\log|x|$

$= -3\log|c|$

$\log|y| + \log|y-x| + \log|x| = \log|c|^{-3}$

$\log\left[y(y-x)x\right] = \log\frac{1}{c^3}$

$xy(y-x) = K$. or

$\boxed{xy^2 - x^2y = K}$

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6) $x\frac{dy}{dx} = y - x\cos^2\left(\frac{y}{x}\right)$

$$x\frac{dy}{dx} = y - x\cos^2\left(\frac{y}{x}\right)$$

$$\frac{dy}{dx} = \frac{y - x\cos^2\left(\frac{y}{x}\right)}{x}$$

$$= \frac{y}{x} - \cos^2\left(\frac{y}{x}\right) \quad \text{—— (1)}$$

This is a homogeneous differential eqn.

put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ —— (2)

Sub (2) in (1)

$$v + x\frac{dv}{dx} = \frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$$

$$v + x\frac{dv}{dx} = v - \cos^2(v)$$

$$x\frac{dv}{dx} = v - \cos^2(v) - v$$

$$x\frac{dv}{dx} = -\cos^2(v)$$

$$\frac{dv}{\cos^2 v} = -\frac{dx}{x}$$

$$\sec^2 v\, dv = -\frac{dx}{x}$$

Integrating,

$$\int \sec^2 v\, dv = -\int \frac{dx}{x}$$

$$\tan v + c = -\log|x| = \log|x|^{-1}$$

$$\tan v + c = \log\frac{1}{|x|}$$

$$\frac{1}{|x|} = e^{\tan v + c}$$

$$b = x\, e^{\tan\left(\frac{y}{x}\right)} \cdot e^{c}$$

$$\frac{1}{e^c} = x\, e^{\tan\left(\frac{y}{x}\right)}$$

$$\therefore x\, e^{\tan\left(\frac{y}{x}\right)} = k.$$

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7) $\left(1+3e^{y/x}\right)dy + 3e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)dx = 0.$

given that $y=0$ when $x=1$

$$\left(1+3e^{\frac{y}{x}}\right)dy = -3e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)dx$$

$$\frac{dy}{dx} = \frac{-3e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)}{1+3e^{y/x}} \quad \text{——} (1)$$

This is homogeneous differential eqn.

put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ —— (2)

sub (2) in (1)

$$v + x\frac{dv}{dx} = \frac{-3e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)}{1+3e^{y/x}}$$

$$v + x\frac{dv}{dx} = \frac{-3e^{v}(1-v)}{1+3e^{v}}$$

$$\therefore x\frac{dv}{dx} = \frac{-3e^{v}+3ve^{v}}{1+3e^{v}} - v$$

$$= \frac{-3e^{v}+3ve^{v}-v-3ve^{v}}{1+3e^{v}}$$

$$= \frac{-(v+3e^{v})}{1+3e^{v}}$$

$$\therefore \frac{1+3e^{v}}{v+3e^{v}}dv = -\frac{dx}{x}$$

Integrating,

$$\int \frac{1+3e^{v}}{v+3e^{v}}dv = -\int \frac{dx}{x}$$

$$\log|v+3e^{v}| = -\log|x| + \log|c|$$

$$\log\left|\frac{y}{x}+3e^{y/x}\right| = -\log|x| + \log|c|$$

$$\log\left|\frac{y+3xe^{y/x}}{x}\right| = -\log|x| + \log|c|$$

$$\log|y+3xe^{y/x}| - \log|x| = -\log|x| + \log|c|$$

$$\log|y+3xe^{y/x}| = \log|c|$$

$$\therefore c = y+3xe^{y/x} \quad \text{——} (3)$$

Given $y=0, x=1$

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$$c = 0 + 3(1)e^{0/1}$$

$$= 3(1)e^{0}$$

$$= 3$$

∴ The required soln is

$$y + 3x e^{y/x} = 3$$

---

8) $(x^2+y^2)\,dy = xy\,dx$. It is given that $y(1) = 1$ and $y(x_0) = e$. Find the value of x_0.

$$(x^2+y^2)\,dy = xy\,dx$$

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2} \quad \text{——— (1)}$$

This is a homogeneous differential eqn.

put $y = vx$, then

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{——— (2)}$$

sub (2) in (1)

$$v + x\frac{dv}{dx} = \frac{xy}{x^2+y^2}$$

$$v + x\frac{dv}{dx} = \frac{x(vx)}{x^2+v^2x^2}$$

$$= \frac{x^2v}{x^2(1+v^2)} = \frac{v}{1+v^2}$$

$$x\frac{dv}{dx} = \frac{v}{1+v^2} - v = \frac{v - v - v^3}{1+v^2}$$

$$x\frac{dv}{dx} = \frac{-v^3}{1+v^2}$$

$$\frac{1+v^2}{v^3}dv = -\frac{dx}{x}$$

Integrating,

$$\int \frac{1+v^2}{v^3}dv = -\int \frac{dx}{x}$$

$$\int \frac{1}{v^3}dv + \int \frac{v^2}{v^3}dv = -\int \frac{dx}{x}$$

$$\int \frac{1}{v^3}dv + \int \frac{1}{v}dv = -\int \frac{dx}{x}$$

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$$\int v^{-3}\,dv + \int \frac{1}{v}\,dv = -\int \frac{dx}{x}$$

$$\frac{v^{-3+1}}{-3+1} + \log v = -\log|x| + \log|c|$$

$$\frac{v^{-2}}{-2} + \log v = -\log|x| + \log|c|$$

$$-\frac{1}{2v^2} + \log\left|\frac{y}{x}\right| = -\log|x| + \log|c|$$

$$-\frac{1}{2\frac{y^2}{x^2}} + \log|y| - \log|x| = -\log|x| + \log|c|$$

$$-\frac{x^2}{2y^2} + \log|y| = \log|c|$$

$$\log|y| = \frac{x^2}{2y^2} + \log|c|$$

$$|y| = e^{\frac{x^2}{2y^2} + \log|c|}$$

$$= e^{\frac{x^2}{2y^2}\log|c|}$$

$$= e^{x^2/2y^2}\cdot e$$

$$y = ce^{x^2/2y^2} \quad \text{——} \quad (3)$$

Given $x = x_0$, $y = e$

$\therefore$ (3) $\Rightarrow$ $e = ce^{\frac{x_0^2}{2e^2}}$ —— (4)

Also given $x = 1$, $y = 1$

(3) $\Rightarrow$ $1 = ce^{\frac{1}{2}}$ —— (5)

$\frac{(4)}{(5)} \Rightarrow e = \dfrac{ce^{\frac{x_0^2}{2e^2}}}{ce^{\frac{1}{2}}}$

$$e\cdot e^{\frac{1}{2}} = e^{\frac{x_0^2}{2e^2}}$$

$$e^{\frac{3}{2}} = e^{\frac{x^2}{2e^2}}$$

$$\Rightarrow \frac{x_0^2}{2e^2} = \frac{3}{2} \Rightarrow 2x_0^2 = 2\times 3e^2$$

$$x_0^2 = 3e^2$$

$\therefore x_0 = \pm\sqrt{3}\,e$

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First order Linear Differential Equations:- Type I

A first order differential eqn of the form

$$\frac{dy}{dx} + py = Q$$

where p and Q are functions of x only.

The Solutions is given by

$$y e^{\int p dx} = \int Q e^{\int p dx} dx + c$$

where $e^{\int p dx}$ = Integrating factor = IF

Type II $\frac{dx}{dy} + px = Q$

This is linear eqn in x

This solutions is

$$x(IF) = \int Q(IF)\, dy$$

where $IF = e^{\int p dy}$

Here p and Q are functions of y only.

EXERCISE 10.7

1) Solve the following linear differential eqns.

$$\cos x \frac{dy}{dx} + y \sin x = 1$$

$$\frac{dy}{dx} + \frac{y \sin x}{\cos x} = \frac{1}{\cos x}$$

$$\frac{dy}{dx} + y \tan x = \sec x \quad \text{---} (1)$$

This is a linear differential eqn.

This is of the form

$$\frac{dy}{dx} + py = Q$$

Here $p = \tan x$, $Q = \sec x$

$$\int p dx = \int \tan x\, dx$$

$$= \log \sec x$$

∴ Solution of eqn (1) is

$$y e^{\int p dx} = \int Q e^{\int p dx} dx$$

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$$y e^{\log \sec x} = \int \sec x \cdot e^{\log \sec x} dx + C$$

$$y \sec x = \int \sec x \cdot \sec x \, dx + c$$

$$y \sec x = \int \sec^2 x \, dx + C$$

$$y \sec x = \tan x + c$$

$$y \cdot \frac{1}{\cos x} = \frac{\sin x}{\cos x} + c$$

$$y = \frac{\sin x}{\cos x} \times \cos x + C \cos x$$

$$y = \sin x + C \cos x$$

which is the required solution.

---

2) $(1-x^2)\frac{dy}{dx} - xy = 1$

$$\frac{dy}{dx} - \frac{xy}{1-x^2} = \frac{1}{1-x^2} \quad \text{——— ①}$$

This is a linear differential eqn.

This is of the form

$$\frac{dy}{dx} + py = Q$$

Here $P = -\frac{x}{1-x^2}$, $Q = \frac{1}{1-x^2}$

$$\int P dx = \int -\frac{x}{1-x^2} dx$$

put $1-x^2 = t$

$-2x\,dx = dt$

$x\,dx = -\frac{1}{2}dt$

$$\int P dx = \int \frac{+\frac{1}{2}dt}{t} = +\frac{1}{2}\int \frac{dt}{t}$$

$$= +\frac{1}{2}\log t$$

$$= +\frac{1}{2}\log|1-x^2|$$

$$= \log(1-x^2)^{\frac{1}{2}}$$

∴ soln of eqn ① is

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

$$y e^{\log(1-x^2)^{\frac{1}{2}}} = \int \frac{1}{1-x^2} e^{\log(1-x^2)^{\frac{1}{2}}} dx + c$$

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$$y\sqrt{1-x^2} = \sin^{-1}x + c$$

$$y = \frac{\sin^{-1}x}{\sqrt{1-x^2}} + \frac{c}{\sqrt{1-x^2}}$$

$$y = \frac{\sin^{-1}x}{\sqrt{1-x^2}} + c(1-x^2)^{-\frac{1}{2}}$$

---

3) $\frac{dy}{dx} + \frac{y}{x} = \sin x$

$$\frac{dy}{dx} + \frac{1}{x}y = \sin x \quad \text{——— (1)}$$

This is linear differential eqn

This is of the form $\frac{dy}{dx} + py = Q$

Here $p = \frac{1}{x}$, $Q = \sin x$

$$\int p\,dx = \int \frac{1}{x}dx = \log x$$

∴ The soln of eqn (1) is

$$y e^{\int p\,dx} = \int Q e^{\int p\,dx} dx + c$$

$$y e^{\log x} = \int \sin x\, e^{\log x} dx + c$$

$$yx = \int \sin x \cdot x\, dx + c$$

$u = x$ | $dv = \sin x\,dx$

$du = dx$ | $\int dv = \int \sin x\,dx$

$v = -\cos x$

WKT $\int u\,dv = uv - \int v\,du$

$$\therefore yx = x(-\cos x) - \int -\cos x\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

$$yx + x\cos x = \sin x + c$$

$$x(y + \cos x) = \sin x + c$$

This is required solution.

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4) $(x^2+1)\frac{dy}{dx} + 2xy = \sqrt{x^2+4}$

$$\frac{dy}{dx} + \frac{2xy}{x^2+1} = \frac{\sqrt{x^2+4}}{x^2+1} \quad \text{——— (1)}$$

This is linear differential eqn.

This is of the form $\frac{dy}{dx} + Py = Q$

Here $P = \frac{2x}{x^2+1}$, $Q = \frac{\sqrt{x^2+4}}{x^2+1}$

$$\int P\,dx = \int \frac{2x}{x^2+1}\,dx = \log(x^2+1)$$

∴ The soln of eqn (1) is

$$y e^{\int P dx} = \int Q e^{\int P dx}\,dx + c$$

$$y e^{\log(x^2+1)} = \int \frac{\sqrt{x^2+4}}{x^2+1} e^{\log(x^2+1)}\,dx + c$$

$$y(x^2+1) = \int \frac{\sqrt{x^2+4}}{x^2+1} \times (x^2+1)\,dx + c$$

$$= \int \sqrt{x^2+4}\,dx + c$$

$$= \int \sqrt{x^2+2^2}\,dx + c$$

$$y(x^2+1) = \frac{x\sqrt{x^2+2^2}}{2} + \frac{2^2}{2}\log\left|x+\sqrt{x^2+2^2}\right| + c$$

$$y(x^2+1) = \frac{x\sqrt{x^2+2^2}}{2} + 2\log\left|x+\sqrt{x^2+2^2}\right| + c$$

---

5) $(2x-10y^3)\,dy + y\,dx = 0$

$$(2x-10y^3)\,dy = -y\,dx$$

$$\frac{2x-10y^3}{-y} = \frac{dx}{dy}$$

$$\frac{dx}{dy} = \frac{2x}{-y} + \frac{10y^3}{y}$$

$$\frac{dx}{dy} = -\frac{2}{y}x + 10y^2$$

$$\frac{dx}{dy} + \frac{2}{y}x = 10y^2 \quad \text{——— (1)}$$

This is a linear differential eqn

This is of the form $\frac{dx}{dy} + Px = Q$

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Here $p = \frac{2}{y}$, $Q = \log y^2$

$$\int p\,dy = \int \frac{2}{y}\,dy = 2\int \frac{1}{y}\,dy$$

$$= 2\log y = \log y^2$$

The soln of eqn ① is

$$x e^{\int p\,dy} = \int Q e^{\int p\,dy}\,.\,dy + c$$

$$x e^{\log y^2} = \int \log y^2 e^{\log y^2}\,dy + c$$

$$xy^2 = 10\int y^2 \cdot y^2\,dy + c$$

$$xy^2 = 10\int y^4\,dy + c$$

$$= 10\,\frac{y^5}{5} + c = 2y^5 + c$$

$\therefore$ $\boxed{xy^2 = 2y^5 + c}$

which is the required eqn

---

6) $x \sin x \frac{dy}{dx} + (x\cos x + \sin x)y = \sin x$

$$\frac{dy}{dx} + \frac{(x\cos x + \sin x)y}{x\sin x} = \frac{\sin x}{x\sin x}$$

$$\frac{dy}{dx} + \left(\frac{x\cos x + \sin x}{x\sin x}\right)y = \frac{1}{x} \quad \text{—— ①}$$

This is a linear differential eqn

This is of the form $\frac{dy}{dx} + py = Q$

Here $p = \frac{x\cos x + \sin x}{x\sin x}$, $Q = \frac{1}{x}$

$$\int p\,dx = \int \left(\frac{x\cos x + \sin x}{x\sin x}\right)dx$$

Let $x\sin x = t$

$(x\cos x + \sin x)\,dx = dt$

$$\therefore \int p\,dx = \int \frac{x\cos x + \sin x}{x\sin x}\,dx$$

$$= \int \frac{dt}{t} = \log t$$

$$= \log(x\sin x)$$

The soln of eqn ① is

$$y e^{\int p\,dx} = \int Q e^{\int p\,dx}\,dx + c$$

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$$y e^{\log(x\sin x)} = \int \frac{1}{x} e^{\log(x\sin x)}\, dx + c$$

$$y(x\sin x) = \int \frac{1}{x} x\sin x\, dx + c$$

$$= \int \sin x\, dx + c$$

$$xy\sin x = -\cos x + c$$

$$xy\sin x + \cos x = c$$

---

7) $\left(y - e^{\sin^{-1}x}\right)\frac{dx}{dy} + \sqrt{1-x^2} = 0$

$$\left(y - e^{\sin^{-1}x}\right)\frac{dx}{dy} = -\sqrt{1-x^2}$$

$$\frac{y - e^{\sin^{-1}x}}{-\sqrt{1-x^2}} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{y}{\sqrt{1-x^2}} + \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} + \frac{1}{\sqrt{1-x^2}}y = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}} \quad \text{— (1)}$$

This is a linear differential eqn

This is of the form

$$\frac{dy}{dx} + Py = Q$$

where $P = \frac{1}{\sqrt{1-x^2}}$, $Q = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

$$\int P\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x$$

Soln of eqn (1) is

$$y e^{\int P\,dx} = \int Q e^{\int P\,dx}\, dx + c$$

$$y e^{\sin^{-1}x} = \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}} e^{\sin^{-1}x}\, dx + c$$

$$y e^{\sin^{-1}x} = \int \left(e^{\sin^{-1}x}\right)^2 \times \frac{1}{\sqrt{1-x^2}}\, dx + c$$

put $\sin^{-1}x = t$

then $\frac{1}{\sqrt{1-x^2}}\,dx = dt$.

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$$ye^{\sin^{-1}x} = \int (e^t)^2 dt + c$$

$$= \int e^{2t} dt + c$$

$$= \frac{e^{2t}}{2} + c$$

$$ye^{\sin^{-1}x} = \frac{e^{2\sin^{-1}x}}{2} + c$$

---

8) $\frac{dy}{dx} + \frac{y}{(1-x)\sqrt{x}} = 1-\sqrt{x}$

$$\frac{dy}{dx} + \frac{1}{(1-x)\sqrt{x}} y = 1-\sqrt{x} \quad \text{——— (1)}$$

This is a linear differential eqn

This is of the form $\frac{dy}{dx} + py = Q$

Here $p = \frac{1}{(1-x)\sqrt{x}}$, $Q = 1-\sqrt{x}$

$$\int p\,dx = \int \frac{1}{(1-x)\sqrt{x}} dx$$

put $x = t^2$

then $dx = 2t\,dt$

$$\therefore \int p\,dx = \int \frac{2t\,dt}{(1-t^2)\sqrt{t^2}}$$

$$= 2\int \frac{t\,dt}{(1-t^2)t}$$

$$= 2\int \frac{dt}{1-t^2}$$

$$= 2 \times \frac{1}{2} \log\left|\frac{1+t}{1-t}\right|$$

$$\therefore \int p\,dx = \log\left|\frac{1+\sqrt{x}}{1-\sqrt{x}}\right|$$

Soln of eqn (1) is

$$ye^{\int p\,dx} = \int Q e^{\int p\,dx} dx + c$$

$$ye^{\log\left|\frac{1+\sqrt{x}}{1-\sqrt{x}}\right|} = \int (1-\sqrt{x}) e^{\log\left|\frac{1+\sqrt{x}}{1-\sqrt{x}}\right|} dx + c$$

$$y\left|\frac{1+\sqrt{x}}{1-\sqrt{x}}\right| = \int (1-\sqrt{x}) \frac{1+\sqrt{x}}{1-\sqrt{x}} dx + c$$

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$$y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = \int (1+\sqrt{x})\,dx + c$$

$$= \int dx + \int \sqrt{x}\,dx + c$$

$$= x + \frac{x^{3/2}}{3/2} + c$$

$$= x + \frac{2}{3}x^{3/2} + c$$

$$\therefore y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = x + \frac{2}{3}x^{3/2} + c.$$

---

9) $(1+x+xy^2)\frac{dy}{dx} + (y+y^3) = 0.$

$$(1+x+xy^2)\frac{dy}{dx} = -(y+y^3)$$

$$\frac{1+x+xy^2}{-(y+y^3)} = \frac{dx}{dy}$$

$$\frac{dx}{dy} = -\frac{[1+x(1+y^2)]}{(y+y^3)}$$

$$\frac{dx}{dy} = -\frac{1}{y+y^3} - \frac{x(1+y^2)}{y(1+y^2)}$$

$$\frac{dx}{dy} = -\frac{1}{y+y^3} - \frac{1}{y}x$$

$$\frac{dx}{dy} + \frac{1}{y}x = -\frac{1}{y(1+y^2)} \quad \text{——(1)}$$

This is a linear differential eqn.

This is of the form $\frac{dx}{dy} + px = Q$

Here $p = \frac{1}{y}$, $Q = -\frac{1}{y(1+y^2)}$

$$\int p\,dy = \int \frac{1}{y}dy = \log y$$

The soln of eqn (1) is

$$x e^{\int p\,dy} = \int Q e^{\int p\,dy} dy + c$$

$$x e^{\log y} = \int -\frac{1}{y(1+y^2)} e^{\log y} dy + c$$

$$xy = \int \frac{-y\,dy}{y(1+y^2)} + c$$

$$= -\int \frac{dy}{1+y^2} + c$$

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$$xy = -\tan^{-1}(y) + c$$

$$xy + \tan^{-1}(y) = c$$

---

10) $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{\sin 2x}{\log x}$

$$\frac{dy}{dx} + \frac{1}{x\log x} \cdot y = \frac{\sin 2x}{\log x} \quad \text{——} (1)$$

This is a linear differential eqn.

This is of the form $\frac{dy}{dx} + py = Q$

Here $p = \frac{1}{x\log x}$, $Q = \frac{\sin 2x}{\log x}$

$$\int p\,dx = \int \frac{1}{x\log x}\,dx$$

$$= \int \frac{1}{\log x} \cdot \frac{1}{x}\,dx$$

put $\log x = t$ then

$\frac{1}{x}\,dx = dt$

$$\therefore \int p\,dx = \int \frac{1}{t}\,dt$$

$$= \log t$$

Soln of eqn (1) is

$$y e^{\int p\,dx} = \int Q e^{\int p\,dx}\,dx + c$$

$$y e^{\log t} = \int \frac{\sin 2x}{\log x} e^{\log t}\,dx + c$$

$$yt = \int \frac{\sin 2x}{\log x} e^{\log(\log x)}\,dx + c$$

$$y\log x = \int \frac{\sin 2x}{\log x} \log x\,dx + c$$

$$y\log x = \int \sin 2x\,dx + c$$

$$y\log x = -\frac{\cos 2x}{2} + c$$

$$y\log x + \frac{\cos 2x}{2} = c$$

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11) $(x+a)\frac{dy}{dx} - 2y = (x+a)^4$

$$\frac{dy}{dx} - \frac{2y}{x+a} = \frac{(x+a)^4}{x+a}$$

$$\frac{dy}{dx} - \frac{2}{x+a}y = (x+a)^3 \quad \text{——— ①}$$

This is a linear differential eqn.

This is of the form $\frac{dy}{dx} + py = Q$

Here $p = -\frac{2}{x+a}$, $Q = (x+a)^3$

$$\int p\,dx = \int -\frac{2}{x+a}\,dx = -2\log(x+a) = \log(x+a)^{-2}$$

∴ Soln of eqn ① is

$$y e^{\int p\,dx} = \int Q e^{\int p\,dx}\,dx + c$$

$$y e^{\log(x+a)^{-2}} = \int (x+a)^2 e^{\log(x+a)^{-2}}\,dx + c$$

$$y(x+a)^{-2} = \int (x+a)^3 (x+a)^{-2}\,dx + c$$

$$\frac{y}{(x+a)^2} = \int (x+a)\,dx + c$$

$$= \frac{(x+a)^2}{2} + c$$

$$\frac{y}{(x+a)^2} = \frac{(x+a)^2}{2} + c = \frac{(x+a)^2 + 2c}{2}$$

$$\therefore 2y = (x+a)^4 + 2c(x+a)^2$$

——— x ——— x ———

12) $\frac{dy}{dx} = \frac{8m^2x}{1+x^3} - \frac{3x^2}{1+x^3}y$

$$\frac{dy}{dx} + \frac{3x^2}{1+x^3}y = \frac{8m^2x}{1+x^3} \quad \text{——— ①}$$

This is a linear differential eqn.

This is of the form $\frac{dy}{dx} + py = Q$

Here $p = \frac{3x^2}{1+x^3}$, $Q = \frac{8m^2x}{1+x^3}$

$$\int p\,dx = \int \frac{3x^2}{1+x^3}\,dx$$

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$\int p\,dx = \log(1+x^3)$

∴ The soln of eqn ① is

$$y e^{\int p\,dx} = \int Q e^{\int p\,dx} dx + c$$

$$y e^{\log(1+x^3)} = \int \frac{\sin^2 x}{1+x^3} e^{\log(1+x^3)} dx + c$$

$$y(1+x^3) = \int \frac{\sin^2 x}{1+x^3}(1+x^3)\,dx + c$$

$$= \int \sin^2 x\,dx + c$$

$$= \int \frac{1-\cos 2x}{2} dx + c$$

$\cos 2\alpha = 1 - 2\sin^2\alpha$

$2\sin^2\alpha = 1 - \cos 2\alpha$

$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$

$$= \int \frac{1}{2} dx - \int \frac{\cos 2x}{2} dx + c$$

$$= \frac{1}{2}x - \frac{1}{2}\int \cos 2x\,dx + c$$

$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + c$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + c$$

$$\therefore y(1+x^3) = \frac{x}{2} - \frac{\sin 2x}{4} + c$$

---

13) $x\frac{dy}{dx} + y = x\log x$

$$\frac{dy}{dx} + \frac{y}{x} = \frac{x\log x}{x}$$

$$\frac{dy}{dx} + \frac{1}{x}y = \log x \quad \text{——} ①$$

This is a linear differential eqn.

This is of the form $\frac{dy}{dx} + py = Q$ where $p = \frac{1}{x}$, $Q = \log x$

$$\int p\,dx = \int \frac{1}{x} dx = \log x$$

The soln of eqn ① is

$$y e^{\int p\,dx} = \int Q e^{\int p\,dx} dx + c$$

$$y e^{\log x} = \int \log x\, e^{\log x} dx + c$$

$$yx = \int x\log x\,dx + c$$

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Using Integration by parts

$u = \log x$ | $dv = x\,dx$

$du = \frac{1}{x}dx$ | $\int dv = \int x\,dx = \frac{x^2}{2}$

$v = \frac{x^2}{2}$

$\therefore xy = \frac{x^2}{2}\log x - \int \frac{x^2}{2} \times \frac{1}{x}\,dx + c$

$xy = \frac{x^2}{2}\log x - \frac{1}{2}\int x\,dx + c$

$xy = \frac{x^2}{2}\log x - \frac{1}{2}\frac{x^2}{2} + c$

$xy = \frac{x^2 \log x}{2} - \frac{x^2}{4} + c$

$xy = \frac{2x^2\log x - x^2 + 4c}{4}$

$4xy = 2x^2\log x - x^2 + 4c$.

---

14) $x\frac{dy}{dx} + 2y - x^2\log x = 0$

$x\frac{dy}{dx} + 2y = x^2\log x$

$\frac{dy}{dx} + \frac{2y}{x} = \frac{x^2\log x}{x}$

$\frac{dy}{dx} + \frac{2}{x}y = x\log x$ ——— ①

This is a linear differential eqn.

This is of the form $\frac{dy}{dx} + py = Q$

Here $p = \frac{2}{x}$, $Q = x\log x$

$\int p\,dx = 2\int \frac{1}{x}dx = 2\log x = \log x^2$

The soln of eqn ① is

$y e^{\int p\,dx} = \int Q e^{\int p\,dx} dx + c$

$y e^{\log x^2} = \int x\log x\, e^{\log x^2} dx + c$

$yx^2 = \int x\log x \cdot x^2\,dx + c$

$yx^2 = \int \log x \cdot x^3\,dx + c$.

Using Integration by parts

Let $u = \log x$ | $dv = x^3\,dx$

$du = \frac{1}{x}dx$ | $\int dv = \int x^3\,dx$

$v = x^4/4$

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$$x^2y = \frac{x^4}{4}\log x - \int \frac{x^4}{4} \times \frac{1}{x}\,dx + c$$

$$= \frac{x^4}{4}\log x - \frac{1}{4}\int x^3\,dx + c$$

$$= \frac{x^4}{4}\log x - \frac{1}{4} \times \frac{x^4}{4} + c$$

$$x^2y = \frac{x^4}{4}\log x - \frac{x^4}{16} + c$$

---

15) $\frac{dy}{dx} + \frac{3y}{x} = \frac{1}{x^2}$, given $y=2$, when $x=1$.

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{1}{x^2} \quad \text{——— (1)}$$

This is a linear differential eqn

This is of the form $\frac{dy}{dx} + py = Q$

Here $p = \frac{3}{x}$, $Q = \frac{1}{x^2}$

$$\int p\,dx = \int \frac{3}{x}\,dx = 3\int \frac{1}{x}\,dx$$

$$= 3\log x = \log x^3$$

The soln of eqn (1)

$$y\,e^{\int p\,dx} = \int Q\,e^{\int p\,dx}\,dx + c$$

$$y\,e^{\log x^3} = \int \frac{1}{x^2}\,e^{\log x^3}\,dx + c$$

$$x^3y = \int \frac{1}{x^2} \cdot x^3\,dx + c$$

$$x^3y = \int x\,dx = \frac{x^2}{2} + c$$

$$x^3y = \frac{x^2}{2} + c \quad \text{——— (2)}$$

Given $y=2$, when $x=1$

$$1(2) = \frac{1}{2} + c \Rightarrow 2 - \frac{1}{2} = c$$

$$\Rightarrow c = 3/2$$

$\therefore$ (2) $\Rightarrow$

$$x^3y = \frac{x^2}{2} + \frac{3}{2}$$

$$x^3y = \frac{x^2+3}{2}$$

$$2x^3y = x^2+3$$

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Applications of first order ordinary Differential Equations :-

10.27 The growth of population is proportional to the number present. If the population of a colony doubles in 50 years, in how many years will the population become triple?

Let A be the population at time 't'

Given that $\frac{dA}{dt}$ is proportional to A

$$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = kA$$

$$\therefore A = ce^{kt} \quad \text{——— (1)}$$

$\frac{dA}{dt} = kA$

$\frac{dA}{A} = k\,dt$

$\int \frac{dA}{A} = k\int dt$

$\log A = kt + c$

$A = e^{kt+c}$

$= e^{kt} \cdot e^{c}$

$= ce^{kt}$

Assume that A_0 be the initial population.

when $t = 0$, $A = A_0$

$\therefore$ (1) $\Rightarrow A_0 = ce^{k(0)}$

$= ce^{(0)}$

$= c$

$\therefore \boxed{A_0 = c}$

when $t = 50$, $A = 2A_0$

(1) $\Rightarrow 2A_0 = A_0 e^{k50}$

$e^{50k} = 2$ ——— (2).

Taking log,

$\log e^{50k} = \log 2$

$\log 2 = 50k$

$k = \frac{\log 2}{50}$

$\therefore$ (1) $\Rightarrow A = A_0 e^{\left(\frac{\log 2}{50}\right)t}$

$= A_0 e^{t/50 \cdot \log 2}$

$= A_0 e^{\log 2^{t/50}}$

$\therefore A = A_0 2^{t/50}$ ——— (3).

when $A = 3A_0$, $t = ?$

(3) $\Rightarrow 3A_0 = A_0 2^{t/50}$

$3 = 2^{t/50}$

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Taking log on both sides

$$\log 3 = \log 2^{t/50}$$

$$\log 3 = \frac{t}{50} \cdot \log 2$$

$$\frac{t}{50} = \frac{\log 3}{\log 2}$$

$$t = \frac{\log 3}{\log 2} \times 50$$

∴ The population is tripled in $50\left(\frac{\log 3}{\log 2}\right)$ years.

10.28 A radioactive isotope has an initial mass 200gms, which two years later is 50mg. Find the expression for the amount of isotope remaining at any time. What is its half-life (half-life means the time taken for the radioactivity of a specified isotope to fall to half its original value).

Let A be mass of radio active isotope at time 't'.

$$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = -kA$$

(Here k is negative since mass of radio active isotope is decreasing)

$$\therefore A = Ce^{-kt} \quad \text{——— (1)}$$

when $t = 0$, $A = 200$

$$\therefore (1) \Rightarrow 200 = Ce^{-k(0)}$$

$$\boxed{200 = C}$$

$$\therefore (1) \Rightarrow A = 200e^{-kt} \quad \text{——— (2)}$$

when $t = 2$, $A = 150$.

$$\therefore (2) \Rightarrow 150 = 200e^{-2k}$$

$$\frac{150}{200} = e^{-2k}$$

$$\frac{3}{4} = e^{-2k}$$

taking log,

$$\log\left(\frac{3}{4}\right) = \log e^{-2k}$$

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$\log\left(\frac{3}{4}\right) = -2k$

$\therefore k = -\frac{1}{2}\log\left(\frac{3}{4}\right)$

$-\left[-\frac{1}{2}\log\left(\frac{3}{4}\right)\right]t$

$\therefore$ (2) $\Rightarrow A = 200\, e^{\frac{t}{2}\log\left(\frac{3}{4}\right)}$

$A = 200\, e^{\frac{t}{2}\log\left(\frac{3}{4}\right)}$ ——(3)

when $t = t_h$ (t_h is half life)

$A = 100$

$\therefore$ (3) $\Rightarrow 100 = 200\, e^{\frac{t_h}{2}\log\left(\frac{3}{4}\right)}$

$\frac{100}{200} = e^{\frac{t_h}{2}\log\left(\frac{3}{4}\right)}$

$\log\left(\frac{1}{2}\right) = \log e^{\frac{t_h}{2}\log\left(\frac{3}{4}\right)}$

$= \frac{t_h}{2}\log\left(\frac{3}{4}\right)$

$2\log\left(\frac{1}{2}\right) = t_h \log\left(\frac{3}{4}\right)$

$\therefore t_h = \frac{2\log\left(\frac{1}{2}\right)}{\log\left(\frac{3}{4}\right)}$ //

10.29 In a murder investigation, a corpse was found by a detective at exactly 8 pm. Being alert, the detective also measured the body temperature and found it to be 70°F. Two hours later the detective measured the body temperature again and found it to be 60°F. If the room temperature is 50°F, and assuming that the body temperature of the person before death was 98.6°F, at what time did the murder occur.
($\log 2.43 = 0.88789$, $\log(0.5) = -0.69315$)

Let T be the temperature of the body at any time t, and S be the Room temperature.

By Newton's Law of cooling

$\frac{dT}{dt} \propto T - S$

Here $S = 50^\circ$

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$\therefore \frac{dT}{dt} = k(T-S)$

$T-S = ce^{kt}$

$T-50 = ce^{kt}$

$T = 50 + ce^{kt}$ ——— (1)

consider the time 8.p.m as $t=0$.

when $t=0$, $T = 70^\circ$

(1) $\Rightarrow$ $70 = 50 + ce^{k(0)}$

$70-50 = ce^{0}$

$\boxed{20 = c}$

$\therefore$ (1) $\Rightarrow$ $T = 50 + 20e^{kt}$ ——— (2).

when $t=2$, $T = 60^\circ$

(2) $\Rightarrow$ $60 = 50 + 20e^{2k}$

$\frac{60-50}{20} = e^{2k}$

$\frac{10}{20} = e^{2k} \Rightarrow \frac{1}{2} = e^{2k}$

Taking log,

$\log\left(\frac{1}{2}\right) = \log e^{2k}$

$\Rightarrow \log\left(\frac{1}{2}\right) = 2k$

$\Rightarrow k = \frac{1}{2}\log\left(\frac{1}{2}\right)$ $\quad \frac{1}{2}\log\left(\frac{1}{2}\right)t$

$\therefore$ (2) $\Rightarrow$ $T = 50 + 20e^{\frac{t}{2}\log\left(\frac{1}{2}\right)}$

$T = 50 + 20e^{\log\left(\frac{1}{2}\right)^{t/2}}$

$T = 50 + 20\left(\frac{1}{2}\right)^{t/2}$ ——— (3)

To find t, when $T = 98.6^\circ F$

(3) $\Rightarrow$

$98.6 = 50 + 20\left(\frac{1}{2}\right)^{t/2}$

$\frac{98.6-50}{20} = \left(\frac{1}{2}\right)^{t/2}$

$\frac{48.6}{20} = \left(\frac{1}{2}\right)^{t/2}$

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$$2.43 = \left(\frac{1}{2}\right)^{t/2}$$

Taking log,

$$\log 2.43 = \log\left(\frac{1}{2}\right)^{t/2} = \frac{t}{2}\log\left(\frac{1}{2}\right)$$

$$= \frac{t}{2}\log(0.5)$$

$$\frac{\log 2.43}{\log 0.5} = t/2$$

$$\Rightarrow t = 2\left(\frac{\log 2.43}{\log 0.5}\right)$$

$$= 2\left[\frac{0.88789}{-0.69315}\right]$$

$$= -2\left[\frac{88789}{69315}\right] = -2(1.28)$$

$$= -2.56$$

∴ The murder happened 2.56 hrs before 8 p.m

∴ The person was murdered at

$= 8 - 2.56$

$= 5.04$ hrs

$= 5 + 0.4 \times 60$

$= 5.24$ pm //

10.30 A tank contains 1000 litres of water in which 100 grams of salt is dissolved Brine. (Brine is high concentration solution of salt in water) runs in a rate of 10 litres per minute, and each litre contains 5 grams of dissolved salt. The mixture of the tank is kept uniform by stirring. Brine runs out at 10 litres per minute. Find the amount of salt at any time t.

Let x denote the amount of salt in the tank at time t.

Rate of change $\frac{dx}{dt}$ = Inflow rate − outflow rate.

Given 1 litre Brine contains 5 gms of salt.

Inflow rate = 10 litres per minute.

∴ Inflow rate of salt = 10×5 = 50 gms of salt.

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out flow rate = $\frac{10}{1000}$ times of x

$= \frac{10x}{1000} = \frac{x}{100}$

$= 0.01x$

∴ Rate of change $\frac{dx}{dt}$ = Inflow rate of salt − outflow rate of salt.

(i) $\frac{dx}{dt} = 50 - 0.01x$

$= \left(50 - \frac{x}{100}\right)$

$= \frac{1}{100}(5000 - x)$

$= -\frac{1}{100}(x - 5000)$

$= -0.01(x - 5000)$

$\frac{dx}{x-5000} = 0.01 \Rightarrow x - 5000 = ce^{-0.01t}$

$x = 5000 + ce^{-0.01t}$ ——— (1)

When $t = 0$, $x = 100$.

∴ (1) ⇒ $100 = 5000 + ce^{-0.01(0)}$

$100 - 5000 = ce^{0}$

$-4900 = c$

∴ $\boxed{c = -4900}$

∴ (1) ⇒ $x = 5000 - 4900e^{-0.01t}$

———— x ————

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EXERCISE 10.8

1) The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given that the number triples in 5 hrs, find how many bacteria will be present after 10 hours?

Let A be the number of bacteria at any time 't'.

Given $\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = kA$

$\Rightarrow A = ce^{kt}$ ——— (1)

When $t = 0$, $A = A_0$

(1) $\Rightarrow A_0 = ce^{k(0)}$

$= ce^0$

$= c(1) = c$

$\therefore \boxed{c = A_0}$

Given $t = 5$, $A = 3A_0$

$\therefore$ (1) $\Rightarrow 3A_0 = A_0 e^{5k}$

$3 = e^{5k}$ ——— (2)

When $t = 10$, $A = ?$

(1) $\Rightarrow A = A_0 e^{10k}$

$= A_0 (e^{5k})^2$

$= A_0 (3)^2$

$= 9A_0$.

Hence, the amount of bacteria will be after 10 hrs is 9 times the initial amount.

---

2) Find the population of a city at any time t, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years the population increased from 3,00,000 to 4,00,000.

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Let A denote the population of city at any time t.

Given $\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = kA$

$\therefore A = ce^{kt}$ ——— (1).

Given when $t = 0$, $A = 3,00,000$

$\therefore$ (1) $\Rightarrow 3,00,000 = ce^{k(0)} = c$

$\therefore \boxed{c = 3,00,000}$

$\therefore$ (1) $\Rightarrow A = 3,00,000\, e^{kt}$ ——— (2).

when $t = 40$, $A = 4,00,000$

$\therefore$ (2) $\Rightarrow 4,00,000 = 3,00,000\, e^{40k}$.

$e^{40k} = \frac{4,00,000}{3,00,000} = \frac{4}{3}$

Taking log,

$\log e^{40k} = \log\left(\frac{4}{3}\right)$

$40k = \log\left(\frac{4}{3}\right)$

$k = \frac{1}{40}\log\left(\frac{4}{3}\right)$

$\therefore$ (1) $\Rightarrow A = 3,00,000\, e^{\frac{1}{40}\log\left(\frac{4}{3}\right)t}$

$= 3\,00000\, e^{\frac{t}{40}\log\left(\frac{4}{3}\right)}$

$= 3,00,000\, e^{\log\left(\frac{4}{3}\right)^{\frac{t}{40}}}$

$= 3,00,000\, e^{t/40}$

$= 3,00,000\left(\frac{4}{3}\right)^{t/40}$

× ——— × ———

4) The engine of a motor boat moving at 10 m/s is shut off. Given that the retardation at any subsequent time (after shutting off the engine) equal to the velocity at that time. Find the velocity after 2 seconds of switching off the engine.

Let v be the velocity and the retardation (negative acceleration) be $-\frac{dv}{dt}$.

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Given $\frac{dV}{dt} = -V$

$\Rightarrow V = ce^{-t}$ ——— (1)

Given, when $t = 0$, $V = 10 m/s$

$\therefore$ (1) $\Rightarrow 10 = ce^{-0}$

$\boxed{10 = c}$

$\therefore$ (1) $\Rightarrow V = 10e^{-t}$ ——— (2)

Given when $t = 2$ sec, $V = ?$

$\therefore$ (2) $\Rightarrow V = 10e^{-2}$

$V = \frac{10}{e^2}$

---

5) Suppose a person deposits 10,000 Indian rupees in a bank account at the rate of 5% p.a compounded continuously. How much money will be in his account 18 months later?

Let p be the principal at time t.

Given $\frac{dp}{dt} = (0.05)p$

$\Rightarrow p = ce^{0.05t}$ ——— (1)

Given when $t = 0$, $p = 10,000$.

$\therefore$ (1) $\Rightarrow 10000 = ce^{(0.05)0}$

$\boxed{10000 = c}$

$\therefore$ (1) $\Rightarrow p = 10000\, e^{0.05t}$ ——— (2).

when $t = 18$ month, $p = ?$

(2) $\Rightarrow p = 10000\, e^{(0.05)\frac{3}{2}}$

$p = 10000\, e^{\frac{0.15}{2}}$

$p = 10000\, e^{0.075}$

$\therefore$ The amount of money in the bank after 18 months is

$p = 10000\, e^{0.075}$ //

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6) Assume that the rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. In a certain sample 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years. what percentage of the original radioactive nuclei will remain after 1000 years?

Let A be the amount of nuclei at time t.

Given $\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = kA$

$\Rightarrow A = ce^{kt}$ —— (1)

when $t = 0$, $A = A_0$

$\therefore$ (1) $\Rightarrow A_0 = ce^{k(0)} = ce^0$

$\therefore \boxed{A_0 = c}$ $(\because e^0 = 1)$

(1) $\Rightarrow A = A_0 e^{kt}$ —— (2).

when $t = 100$, $A = \frac{9}{10} A_0$

$\therefore$ (2) $\Rightarrow \frac{9}{10} A_0 = A_0 e^{100k}$

$\boxed{e^{100k} = \frac{9}{10}}$

when $t = 1000$, to find A.

(2) $\Rightarrow A = A_0 e^{k(1000)}$

$= A_0 e^{(100k)^{10}}$

$= A_0 \left(\frac{9}{10}\right)^{10}$

$= A_0 \frac{9^{10}}{10^{10}} \times 100\%$..

$= \frac{9^{10}}{10^8} \times A_0 \%$

$\therefore \frac{9^{10}}{10^8}\%$ of the radioactive element will remain after 1000 yrs.

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7) Water at temperature 100°c cools in 10 min to 80°c in a room temperature of 25°. find

(i) The temperature of water after 20 min.

(ii) The time when the temperature is 40°c

Let T be the temperature of water at any time t, and S be the Room Temperature.

By Newton's Law of cooling

$\frac{dT}{dt} \propto T-S$, Here $S = 25°$.

$\frac{dT}{dt} = k(T-S)$

$\Rightarrow T-S = ce^{kt}$

$T-25° = ce^{kt}$

$\therefore T = 25 + ce^{kt}$ ——— (1)

Given when $t=0$, $T=100$

(1) $\Rightarrow 100 = 25 + ce^{k(0)}$

$75 = ce^{0} \Rightarrow \boxed{c = 75}$

$\therefore$ (1) $\Rightarrow T = 25 + 75e^{kt}$ ——— (2)

when $t=10$, $T=80°$

$\therefore$ (2) $\Rightarrow 80 = 25 + 75e^{10k}$

$55 = 75e^{10k}$

$\therefore e^{10k} = \frac{55}{75} = \frac{11}{15}$

$\boxed{e^{10k} = \frac{11}{15}}$ ——— (3)

(i) When $t=20$ to find T

(2) $\Rightarrow T = 25 + 75e^{20k}$

$= 25 + 75(e^{10k})^2$

$= 25 + 75\left(\frac{11}{15}\right)^2$

$= 25 + \frac{\cancel{15}^{5}}{\cancel{75}} \times \frac{11}{\cancel{15}} \times \frac{11}{15_3}$

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$$T = 25 + \frac{121}{3}$$

$$= \frac{75+121}{3} = \frac{196}{3}$$

$$= \boxed{65.33^\circ C}$$

(iv). When $T = 40^\circ$ to find 't'.

③ $\Rightarrow e^{10k} = \frac{11}{15}$

Taking log,

$$\log e^{10k} = \log\left(\frac{11}{15}\right)$$

$$10k = \log\left(\frac{11}{15}\right)$$

$$10k = -0.3101$$

$$k = \frac{-0.3101}{10} = -0.03101$$

$$\therefore \boxed{k = -0.03101}$$

② $\Rightarrow T = 25 + 75e^{-0.03101t}$

Now $40 = 25 + 75e^{-0.03101t}$

$$15 = 75e^{-0.03101t}$$

$$\frac{15}{75} = e^{-0.03101t}$$

$$\frac{1}{5} = e^{-0.03101t} = \frac{1}{e^{0.03101t}}$$

$$\Rightarrow e^{0.03101t} = 5$$

Taking log,

$$\log e^{0.03101t} = \log 5$$

$$0.03101t = 1.6094 \text{ (given)}$$

$$t = \frac{1.6094}{0.03101}$$

$$= \frac{160940}{3101}$$

$$= 51.89 \text{ min}.$$

%
51.89
3101 | 160940
15505
5890
3101
27890
24808
30820

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8) At 10.00 AM, a woman took a cup of hot instant coffee from her microwave oven and placed it on a nearby kitchen counter to cool. At this instant the temp of the coffee was 180°F, and 10 min later it was 160°F. Assume that constant temperature of the kitchen was 70°F

a) what was the temperature of the coffee at 10.15 A.M?

b) The woman likes to drink coffee when its temperature is betn 130°F and 140°F. between what times should she have drunk the coffee?

Let T be the Temperature of the coffee at time t and S be the room temperature.

By Newton's Law of cooling

$$\frac{dT}{dt} \propto T-S$$

$$\frac{dT}{dt} = k(T-S)$$

$$\Rightarrow T-S = ce^{kt}, \text{ Here } S = 70.$$

$$T-70 = ce^{kt} \quad \text{——— (1)}$$

when $t=0$ (10.AM), $T = 180^\circ$

(1) $\Rightarrow T-70 = ce^0$

$180-70 = c \Rightarrow \boxed{c = 110}$

$\therefore$ (1) $\Rightarrow T-70 = 110e^{kt}$ ——— (2).

when $t = 10$, $T = 160^\circ$

(2) $\Rightarrow 160-70 = 110e^{10k}$

$90 = 110e^{10k}$

$\frac{90}{110} = e^{10k}$

$\boxed{\frac{9}{11} = e^{10k}}$ ——— (3).

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(i) when $t = 15$, to find T

(2) $T - 70 = 110\, e^{15k}$

$T = 70 + 110\, e^{\frac{30}{2}k}$.

$= 70 + 110 \left(e^{10k}\right)^{\frac{3}{2}}$

$= 70 + 110 \left(\frac{9}{11}\right)^{3/2}$

$= 70 + 110 \times \frac{9}{11} \times \sqrt{\frac{9}{11}}$

$= 70 + 110 \times \frac{9}{11} \times 0.9$

$= 70 + 10 \times 9 \times 0.9$

$= 70 + 9 \times 9 = 70 + 81$

$= \boxed{151^\circ F.}$

(ii) (3) $\Rightarrow e^{10k} = \frac{9}{11}$

Taking log, $\log e^{10k} = \log\left(\frac{9}{11}\right)$

$10k = \log\left(\frac{9}{11}\right)$

$10k = \log 9 - \log 11$

$= 0.$

$\boxed{k = -0.0087}$

when $T = 130^\circ$ to find t.

(2) $\Rightarrow T - 70 = 110\, e^{kt}$

$130 - 70 = 110\, e^{-0.0087t}$

$e^{-0.0087t} = \frac{60}{110} = \frac{6}{11}$

Taking log,

$\log e^{-0.0087t} = \log\left(\frac{6}{11}\right)$

$-0.0087t = -0.2632$

$t = \frac{0.2632}{0.0087}$

$= \frac{2632}{87} = 30.2$

$\therefore t \simeq 30$ min.

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when $T = 140^\circ$ to find t.

② $\Rightarrow T - 70 = 110 e^{kt}$

$140 - 70 = 110 e^{-0.0087t}$

$\frac{70}{110} = e^{-0.0087t}$

$e^{-0.0087t} = \frac{7}{11}$

Taking log,

$\log e^{-0.0087t} = \log\left(\frac{7}{11}\right)$

$-0.0087t = \log\left(\frac{7}{11}\right)$

$-0.0087t = -0.1963$

$t = \frac{0.1963}{0.0087}$

$= \frac{1963}{87} = 22.4 \Rightarrow t \simeq 22$ min.

∴ She drunk the coffee between 10.22 and 10.30 respectively.

9. A bot of boiling water at 100°C is removed from a stove at time $t=0$ and left to cool in the kitchen. After 5 min, the water temperature has decreased to 80°C, and another 5 min later it has dropped to 65°C. Determine the temperature of the kitchen.

Let T represent the temperature of the boiling water and S represents the temperature of kitchen.

By Newton's Law of cooling,

$\frac{dT}{dt} \propto (T-S) \Rightarrow \frac{dT}{dt} = K(T-S)$

$\therefore T - S = ce^{kt}$ ——— ①

Given when $t=0$, $T = 100^\circ$

$\therefore$ ① $\Rightarrow 100 - S = ce^{k(0)}$

$\boxed{100 - S = c}$

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$\therefore$ ① $\Rightarrow$ $T - S = (100 - S)e^{kt}$ ——— ②

when $t = 5$, $T = 80^\circ C$

② $\Rightarrow$ $80 - S = (100 - S)e^{5k}$

$\therefore$ $e^{5k} = \frac{80 - S}{100 - S}$

When $t = 10$, $T = 65^\circ$

② $\Rightarrow$ $T - S = (100 - S)e^{10k}$

$65 - S = (100 - S)(e^{5k})^2$

$= (100 - S)\left(\frac{80 - S}{100 - S}\right)^2$

$= \frac{(100 - S)(80 - S)^2}{(100 - S)^2}$

$= \frac{(80 - S)^2}{100 - S}$

$(100 - S)(65 - S) = (80 - S)^2$

$6500 - 65S - 100S + S^2 = 6400 + S^2 - 160S$

$6500 - 165S = 6400 - 160S$

$6500 - 6400 = 165S - 160S$

$100 = 5S$

$\therefore$ $S = \frac{100}{5} = 20^\circ$

$\therefore$ The kitchen temperature is $20^\circ C$

——— x ——— x ———

10) A tank initially contains 50 litres of pure water. starting at time $t = 0$ a brine containing with 2 gms of dissolved salt per litre flows into the tank at the rate of 3 litres per minutes. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate. Find the amount of salt present in the tank at any time $t > 0$.

Let x denote the amount of salt in the tank at time 't'.

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Rate of change = Inflow rate − outflow rate.

1 litre Brine contains 2 gms of salt.

Inflow rate = 3 litre per minute

Inflow rate of salt = 3 × 2 gms

= 6 gms of salt.

outflow rate of salt = $\frac{3}{50}$ times x

= $\frac{3x}{50}$.

∴ Rate of change = $\frac{dx}{dt} = 6 - \frac{3x}{50}$

$$\frac{dx}{dt} = \frac{300 - 3x}{50} = \frac{3(100 - x)}{50}$$

$$\frac{dx}{100 - x} = \frac{3}{50} dt$$

$$\frac{dx}{x - 100} = -\frac{3}{50} dt$$

$$\log(x - 100) = -\frac{3}{50}t + c$$

$$x - 100 = ce^{-\frac{3}{50}t} \quad \text{——— (1)}$$

When $t = 0$, $x = 0$

$$0 - 100 = ce^{-\frac{3}{50}(0)}$$

$$-100 = ce^{0}$$

$$\therefore \boxed{c = -100}$$

∴ (1) ⇒ $x - 100 = -100e^{-\frac{3}{50}t}$

$$x = 100 - 100e^{-\frac{3}{50}t} \quad \text{——— (2)}$$

$$x = 100\left(1 - e^{-\frac{3}{50}t}\right)$$

Hence, the amount of salt in the tank at time t. is

$$x = 100\left(1 - e^{-\frac{3}{50}t}\right)$$

———— x ———— x ————

3) The eqn of electromotive force for an electric circuit containing resistance and self inductance is $E = Ri + L\frac{di}{dt}$, where E is the electromotive force is given to the circuit, R the resistance

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and L, the coefficient of induction. Find the current 'i' at time t when E = 0.

Given $E = Ri + L\frac{di}{dt}$

$$L\frac{di}{dt} + Ri = E \Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

This is linear eqn in i

Here $P = \frac{R}{L}$ and $Q = \frac{E}{L}$

where $I.F = e^{\int P dt}$

$$= e^{\int \frac{R}{L} dt} \quad = e^{\frac{R}{L}\int dt}$$

$$= e^{\frac{R}{L}(t)}$$

$$= e^{Rt/L}$$

$$\boxed{IF = e^{Rt/L}}$$

∴ The solution is

$$i\,IF = \int Q(IF)\,dt$$

$$i e^{Rt/L} = \int \frac{E}{L} e^{\frac{Rt}{L}} dt$$

$$= \frac{E}{L}\int e^{\frac{Rt}{L}} dt + c$$

$$\therefore i e^{\frac{Rt}{L}} = \frac{E}{L} \frac{e^{Rt/L}}{\frac{R}{L}} + c$$

$$= \frac{E}{L} \times \frac{L}{R} e^{\frac{Rt}{L}} + c$$

$$= \frac{E}{R} e^{\frac{Rt}{L}} + c \quad \text{— (1)}$$

when E = 0

(1) $\Rightarrow$ $i e^{\frac{Rt}{L}} = 0 + c$

$$\therefore i e^{\frac{Rt}{L}} = c$$

$$i = \frac{c}{e^{\frac{Rt}{L}}}$$

$$= c e^{-\frac{Rt}{L}}$$

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Chapter 11 – Probability Distributions

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CHAPTER - 11

PROBABILITY DISTRIBUTIONS

Random Variable:- A random variable x is a function defined on a sample space S into the real numbers R such that the inverse image of points or subset or interval of R is an event in S, for which the probability is assigned.

EX 11.1 Suppose two coins are tossed once. If x denotes the number of tails, (i) write down the sample space (ii) find the inverse image of 1 (iii) The value of random variable and number of elements in its inverse images.

Two coins are tossed once.

(i) ∴ Sample space $S = \{H,T\} \times \{H,T\}$

$$S = \{(H,H), (H,T), (T,H), (T,T)\}$$

$$n(S) = 4$$

(ii) Let x denote the no of tails.

Clearly x can takes values 0, 1, 2.

$x(HH) = 0$ (no tails)

$x(TH) = 1$ (1 tail)

$x(HT) = 1$ (1 tail)

$x(TT) = 2$ (2 tails)

Let $x(w)$ denote the no of tails

$$\therefore x(w) = \begin{cases} 2 & \text{if } w = TT \\ 1 & \text{if } w = HT, TH \\ 0 & \text{if } w = HH \end{cases}$$

The inverse image of 1 is $\{TH, HT\}$

i.e. $x^{-1}(1) = \{TH, HT\}$

(iii) Number of elements in inverse images are

Value of the Random variable	0	1	2	Total
No of elements in inverse image	1	2	1	4

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Ex 11.2 Suppose a pair of unbiased dice is rolled once. If X denotes the total score of two dice, write down (i) the sample space. (ii) the values taken by the random variable X. (iii) the inverse image of 10, and (iv) the number of elements in inverse image of X.

Given A pair of unbiased dice is rolled.

(i) Sample space S

$$= \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$$
$$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$$
$$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$$
$$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$$
$$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$$
$$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$$

$n(S) = 36$.

(ii) Let the random variable X denote the sum of the values on each die.

Clearly X takes values 2 to 12.

$X(1,1) = 2$

$X(1,2) = X(2,1) = 3$

$X(1,3) = X(3,1) = X(2,2) = 4$

$X(1,4) = X(4,1) = X(2,3) = X(3,2) = 5$

$X(1,5) = X(5,1) = X(2,4) = X(4,2) = X(3,3) = 6$

$X(1,6) = X(6,1) = X(2,5) = X(5,2) = X(3,4) = X(4,3) = 7$

$X(2,6) = X(6,2) = X(3,5) = X(5,3) = X(4,4) = 8$

$X(3,6) = X(6,3) = X(4,5) = X(5,4) = 9$

$X(4,6) = X(6,4) = X(5,5) = 10$

$X(5,6) = X(6,5) = 11$

$X(6,6) = 12$

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(iii) The inverse image of 10 is

$\{(4,6), (6,4), (5,5)\}$

ie $\bar{x}^{-1}(10) = \{(4,6), (6,4), (5,5)\}$

(iv)

Values of Random variable	2	3	4	5	6	7	8	9	10	11	12
No of elements in inverse image	1	2	3	4	5	6	5	4	3	2	1

Ex: 11.3 An urns contains 2 white balls and 3 red balls. A sample of 3 balls are chosen at random from the urn. If x denotes the number of red balls chosen, find the values taken by the random variable x and its number of inverse images.

No of balls in urn $= 2W + 3R = 5$

3 balls chosen at random from the urn.

$\therefore$ Sample space $= {}^5C_3 = {}^5C_2$

$= \frac{5 \times 4}{1 \times 2} = 10.$

Let us denote white and red balls as w_1, w_2, r_1, r_2, r_3

$\therefore$ Sample space S

$= \{w_1 r_1 r_2, \; w_1 r_2 r_3, \; w_1 r_1 r_3$
$w_2 r_1 r_2, \; w_2 r_2 r_3, \; w_2 r_1 r_3$
$w_1 w_2 r_1, \; w_1 w_2 r_2, \; w_1 w_2 r_3$
$r_1 r_2 r_3\}$

Let x denote the number of red balls chosen

clearly x takes values 1, 2, 3

x (getting 1 Red) $= \{w_1 w_2 r_1, \; w_1 w_2 r_2, \; w_1 w_2 r_3\} = 3$

x (getting 2 Red) $= \{w_1 r_1 r_2, \; w_1 r_2 r_3, \; w_1 r_1 r_3, \; w_2 r_1 r_2, \; w_2 r_2 r_3, \; w_2 r_1 r_3\} = 6$

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X (getting 3 Red) $= \{R_1 R_2 R_3\} = 1$

Value of the Random Variable x	1	2	3	Total
Number of elements in inverse image	3	6	1	10

Ex 11.4 Two balls are chosen randomly from an urn containing 6 white and 4 black balls. Suppose that we win ₹30 for each black ball selected and lose ₹20 for each white ball selected. If x denote the winning amount, then find the values of x and number of points in its inverse images.

white black Total

6 4 10

$\therefore n(S) = {}^{10}C_2 = \frac{10 \times 9}{1 \times 2} = \boxed{45}$

Let the random variable x denote the winning amount.

x (both white hall) $= -20 - 20 = -40$

x (one white and one black) $= -20 + 30 = 10$

x (both black) $= 30 + 30 = 60$

$$x(w) = \begin{cases} -40 & \text{if } w = \text{both white} \\ 10 & \text{if } w = 1 \text{ white \& } 1 \text{ black} \\ 60 & \text{if } w = 2 \text{ black.} \end{cases}$$

x (both are white) $= {}^6C_2 \times {}^4C_0$

$= \frac{6 \times 5}{1 \times 2} \times 1 = 15$

x (one white, one black) $= {}^6C_1 \times {}^4C_1$

$= 6 \times 4 = 24$

x (both are black) $= {}^4C_2 = \frac{4 \times 3}{1 \times 2}$

$= 6$

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Values of the random variable x	−40	10	60	Total
Number of elements in inverse images	15	24	6	45

EXERCISE 11.1

1) Suppose x is the number of tails occured when three fair coins are tossed once simultaneously. Find the values of the random variable x and number of points in its inverse images.

Given three coins are tossed once.

Sample space

$S = \{HHH, HHT, THH, HTH, HTT, THT, TTH, TTT\}$

$n(S) = 8$

Let x denote the number tails occured

clearly x takes values 0, 1, 2, 3.

$x(HHH) = 0$

$x(HHT) = 1$

$x(THH) = 1$

$x(HTH) = 1$

$x(HTT) = 2$

$x(THT) = 2$

$x(TTH) = 2$

$x(TTT) = 3$

Values of the random variable X	0	1	2	3	Total
Number of points in inverse images	1	3	3	1	8

2) In a pack of 52 playing cards, two cards are drawn at random simultaneously. If the number of black cards drawn is a random variables, find the values of the random variables and number of points in its inverse images.

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Total cards = 52

No of black cards = 26

No of Red cards = 26

Let X denote the number of black cards drawn.

clearly X can take values 0, 1, 2.

X (no black cards) = 0

X (one black and one red) = 1

X (two black cards) = 2

$$X(w) = \begin{cases} 0 & \text{if no black} \\ 1 & \text{if one black} \\ 2 & \text{if 2 black.} \end{cases}$$

X (no black cards) $= {}^{26}C_2 \times {}^{26}C_0$

$= \frac{26 \times 25}{2} \times 1$

$= 13 \times 25 \times 1$

$= 325$

X (one black and one red) $= {}^{26}C_1 \times {}^{26}C_1 = 26 \times 26$

$= 676$

X (two black cards) $= {}^{26}C_2 \times {}^{26}C_0$

$= \frac{26 \times 25}{2}$

$= 13 \times 25$

$= 325$

∴ Values of Random Variables	0	1	2	Total
No of elements in inverse image	325	676	325	1326

3) An urn contains 5 mangoes and 4 apples. Three fruits are taken at random. If the number of apples taken is a random variable the find the values of the random variable and number of points in its inverse images.

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Apple	Mango	Total
4	5	9

$$n(S) = {}^9C_3 = \frac{9\times8\times7}{1\times2\times3}$$

$$= 3\times4\times7$$

$$= 84$$

Let the random variable x denote no of apples in three drawn.

clearly, x takes values 0, 1, 2, 3.

x (no apple) = 0

x (one apple and two mangoes) = 1

x (2 apples and one mango) = 2

x (3 apples and no mangoes) = 3

$$X(w) = \begin{cases} 0, & \text{if no apple} \\ 1 & \text{if one apple, two mango} \\ 2 & \text{if two apple, one mango} \\ 3 & \text{if three apple} \end{cases}$$

$$X(\text{no apple}) = {}^4C_0 \times {}^5C_3$$

$$= 1 \times {}^5C_2$$

$$= 1 \times \frac{5\times4}{1\times2} = \boxed{10}$$

$$X(\text{one apple \& two mangoes}) = {}^4C_1 \times {}^5C_2$$

$$= 4 \times 10 = \boxed{40}$$

$$X(\text{two apple and one mango}) = {}^4C_2 \times {}^5C_1$$

$$= \frac{4\times3}{1\times2} \times 5 = 2\times3\times5$$

$$= \boxed{30}$$

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X (Three apple) $= {}^4C_3 \times {}^5C_0$

$= {}^4C_1 \times 1$

$= 4 \times 1 = \boxed{4}$

Values in random variable	0	1	2	3	Total
No of elements in inverse images	10	40	30	4	84

4) Two balls are chosen randomly from an urn containing 6 red and 8 black balls. Suppose that we win ₹15 for each red ball selected and we lose ₹10 for each black ball selected. X denotes the winning amount, then find the values of X and number of points in its inverse images.

Red	black	Total
6	8	14

$n(S) = {}^{14}C_2 = \frac{14 \times 13}{1 \times 2}$

$= \boxed{91}$

The random variable X denote the winning amount.

X (Both black) $= -10 - 10 = \boxed{-20}$

X (one Red, one black) $= 15 - 10 = \boxed{5}$

X (Both Red) $= 15 + 15$

$= \boxed{30}$

$$X(\omega) = \begin{cases} -20 & \text{if both black} \\ 5 & \text{if one Red \& one black} \\ 30 & \text{if Both Red} \end{cases}$$

X (Both are black) $= {}^8C_2 \times {}^6C_0$

$= \frac{8 \times 7}{1 \times 2} \times 1$

$= 4 \times 7 \times 1 = \boxed{28}$

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$\times$ (one Red, one black) $= {}^6C_1 \times {}^8C_1$

$= 6 \times 8$

$= \boxed{48}$

$\times$ (Both are Red) $= {}^6C_2 \times {}^8C_0$

$= \frac{6 \times 5}{1 \times 2} \times 1$

$= \boxed{15}$

Values of random variable	-20	5	30	Total
No of elements in inverse image	28	48	15	91

5) A six sided die is marked '2' on one face, '3' on two of its faces, and '4' one remaining three faces. The die is thrown twice. If X denote the total score in two throws, find the values of the random variable and no of points in its inverse images.

The die is thrown twice.

∴ The sample space is

$S = \{2,3,3,4,4,4\} \times \{2,3,3,4,4,4\}$

$= \begin{Bmatrix} (2,2),(2,3),(2,3),(2,4),(2,4),(2,4) \\ (3,2),(3,3),(3,3),(3,4),(3,4),(3,4) \\ (3,2),(3,3),(3,3),(3,4),(3,4),(3,4) \\ (4,2),(4,3),(4,3),(4,4),(4,4),(4,4) \\ (4,2),(4,3),(4,3),(4,4),(4,4),(4,4) \\ (4,2),(4,3),(4,3),(4,4),(4,4),(4,4) \end{Bmatrix}$

$n(S) = 36$.

Let the random variable X denote the total score in two throws.

clearly X takes values 4, 5, 6, 7, 8

Values of random variable	4	5	6	7	8	Total
No of elements in inverse image	1	4	10	12	9	36

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Discrete random variable:-

A random variable X is defined on a sample space S into the real nos R is called discrete random variable if the range of X is countable, ie, it can assume only a finite or countably infinite number of values, where every value in the set S has positive probability with total one.

probability Mass function:

If X is discrete random variable with discrete values $x_1, x_2 \ldots x_n \ldots$ then the function denoted by $f(\cdot)$ or $p(\cdot)$ and defined by

$f(x_k) = p(X = x_k)$ for $k = 1, 2, \ldots$

is called probability mass function of X.

Theorem: The function $f(x)$ is a probability mass function iff it satisfies the following properties for the set of real nos $x_1, x_2, \ldots x_n, \ldots$

(i) $f(x_k) \geq 0$ for $k = 1, 2, 3 \ldots$

(ii) $\sum f(x_k) = 1$.

Ex 11.5 Two fair coins are tossed simultaneously (equivalent to a fair coin is tossed twice). Find the probability mass function for number of heads occured.

The sample space $S = \{H, T\} \times \{H, T\}$

$= \{HH, HT, TH, TT\}$

Let X be the random variable denoting the number of heads.

clearly X can takes values 0, 1, 2

$\therefore$ $X(HH) = 2$

$X(HT) = 1$

$X(TH) = 1$

$X(TT) = 0$

Values of random Variable	0	1	2	Total
No of elements in inverse image	1	2	1	4

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The probabilities are

$f(0) = P(X=0) = \frac{1}{4}$

$f(1) = P(X=1) = \frac{2}{4} = \frac{1}{2}$

$f(2) = P(X=2) = \frac{1}{4}$

The function $f(x)$ satisfies

(i) $f(x) \geq 0$ for $x = 0, 1, 2 \ldots$

(ii) $\sum f(x) = \sum_{x=0}^{2} f(x) = f(0)+f(1)+f(2)$

$= \frac{1}{4} + \frac{1}{2} + \frac{1}{4}$

$= 1$

$\therefore$ The probability mass fun is

x	0	1	2
$f(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

or

$$f(x) = \begin{cases} \frac{1}{4} & \text{for } x = 0 \\ \frac{1}{2} & \text{for } x = 1 \\ \frac{1}{4} & \text{for } x = 2 \end{cases}$$

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Ex 11.6 A pair of fair dice is rolled once. Find the probability mass function to get the number of fours.

A pair of fair dice is rolled once.

Spample space

$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$n(S) = 36$

Let the random variable X denotes to get the number four

clearly X takes values 0, 1, 2.

if $X = 1$; $\{(1,4)\ (2,4), (3,4), (5,4), (6,4)$
$(4,1), (4,2)\ (4,3), (4,5)$
$(4,6)\}$

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$X = 2$ if $(4,4)$ — 1

$X = 0$, (X = no four)

$= 36 - 11 = 25$.

Values of random variable X	0	1	2	Total
No of elements in inverse images	25	10	1	36

∴ The propabilities are

$f(0) = p(X=0) = 25/36$

$f(1) = p(X=1) = 10/36$

$f(2) = p(X=2) = 1/36$

clearly $f(x)$ satifies the condition

(i) $f(x) \geq 0$ for $x = 0, 1, 2 \ldots$

(ii) $\sum f(x) = \sum_{x=0}^{2} f(x)$

$= f(0) + f(1) + f(2)$

$= \frac{25}{36} + \frac{10}{36} + \frac{1}{36}$.

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(or)

$$f(x) = \begin{cases} \frac{25}{36}, & \text{if } x = 0 \\ \frac{10}{36}, & \text{if } x = 1 \\ \frac{1}{36}, & \text{if } x = 2 \end{cases}$$

Cumulative Distribution function.

The cumulative distribution function $F(x)$ of a discrete radom variable X, taking the values $x_1, x_2, \ldots$ such that $x_1 < x_2 < x_3 \ldots$ with probability mass function $f(x_i)$ is

$$F(x) = p(X \leq x) = \sum_{i=1}^{n} f(x_i) \quad x \in R$$

EX: 11.7 If the probability mass function $f(x)$ of a random variable X is

x	1	2	3	4
$f(x)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

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find (i) its cummative distribution function, hence find (ii) $P(X \leq 3)$ and (iii) $P(X \geq 2)$

(i) Cumulative distribution function

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} P(x = x_i)$$

$P(X \leq x) = 0$ for $-\infty < x < 1$

$$F(1) = P(x \leq 1)$$
$$= P(x < 1) + P(X = 1)$$
$$= 0 + \frac{1}{12} = \boxed{\frac{1}{12}}$$

$$F(2) = P(X \leq 2)$$
$$= P(X < 1) + P(x = 1) + P(X = 2)$$
$$= 0 + \frac{1}{12} + \frac{5}{12}$$
$$= \frac{6}{12} = \boxed{\frac{1}{2}}$$

$$F(3) = P(X \leq 3)$$
$$= P(X < 1) + P(X = 1) + P(X = 2) + P(X = 3)$$
$$= 0 + \frac{1}{12} + \frac{5}{12} + \frac{5}{12}$$
$$= \boxed{\frac{11}{12}}$$

$$F(4) = P(X \leq 4)$$
$$= P(X < 1) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$
$$= 0 + \frac{1}{12} + \frac{5}{12} + \frac{5}{12} + \frac{1}{12}$$
$$= \frac{12}{12} = \boxed{1}$$

Cumulative distn function is

$$F(x) = \begin{cases} 0, & -\infty < x < 1 \\ \frac{1}{12}, & 1 \leq x < 2 \\ \frac{1}{2}, & 2 \leq x < 3 \\ \frac{11}{12}, & 3 \leq x < 4 \\ 1 & 4 \leq x < \infty \end{cases}$$

(ii) $P(X \leq 3) = F(3) = \boxed{\frac{11}{12}}$

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(iii) $P(X \geq 2) = 1 - P(X < 2)$

$= 1 - P(X \leq 1)$

$= 1 - F(1)$

$= 1 - \frac{1}{12} = \boxed{\frac{11}{12}}$

Ex 11.8 A Six sided die is marked '1' on one face, '2' on two of its faces, and '3' on remaining three faces. The die is rolled twice. If X denotes the total score in two throws.

(i) Find the probability mass function

(ii) Find the cumulative distn function

(iii) Find $P(3 \leq X < 6)$

(iv) Find $P(X \geq 4)$

Given, the die is rolled twice.

Let X denote the total score is 2 in two throws.

clearly X takes the values 2, 3, 4, 5 and 6.

Sample space:-

	1	2	2	3	3	3
1	2	3	3	4	4	4
2	3	4	4	5	5	5
2	3	4	4	5	5	5
3	4	5	5	6	6	6
3	4	5	5	6	6	6
3	4	5	5	6	6	6

Value of Random Variable	2	3	4	5	6	Total
No of elements in the inverse image	1	4	10	12	9	36

probabilities are

$P(X=2) = 1/36$

$P(X=3) = 4/36$

$P(X=4) = 10/36$

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$P(X=5) = 12/36$

$P(X=6) = 9/36$

(i) Probability Mass function:

x	2	3	4	5	6
$f(x)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$

(ii) cumulative distn function

$F(x) = P(X \le x) = \sum_{x_i \le x} P(X = x_i)$

$P(X < x) = 0$ for $-\infty < x < 2$

$F(2) = P(X \le 2) = P(X<2) + P(X=2)$

$= 0 + \frac{1}{36} = \boxed{\frac{1}{36}}$

$F(3) = P(X \le 3) = P(X<2) + P(X=2) + P(X=3)$

$= 0 + \frac{1}{36} + \frac{4}{36} = \boxed{\frac{5}{36}}$

$F(4) = P(X \le 4) = P(X<2) + P(X=2) + P(X=3) + P(X=4)$

$= 0 + \frac{1}{36} + \frac{4}{36} + \frac{10}{36} = \boxed{\frac{15}{36}}$

$F(5) = P(X \le 5)$

$= P(X<2) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$= 0 + \frac{1}{36} + \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \boxed{\frac{27}{36}}$

$F(6) = P(X \le 6)$

$= P(X<2) + P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6)$

$= 0 + \frac{1}{36} + \frac{4}{36} + \frac{10}{36} + \frac{12}{36} + \frac{9}{36}$

$= \boxed{\frac{36}{36}} = \boxed{1}$

∴ The cumulative distn function is

$$f(x) = \begin{cases} 0, & \text{for } -\infty < x < 2 \\ \frac{1}{36}, & \text{for } 2 \le x < 3 \\ \frac{5}{36}, & \text{for } 3 \le x < 4 \\ \frac{15}{36}, & \text{for } 4 \le x < 5 \\ \frac{27}{36}, & \text{for } 5 \le x < 6 \\ 1 & \text{for } 6 \le x < \infty \end{cases}$$

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(iii) $P(3 \le x < 6) = P(x=3) + P(x=4) + P(x=5)$

$= \frac{4}{36} + \frac{10}{36} + \frac{12}{36}$

$= \boxed{\frac{26}{36}}$

(iv) $P(x \ge 4) = P(x=4) + P(x=5) + P(x=6)$

$= \frac{10}{36} + \frac{12}{36} + \frac{9}{36}$

$= \boxed{\frac{31}{36}}$

Cumulative distribution fun (cdf)

$$F(x) = P(X \le x) = \sum_{x_i \le x} f(x_i)$$

Properties:-

1) $F(x)$ is non descending fun
2) $0 \le f(x) \le 1$ for all $x \in R$
3) $\lim_{x \to a^+} F(x) = 0$
4) $F(-\infty) = 0$
5) $F(\infty) = 1$
6) $P(x_1 < x \le x_2) = F(x_2) - F(x_1)$
7) $P(X \ge x) = 1 - P(X < x)$
   $= 1 - F(x)$
8) $P(X = x_k) = F(x_k) - F(x_{k-1})$

Probability Mass function from cummulative distribution function.

$$f(x_i) = F(x_i) - F(x_{i-1})$$

Ex 11.9 Find the probability mass function $f(x)$ of the discrete random variable X whose cummulative distribution function $F(x)$ is given by

$$F(x) = \begin{cases} 0 & -\infty < x < -2 \\ 0.25 & -2 \le x < -1 \\ 0.60 & -1 \le x < 0 \\ 0.90 & 0 \le x < 1 \\ 1 & 1 \le x < \infty \end{cases}$$

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Also find (i) $p(x<0)$ and (ii) $p(x \geq -1)$.

Soln:- Let x be the discrete random variable.

clearly x can take values $-2, -1, 0, 1$,

WKT $f(x) = F(x) - F(x-1)$

$f(-2) = F(-2) - F(-3)$
$= 0.25 - 0 = \boxed{0.25}$

$f(-1) = F(-1) - F(-2)$
$= 0.60 - 0.25 = \boxed{0.35}$

$f(0) = F(0) - F(-1)$
$= 0.90 - 0.60 = \boxed{0.30}$

$f(1) = F(1) - F(0)$
$= 1 - 0.90 = \boxed{0.10}$

∴ probability Mass function is

x	-2	-1	0	1
$f(x)$	0.25	0.35	0.30	0.10

(ii) $p(x<0) = p(-1) + p(-2)$
$= 0.35 + 0.25$
$= \boxed{0.60}$

(iii) $p(x \geq -1) = p(x=-1) + p(x=0) + p(x=1)$
$= 0.35 + 0.30 + 0.10$
$= \boxed{0.75}$

Ex 11.10 A random variable x has the following probability mass function

x	1	2	3	4	5	6
$f(x)$	K	2k	6k	5k	6k	10k

Find (i) $p(2<x<6)$
(ii) $p(2 \leq x < 5)$
(iii) $p(x \leq 4)$
(iv) $p(3<x)$

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Given, The function is probability mass function.

∴ The total probability is one.

i.e. $\sum_{x=1}^{6} f(x) = 1$

i.e. $f(1) + f(2) + f(3) + f(4) + f(5) + f(6) = 1$

i.e. $k + 2k + 6k + 5k + 6k + 10k = 1$

$30k = 1$

$\Rightarrow \boxed{k = \frac{1}{30}}$

∴ The probability Mass function is

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{30}$	$\frac{2}{30}$	$\frac{6}{30}$	$\frac{5}{30}$	$\frac{6}{30}$	$\frac{10}{30}$

(i) $P(2 < x < 6) = f(3) + f(4) + f(5)$

$= \frac{6}{30} + \frac{5}{30} + \frac{6}{30}$

$= \boxed{\frac{17}{30}}$

(ii) $P(2 \leq x < 5) = f(2) + f(3) + f(4)$

$= \frac{2}{30} + \frac{6}{30} + \frac{5}{30}$

$= \boxed{\frac{13}{30}}$

(iii) $P(x \leq 4) = f(1) + f(2) + f(3) + f(4)$

$= \frac{1}{30} + \frac{2}{30} + \frac{6}{30} + \frac{5}{30}$

$= \boxed{\frac{14}{30}}$

(iv) $P(3 < x) = P(x > 3)$

$= f(4) + f(5) + f(6)$

$= \frac{5}{30} + \frac{6}{30} + \frac{10}{30}$

$= \boxed{\frac{21}{30}}$

EXERCISE 11.2

1) Three fair coins are tossed simultaneously. find the probability mass function for number number of heads occurred.

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Soln: Given three coins are tossed simultaneously.

Sample space $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

Let X denote the no of heads occured.

$\therefore$ X takes values 0, 1, 2, 3.

$P(X=0) = P(\text{no head}) = P(TTT)$

$= P(T) \times P(T) \times P(T)$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= \boxed{\frac{1}{8}}$

$P(X=1) = P(\text{one head}) = P(HTT, THT, TTH)$

$= \boxed{\frac{3}{8}}$

$P(X=2) = P(\text{two head})$

$= P(HHT, HTH, THH) = \boxed{\frac{3}{8}}$

$P(X=3) = P(\text{three head})$

$= P(HHH)$

$= \boxed{\frac{1}{8}}$

$\therefore$ The probability mass function is

x	0	1	2	3
f(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

2) A six sided die is marked '1' on one face, 3 on two of its faces, and 5 on remaining three faces. The die is thrown twice. If X denotes the total score in two throws, find

(i) the probability mass function

(ii) the cumulative distn function

(iii) $P(4 \leq X \leq 10)$ (iv) $P(X \geq 6)$

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The numbers on dice are

1, 3, 3, 5, 5, 5

Given X denote the total score in two throws.

Sample space S is

(+)	1	3	3	5	5	5
1	2	4	4	6	6	6
3	4	6	6	8	8	8
3	4	6	6	8	8	8
5	6	8	8	10	10	10
5	6	8	8	10	10	10
5	6	8	8	10	10	10

clearly X takes the values

2, 4, 6, 8, 10.

probabilities are

$P(X=2) = \frac{1}{36}$, $P(X=4) = \frac{4}{36}$

$P(X=6) = \frac{10}{36}$, $P(X=8) = \frac{12}{36}$

$P(X=10) = \frac{9}{36}$

The probability mass function is

x	2	4	6	8	10
f(x)	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$

(ii) cumulative distn function:-

$F(x) = P(X \le x)$

$F(2) = P(X \le 2) = P(X=2) = \boxed{\frac{1}{36}}$

$F(4) = P(X \le 4) = P(X=2) + P(X=4)$

$= \frac{1}{36} + \frac{4}{36} = \boxed{\frac{5}{36}}$

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$F(6) = P(X \le 6)$

$= P(X=2) + P(X=4) + P(X=6)$

$= \frac{1}{36} + \frac{4}{36} + \frac{10}{36} = \boxed{\frac{15}{36}}$

$F(8) = P(X \le 8)$

$= P(X=2) + P(X=4) + P(X=6) + P(X=8)$

$= \frac{1}{36} + \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \boxed{\frac{27}{36}}$

$F(10) = P(X \le 10)$

$= P(X=2) + P(X=4) + P(X=6) + P(X=8) + P(X=10)$

$= \frac{1}{36} + \frac{4}{36} + \frac{10}{36} + \frac{12}{36} + \frac{9}{36}$

$= \frac{36}{36} = \boxed{1}$

∴ The cumulative distn function is

$$F(x) = \begin{cases} 0 & \text{for } x < 2 \\ \frac{1}{36} & \text{for } x \le 2 \\ \frac{5}{36} & \text{for } x \le 4 \\ \frac{15}{36} & \text{for } x \le 6 \\ \frac{27}{36} & \text{for } x \le 8 \\ 1 & \text{for } x \le 10 \end{cases}$$

(iii) $P(4 \le x < 10) = P(X=4) + P(X=6) + P(X=8)$

$= \frac{4}{36} + \frac{10}{36} + \frac{12}{36}$

$= \frac{26}{36} = \boxed{\frac{13}{18}}$

(iv) $P(X \ge 6) = P(X=6) + P(X=8) + P(X=10)$

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$= \frac{10}{36} + \frac{12}{36} + \frac{9}{36} = \boxed{\frac{31}{36}}$

---

3) Find the probability mass function and cumulative distn function of number of girl child in families with 4 children assuming equal probabilities for boys and girls.

$n(S) = 2^4 = 16$.

Let the random variable X denote the no of girl child in a family of 4 children.

clearly x takes values 0, 1, 2, 3, 4

X(no girl) $= {}^4C_0 = 1$

X(1 girl) $= {}^4C_1 = 4$

X(2 girls) $= {}^4C_2 = \frac{4 \times 3}{1 \times 2} = 6$

X(3 girls) $= {}^4C_3 = {}^4C_1 = 4$

X(4 girls) $= {}^4C_4 = 1$

∴ The probabilities are

$P(X=0) = \frac{1}{16}$

$P(X=1) = \frac{4}{16}$

$P(X=2) = \frac{6}{16}$

$P(X=3) = \frac{4}{16}$

$P(X=4) = \frac{1}{16}$

∴ The probability mass function is

x	0	1	2	3	4
$f(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

(ii) cumulative distn function

$F(0) = P(X=0) = \frac{1}{16}$

$F(1) = P(X \leq 1) = P(X=0) + P(X=1)$

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$= \frac{1}{16} + \frac{4}{16} = \boxed{\frac{5}{16}}$

$F(2) = P(X \le 2)$

$= P(X=0) + P(X=1) + P(X=2)$

$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} = \boxed{\frac{11}{16}}$

$F(3) = P(X \le 3)$

$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} = \boxed{\frac{15}{16}}$

$F(4) = P(X \le 4)$

$= P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16}$

$= \frac{16}{16} = \boxed{1}$

∴ The cumulative distn function is

$$F(x) = \begin{cases} \frac{0}{16} & \text{for } x < 0 \\ \frac{1}{16} & \text{for } x \le 0 \\ \frac{5}{16} & \text{for } x \le 1 \\ \frac{11}{16} & \text{for } x \le 2 \\ \frac{15}{16} & \text{for } x \le 3 \\ 1 & \text{for } x \le 4 \end{cases}$$

4) Suppose a discrete random variable can only take the values 0, 1 and 2. The probability mass function is defined by

$$f(x) = \begin{cases} \frac{x^2+1}{K} & \text{for } x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

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Find (i) the value of k.

(ii) cumulative distn function

(iii) $P(X \geq 1)$

Given $f(x) = \begin{cases} \frac{x^2+1}{k}, & x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$

$f(x)$ is probability mass function

$\Rightarrow \sum_{i=0}^{2} f(x_i) = 1$

$\Rightarrow f(0) + f(1) + f(2) = 1$

$\Rightarrow \frac{0+1}{k} + \frac{1+1}{k} + \frac{4+1}{k} = 1$

$\frac{1}{k} + \frac{2}{k} + \frac{5}{k} = 1 \Rightarrow \frac{8}{k} = 1$

$\Rightarrow \boxed{k = 8}$

The probability mass function is

$f(0) = \frac{1}{k} = \frac{1}{8}$

$f(1) = \frac{2}{k} = \frac{2}{8}$

$f(2) = \frac{5}{k} = \frac{5}{8}$

$\therefore$ probability Mass function is

x	0	1	2	Total
$f(x)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{5}{8}$	1

(ii) cumulative distn function

$F(0) = P(X \leq 0)$

$= P(X = 0) = \boxed{\frac{1}{8}}$

$F(1) = P(X \leq 1)$

$= P(X = 0) + P(X = 1)$

$= \frac{1}{8} + \frac{2}{8} = \boxed{\frac{3}{8}}$

$F(2) = P(X \leq 2)$

$= P(X = 0) + P(X = 1) + P(X = 2)$

$= \frac{1}{8} + \frac{2}{8} + \frac{5}{8} = \frac{8}{8} = \boxed{1}$

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cumulative function is

$$F(x) = \begin{cases} \frac{1}{8} & \text{for } x \le 0 \\ \frac{3}{8} & \text{for } x \le 1 \\ 1 & \text{for } x \le 2 \end{cases}$$

(iii) $P(X \ge 1) = P(X=1) + P(X=2)$

$= \frac{2}{8} + \frac{5}{8} = \boxed{\frac{7}{8}}$

---

5) The cumulative distribution of a discrete random variable is given by

$$F(x) = \begin{cases} 0 & , -\infty < x < -1 \\ 0.15 & , -1 \le x < 0 \\ 0.35 & , 0 \le x < 1 \\ 0.60 & , 1 \le x < 2 \\ 0.85 & , 2 \le x < 3 \\ 1 & , 3 \le x < \infty \end{cases}$$

Find (i) the probability mass function (ii) $P(X<1)$, (iii) $P(X \ge 2)$.

Since X is discrete random variable, clearly X takes values $-1, 0, 1, 2, 3$.

WKT, $f(x_i) = F(i) - F(x_{i-1})$

$f(-1) = F(-1) - F(0)$

$= 0.15 - 0 = \boxed{0.15}$

$f(0) = F(0) - F(-1)$

$= 0.35 - 0.15 = \boxed{0.20}$

$f(1) = F(1) - F(0)$

$= 0.60 - 0.35 = \boxed{0.25}$

$f(2) = F(2) - F(1)$

$= 0.85 - 0.60 = \boxed{0.25}$

$f(3) = f(3) - f(2)$

$= 1 - 0.85 = \boxed{0.15}$

∴ The probability mass fun D

x	-1	0	1	2	3
$f(x)$	0.15	0.20	0.25	0.25	0.15

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(ii) $P(X<1) = P(X=-1) + P(X=0)$

$= 0.15 + 0.25$

$= \boxed{0.35}$

(iii) $P(X \geq 2) = P(X=2) + P(X=3)$

$= 0.25 + 0.15$

$= \boxed{0.40}$

---

6) A random variable X has the following probability mass function

x	1	2	3	4	5
$f(x)$	k^2	$2k^2$	$3k^2$	$2k$	$3k$

Find (i) The value of k (ii) $P(2 \leq X < 5)$ (iii) $P(3 < X)$.

Soln:-

Since $f(x)$ is a probability mass function.

$$\sum_{i=1}^{5} f(x_i) = 1$$

$$f(1) + f(2) + f(3) + f(4) + f(5) = 1$$

$$\Rightarrow k^2 + 2k^2 + 3k^2 + 2k + 3k = 1$$

$$6k^2 + 5k = 1$$

$$6k^2 + 5k - 1 = 0$$

$$(k+1)(6k-1) = 0$$

$$k = -1 \text{ or } k = \frac{1}{6}$$

-6: $6, -1$; $\frac{6}{6}, \frac{-1}{6}$; $1, -\frac{1}{6}$

$k = -1$ is not possible

$\therefore \boxed{k = \frac{1}{6}}$

(ii) $P(2 \leq X < 5) = P(X=2) + P(X=3) + P(X=4)$

$= 2k^2 + 3k^2 + 2k$

$= 5k^2 + 2k$

$= 5\left(\frac{1}{6}\right)^2 + 2\left(\frac{1}{6}\right)$

$= \frac{5}{36} + \frac{2}{6}$

$= \frac{5+12}{36} = \boxed{\frac{17}{36}}$

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(iii) $P(3<X)$

$P(3<X) = P(X>3)$

$= P(X=4) + P(X=5)$

$= 2k + 3k = 5k$

$= \boxed{\frac{5}{6}}$

---

7) The cumulative distribution function of a discrete random variable is given by.

$$F(x) = \begin{cases} 0 & \text{for } -\infty < x < 0 \\ \frac{1}{2} & \text{for } 0 \le x < 1 \\ \frac{3}{5} & \text{for } 1 \le x < 2 \\ \frac{4}{5} & \text{for } 2 \le x < 3 \\ \frac{9}{10} & \text{for } 3 \le x < 4 \\ 1 & \text{for } 4 \le x < \infty \end{cases}$$

Find (i) The probability mass function (ii) $P(X<3)$ and (iii) $P(X \ge 2)$.

X is discrete random variable.

clearly x takes values 0, 1, 2, 3, 4.

$f(x) = P(X=x)$

$f(0) = F(0) = \frac{1}{2}$

$f(1) = F(1) - F(0) = \frac{3}{5} - \frac{1}{2}$

$= \frac{6-5}{10} = \boxed{\frac{1}{10}}$

$f(2) = F(2) - F(1)$

$= \frac{4}{5} - \frac{3}{5} = \boxed{\frac{1}{5}}$

$f(3) = F(3) - F(2)$

$= \frac{9}{10} - \frac{4}{5} = \frac{9-8}{10} = \boxed{\frac{1}{10}}$

$f(4) = F(4) - F(3)$

$= 1 - \frac{9}{10} = \frac{10-9}{10} = \boxed{\frac{1}{10}}$

∴ The probability mass fun is

x	0	1	2	3	4
$f(x)$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

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(ii) $P(X < 3) = P(X=0) + P(X=1) + P(X=2)$

$= \frac{1}{2} + \frac{1}{10} + \frac{1}{5}$

$= \frac{5+1+2}{10} = \frac{8}{10}$

$= \boxed{\frac{4}{5}}$

(iii) $P(X \geq 2) = P(X=2) + P(X=3) + P(X=4)$

$= \frac{1}{5} + \frac{1}{10} + \frac{1}{10}$

$= \frac{2+1+1}{10} = \boxed{\frac{4}{10}}$

$= \boxed{\frac{2}{5}}$

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probability density function:-

A function $f(x)$ is a probability function for some continuous random variable x if and only if it satisfies the following properties

(i) $f(x) \geq 0$, for every x

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Note: 1) $P(a \leq x \leq b) = \int_a^b f(x)\,dx$

2) $P(x = a) = \int_a^a f(x)\,dx = 0$

3) $P(a < x < b) = P(a \leq x \leq b)$
$= P(a < x \leq b)$
$= P(a \leq x < b)$

Distribution function: (Cumulative distn function)

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\,du.$$

properties:

(1) $F(x)$ is non-decreasing function

2) $0 \leq F(x) \leq 1$

3) $F(-\infty) = 0$

4) $F(\infty) = 1$

5) $P(a < x < b) = F(b) - F(a)$

6) $P(x > x) = 1 - P(X \leq x) = 1 - F(x)$

Ex 11.11 Find the constant c such that the function

$$f(x) = \begin{cases} cx^2 & 1 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

is a density function, and compute

(i) $P(1.5 < x < 3.5)$ (ii) $P(X \leq 2)$

(iii) $P(3 < x)$

Since $f(x)$ is probability density function

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

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$$\int_{-\infty}^{1} f(x)dx + \int_{1}^{4} f(x)dx + \int_{4}^{\infty} f(x)dx = 1$$

$$= 0 + \int_{1}^{4} cx^2 dx + 0 = 1$$

$$c\int_{1}^{4} x^2 dx = 1 \Rightarrow c\left[\frac{x^3}{3}\right]_1^4 = 1$$

$$\Rightarrow c\left[\frac{64}{3} - \frac{1}{3}\right] = 1$$

$$\Rightarrow c\left[\frac{63}{3}\right] = 1$$

$$\Rightarrow c = \frac{3}{63} = \boxed{\frac{1}{21}}$$

∴ The probability density function is

$$f(x) = \begin{cases} \frac{1}{21}x^2, & 1 < x < 4 \\ 0 \quad \text{otherwise} \end{cases}$$

(i) $P(1.5 < x < 3.5) = \int_{1.5}^{3.5} f(x)dx$

$$= \int_{1.5}^{3.5} \frac{x^2}{21} dx$$

$$= \frac{1}{21}\int_{1.5}^{3.5} x^2 dx \Rightarrow \frac{1}{21}\left[\frac{x^3}{3}\right]_{1.5}^{3.5}$$

$$= \frac{1}{21\times 3}\left[3.5^3 - 1.5^3\right]$$

$$= \frac{1}{63}\left[\left(\frac{7}{2}\right)^3 - \left(\frac{3}{2}\right)^3\right]$$

$$= \frac{1}{63}\left[\frac{343}{8} - \frac{27}{8}\right] = \frac{1}{63}\left[\frac{343-27}{8}\right]$$

$$= \frac{1}{63}\left[\frac{316}{8}\right] = \frac{1}{63}\left[\frac{79}{2}\right]$$

$$= \boxed{\frac{79}{126}}$$

(ii) $P(x \le 2) = \int_{-\infty}^{2} f(x)dx$

$$= \int_{-\infty}^{1} f(x)dx + \int_{1}^{2} f(x)dx$$

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$$= 0 + \int_1^2 f(x)\,dx$$

$$= \int_1^2 \frac{x^2}{21}\,dx = \frac{1}{21}\int_1^2 x^2\,dx$$

$$= \frac{1}{21}\left[\frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{21\times 3}\left[2^3 - 1^3\right] = \frac{1}{63}\left[8-1\right] = \frac{7}{63} = \boxed{\frac{1}{9}}$$

(iii) $P(3<x) = \int_3^\infty f(x)\,dx$

$$= \int_3^4 f(x)\,dx + \int_4^\infty f(x)\,dx$$

$$= \int_3^4 f(x)\,dx + 0$$

$$= \int_3^4 \frac{x^2}{21}\,dx = \frac{1}{21}\int_3^4 x^2\,dx$$

$$= \frac{1}{21}\left[\frac{x^3}{3}\right]_3^4$$

$$= \frac{1}{21\times 3}\left[4^3 - 3^3\right] = \frac{1}{63}\left[64-27\right]$$

$$= \frac{1}{63}\left[37\right] = \boxed{\frac{37}{63}}$$

Ex 11.12 If X is the random variable with probability density function $f(x)$ is given by

$$f(x) = \begin{cases} x-1, & 1 \le x < 2 \\ -x+3, & 2 \le x < 3 \\ 0, & \text{otherwise} \end{cases}$$

find (i) the distn function $F(x)$

(ii) $P(1.5 \le x \le 2.5)$

(i) By defn $F(x) = P(X \le x)$

$$= \int_{-\infty}^{x} f(u)\,du$$

when $x < 1$, $F(x) = \int_{-\infty}^{x} f(u)\,du = 0$

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When $1 \le x < 2$

$$F(x) = \int_{-\infty}^{x} f(u)\,du = \int_{-\infty}^{1} f(u)\,du + \int_{1}^{x} f(u)\,du$$

$$= 0 + \int_{1}^{x} (u-1)\,du$$

$$= 0 + \left[\frac{(u-1)^2}{2}\right]_1^x = \boxed{\frac{(x-1)^2}{2}}$$

when $2 \le x < 3$

$$F(x) = \int_{-\infty}^{x} f(u)\,du = \int_{-\infty}^{1} f(u)\,du + \int_{1}^{2} f(u)\,du + \int_{2}^{x} f(u)\,du$$

$$= 0 + \int_{1}^{2} (u-1)\,du + \int_{2}^{x} (-u+3)\,du$$

$$= \left[\frac{(u-1)^2}{2}\right]_1^2 + \left[\frac{(-u+3)^2}{-2}\right]_2^x$$

$$\left[\frac{(2-1)^2}{2} - 0\right] - \left[\frac{3-x}{2} - \frac{3-2}{2}\right]^2$$

$$\frac{1}{2} - \left(\frac{3-x}{2}\right)^2 + \frac{1}{2} = \boxed{1 - \frac{(3-x)^2}{2}}$$

When $x \ge 3$

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

$$= \int_{-\infty}^{1} f(u)\,du + \int_{1}^{2} f(u)\,du + \int_{2}^{3} f(u)\,du + \int_{3}^{x} f(u)\,du$$

$$= 0 + \int_{1}^{2} (u-1)\,du + \int_{2}^{3} (3-u)\,du + 0$$

$$= \left[\frac{(u-1)^2}{2}\right]_1^2 + \left[\frac{(3-u)^2}{-2}\right]_2^3$$

$$= \left[\frac{(u-1)^2}{2}\right]_1^2 - \left[\frac{(3-u)^2}{2}\right]_2^3$$

$$= \left(\frac{1}{2} - 0\right) - \left(0 - \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2}$$

$$= \boxed{1}$$

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∴ Distribution function

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{(x-1)^2}{2}, & 1 \le x < 2 \\ 1 - \frac{(3-x)^2}{2}, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

(ii) $P(1.5 \le x \le 2.5) = F(2.5) - F(1.5)$

$$= 1 - \frac{(3-2.5)^2}{2} - \frac{(1.5-1)^2}{2}$$

$$= 1 - \frac{(0.5)^2}{2} - \frac{(0.5)^2}{2}$$

$$= 1 - \frac{0.25}{2} - \frac{0.25}{2}$$

$$= 1 - \frac{0.50}{2} = 1 - 0.25$$

$$= \boxed{0.75}$$

∴ $P(1.5 \le x \le 2.5) = \boxed{0.75}$

probability density function from probability distribution function.

Suppose $F(x)$ is the distribution function of a continuous random variable X. Then the probability density function $f(x)$ is given by

$$f(x) = \frac{dF(x)}{dx} = F'(x)$$

whenever derivative exists

Ex 11.13 If X is the random variable with distribution function $F(x)$ is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x \end{cases}$$

then find (i) the probability density function $f(x)$.

(ii) $P(0.2 \le x \le 0.7)$

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WKT, (i).

probability density function

$$f(x) = \frac{d}{dx}(F(x))$$

∴ The probability function is

$$f(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(ii) $P(0.2 \le x \le 0.7) = F(0.7) - F(0.2)$

$= 0.7 - 0.2$

$= 0.5$

∴ $\boxed{P(0.2 \le x \le 0.7) = 0.5}$

Ex 11.14 The probability density function of radom variable X is given by

$$f(x) = \begin{cases} K, & 1 \le x \le 5 \\ 0, & \text{otherwise} \end{cases}$$

. find

(i) Distribution function (ii) $P(X < 3)$

(iii) $P(2 < x < 4)$ (iv) $P(3 \le x)$

Since, $f(x)$ is probability density function

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{5} f(x)\,dx + \int_{5}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{1}^{5} f(x)\,dx + 0 = 1$$

$$\Rightarrow \int_{1}^{5} f(x)\,dx = 1 \Rightarrow \int_{1}^{5} K\,dx = 1$$

$$K \int_{1}^{5} dx = 1$$

$$\Rightarrow K[x]_1^5 = 1$$

$$\Rightarrow K[5-1] = 1 \Rightarrow 4K = 1$$

$$\boxed{K = \frac{1}{4}}$$

∴ The probability density function is

$$f(x) = \begin{cases} \frac{1}{4}, & 1 \le x \le 5 \\ 0, & \text{otherwise.} \end{cases}$$

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(i) Distribution function:-

The distribution function $F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\,du$

When $x < 1$

$$F(x) = \int_{-\infty}^{x} f(u)\,du = 0$$

When $1 \le x < 5$

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

$$= \int_{-\infty}^{1} f(u)\,du + \int_{1}^{x} f(u)\,du$$

$$= 0 + \int_{1}^{x} \frac{1}{4}\,dx = \frac{1}{4}\int_{1}^{x} dx$$

$$= \frac{1}{4}[x]_1^x$$

$$= \frac{1}{4}[x-1]$$

$$\boxed{F(x) = \frac{x-1}{4}}$$

When $x > 5$

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

$$= \int_{-\infty}^{1} f(u)\,du + \int_{1}^{5} \frac{1}{4}\,du + \int_{5}^{\infty} f(u)\,du =$$

$$= 0 + \int_{1}^{5} \frac{1}{4}\,du + 0 = \frac{1}{4}\int_{1}^{5} du$$

$$= \frac{1}{4}[u]_1^5$$

$$= \frac{1}{4}[5-1]$$

$$= \frac{4}{4} = \boxed{1}$$

$\therefore \boxed{F(x) = 1}$

$\therefore$ Distribution function is

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{x-1}{4}, & 1 \le x \le 5 \\ 1, & x > 5 \end{cases}$$

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(II) $P(X<3) = P(X\leq 3)$

$= F(3)$

$= \frac{x-1}{4} = \frac{3-1}{4} = \frac{2}{4} = \boxed{\frac{1}{2}}$

(III) $P(2<X<4) = F(4) - F(2)$

$= \frac{4-1}{4} - \frac{2-1}{4}$

$= \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \boxed{\frac{1}{2}}$

(IV) $P(3\leq x) = 1 - P(X<3)$

$= 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$

Ex 11.15 Let X be a random variable denoting the life time of an electrical equipment having probability density function

$$f(x) = \begin{cases} ke^{-2x}, & \text{for } x>0 \\ 0, & \text{for } x\leq 0 \end{cases}$$

Find (i) The value of k

(II) Distribution function

(III) $P(X<2)$

(IV) calculate the probability that X is atleast for four unit of time.

(V) $P(X=3)$

Soln:-

$$f(x) = \begin{cases} ke^{-2x}, & \text{for } x>0 \\ 0, & \text{for } x\leq 0. \end{cases}$$

Since $f(x)$ is probability density function

i $\int_{-\infty}^{\infty} f(x)\,dx = 1$

i $\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$

$0 + \int_{0}^{\infty} f(x)\,dx = 1$

i $\int_{0}^{\infty} ke^{-2x}\,dx = 1$

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$$\Rightarrow k\int_0^\infty e^{-2x}\,dx = 1$$

$$\Rightarrow k\left[\frac{e^{-2x}}{-2}\right]_0^\infty = 1$$

$$\Rightarrow k\left[\frac{e^{-\infty}}{-2} - \frac{e^{0}}{-2}\right] = 1$$

$$\Rightarrow k\left[0 - \frac{1}{-2}\right] = 1 \Rightarrow \frac{k}{2} = 1$$

$$\Rightarrow \boxed{k = 2}$$

∴ The probability density function is

$$f(x) = \begin{cases} 2e^{-2x}, & \text{for } x > 0 \\ 0, & \text{for } x \le 0 \end{cases}$$

(ii) Distribution function:-

By definition, $F(x) = p(X \le x) = \int_{-\infty}^{x} f(u)\,du$.

When $x \le 0$, $F(x) = \int_{-\infty}^{x} f(u)\,du = \boxed{0}$

When $x > 0$,

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

$$= \int_{-\infty}^{0} f(u)\,du + \int_0^x 2e^{-2u}\,du$$

$$= 0 + \left[\frac{2e^{-2u}}{-2}\right]_0^x$$

$$= \left[-e^{-2u}\right]_0^x = -\left[e^{-2x} - e^{-0}\right]$$

$$= -\left[e^{-2x} - 1\right]$$

$$= 1 - e^{-2x}$$

∴ Distribution function

$$F(x) = \begin{cases} 0, & \text{for } x \le 0 \\ 1 - e^{-2x} & \text{for } x > 0 \end{cases}$$

(iii) $p(X < 2) = p(X \le 2)$

$$= F(2) = 1 - e^{-2(2)}$$

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$= 1 - e^{-4}$

(iv) $P(X \geq 4) = 1 - P(X < 4)$

$= 1 - \left[1 - e^{-2(4)}\right]$

$= 1 - \left[1 - e^{-8}\right]$

$= 1 - 1 + e^{-8}$

$= \boxed{e^{-8}}$

(v) $P(X = 3) = \int_{3}^{3} f(x)\,dx = 0$

EXERCISE 11.3

1) The probability density function of X is given by

$$f(x) = \begin{cases} kxe^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Find the value of k.

soln:- Since $f(x)$ is probability density function

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{0}^{\infty} kxe^{-2x}\,dx = 1$$

By Gama Integral

$$\int_{0}^{\infty} x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$

$$\Rightarrow k\left[\frac{1!}{(-2)^{1+1}}\right] = 1$$

$$\Rightarrow k\left[\frac{1}{4}\right] = 1 \Rightarrow \boxed{k = 4}$$

---

2) The probability density function of X is

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}$$

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Find (i) $P(0.2 \le x < 0.6)$

(ii) $P(1.2 \le x < 1.8)$

(iii) $P(0.5 \le x < 1.5)$

Given $f(x)$ is a probability density function

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 \le x < 2 \\ 0, & \text{otherwise} \end{cases}$$

(i) $P(0.2 \le x < 0.6) = \int_{0.2}^{0.6} f(x)\,dx$

$= \int_{0.2}^{0.6} x\,dx = \left[\frac{x^2}{2}\right]_{0.2}^{0.6}$

$= \frac{1}{2}\left[(0.6)^2 - (0.2)^2\right]$

$= \frac{1}{2}\left[0.36 - 0.04\right]$

$= \frac{1}{2}\left[0.32\right]$

$= \boxed{0.16}$

$\therefore P(0.2 \le x < 0.6) = 0.16$

(ii) $P(1.2 \le x < 1.8) = \int_{1.2}^{1.8} f(x)\,dx$

$= \int_{1.2}^{1.8} (2-x)\,dx$

$= \left[2x - \frac{x^2}{2}\right]_{1.2}^{1.8}$

$= \left[\left(2(1.8) - \frac{(1.8)^2}{2}\right) - \left(2(1.2) - \frac{(1.2)^2}{2}\right)\right]$

$= \left[\left(3.6 - \frac{3.24}{2}\right) - \left(2.4 - \frac{1.44}{2}\right)\right]$

$= \left[(3.6 - 1.62) - (2.4 - 0.72\right]$

$= \left[1.98 - 1.68\right] = \boxed{0.3}$

(iii) $P(0.5 \le x < 1.5) = \int_{0.5}^{1.5} f(x)\,dx$

$= \int_{0.5}^{1} f(x)\,dx + \int_{1}^{1.5} f(x)\,dx$

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$$= \int_{0.5}^{1} x\,dx + \int_{1}^{1.5} (2-x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_{0.5}^{1} + \left[2x - \frac{x^2}{2}\right]_{1}^{1.5}$$

$$= \frac{1}{2}\left[1 - (0.5)^2\right] + \left[\left(2(1.5) - \frac{(1.5)^2}{2}\right) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[1 - 0.25\right] + \left[\left(3 - \frac{2.25}{2}\right) - \left(\frac{3}{2}\right)\right]$$

$$= \frac{1}{2}\left[0.75\right] + \left[\left(3 - 1.125\right] - \left(1.5\right)\right]$$

$$= 0.375 + 1.875 - 1.5$$

$$= 2.250 - 1.5$$

$$= 0.750 = \boxed{0.75}$$

3) Suppose the amount of milk sold daily at a milk booth is distributed with a minimum of 200 litres and a maximum of 600 litres with probability density function

$$f(x) = \begin{cases} K, & 200 < x \leq 600 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) the value of K

(ii) the distribution function

(iii) the probability that daily sales will fall betn 300 litres and 500 litres?

Given $f(x)$ is probability density function

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\text{i.e., } \int_{200}^{600} K\,dx = 1 \Rightarrow K\left[x\right]_{200}^{600} = 1$$

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$k[600-200] = 1 \Rightarrow 400k = 1$

$\Rightarrow \boxed{k = \frac{1}{400}}$

(ii) The destribution function:-

$F(x) = p(X \le x) = \int_{-\infty}^{x} f(u)du$

when $x < 200$ $\quad F(x) = \int_{-\infty}^{200} f(u)du$

$\boxed{F(x) = 0}$

when $200 \le x \le 600$

$F(x) = \int_{-\infty}^{200} f(u)du + \int_{200}^{x} f(u)du$

$= 0 + \int_{200}^{x} k\,dx$

$= k[x]_{200}^{x}$

$= k[x-200]$

$= kx - 200k$

$= \frac{x}{400} - \frac{200}{400}$

$= \frac{x}{400} - \frac{1}{2}$ //

when $x > 600$ $\quad F(x) = \int_{600}^{\infty} f(x)dx$

$= 0.$

∴ Distribution function

$$F(x) = \begin{cases} 0 & x < 200 \\ \frac{x}{400} - \frac{1}{2}, & 200 \le x \le 600 \\ 0, & x > 600. \end{cases}$$

(iii) $P(300 < X < 500) = F(500) - F(300)$

$= \left(\frac{500}{400} - \frac{1}{2}\right) - \left(\frac{300}{400} - \frac{1}{2}\right)$

$= \frac{5}{4} - \frac{1}{2} - \frac{3}{4} + \frac{1}{2}$

$= \frac{5-3}{4} = \frac{2}{4} = \boxed{\frac{1}{2}}$

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4) The probability density function of x is given by

$$f(x) = \begin{cases} ke^{-\frac{x}{3}}, & \text{for } x > 0 \\ 0, & \text{for } x \le 0 \end{cases}$$

Find (i) The value of k
(ii) The distribution function
(iii) $P(X < 3)$
(iv) $P(5 \le x)$ (v) $P(x \le 4)$

Given $f(x)$ is probability density function

$$\therefore \int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = 1$$

$$0 + \int_{0}^{\infty} ke^{-\frac{x}{3}}dx = 1$$

$$k\left[\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}}\right]_0^{\infty} = 1$$

$$-3k\left[e^{-\frac{x}{3}}\right]_0^{\infty} = 1$$

$$\text{i.e. } -3k\left[e^{-\infty} - e^{-0}\right] = 1$$

$$-3k[0-1] = 1$$

$$3k = 1 \Rightarrow \boxed{k = \frac{1}{3}}$$

(ii) The distribution function

$$F(x) = P(x \le x) = \int_{-\infty}^{x} f(u)du$$

when $x < 0$

$$F(x) = \int_{-\infty}^{0} f(u)du = 0.$$

when $x > 0$

$$F(x) = \int_{-\infty}^{x} f(u)du$$

$$= \int_{-\infty}^{0} f(u)du + \int_{0}^{x} f(u)du$$

$$= 0 + \int_{0}^{x} f(u)du$$

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$$= \int_0^x k e^{-\frac{u}{3}} du = \frac{1}{3}\int_0^x e^{-\frac{u}{3}} du$$

$$= \frac{1}{3}\left[\frac{e^{-u/3}}{-\frac{1}{3}}\right]_0^x$$

$$= -\left[e^{-u/3}\right]_0^x = -\left[e^{-\frac{x}{3}} - e^0\right]$$

$$= -e^{-\frac{x}{3}} + e^0$$

$$= 1 - \frac{1}{e^{x/3}} //$$

$$\therefore F(x) = \begin{cases} 0 & x \le 0 \\ 1 - e^{-x/3}, & x > 0 \end{cases}$$

(iii) $P(X<3) = \int_0^3 f(x)\,dx$

$$= \int_0^3 k e^{-\frac{x}{3}} dx$$

$$= \frac{1}{3}\int_0^3 e^{-x/3} dx$$

$$= \frac{1}{3}\left[\frac{e^{-x/3}}{-\frac{1}{3}}\right]_0^3$$

$$= \left[-e^{-x/3}\right]_0^3$$

$$= -\left[e^{-3/3} - e^0\right]$$

$$= -\left[e^{-1} - 1\right]$$

$$= \boxed{1 - \frac{1}{e}} \text{ (OR)}$$

$P(X<3) = P(X \le 3)$

$= F(3)_{-3/3}$

$= 1 - e$

$= 1 - e^{-1} = \boxed{1 - \frac{1}{e}}$

(iv) $P(5 \le X) = P(X \ge 5)$

$= 1 - P(X<5)$

$= 1 - F(5)$

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$= 1-(1-e^{-5/3})$

$= 1-1+e^{-5/3} = \boxed{e^{-5/3}}$

(v) $P(X \le 4) = F(4)$

$= 1-e^{-4/3}$

---

5) If X is the random variable with probability density function f(x) given by

$$f(x) = \begin{cases} x+1, & -1 \le x < 0 \\ -x+1, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

then find (i) the distribution function F(x) (ii) $P(0.5 \le X \le 0.5)$

Given f(x) is probability density function defined by

$$f(x) = \begin{cases} x+1, & -1 \le x < 0 \\ -x+1, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(i) The distribution function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\,du$$

when $x \le -1$

$$F(x) = \int_{-\infty}^{x} f(u)\,du = 0$$

$$\boxed{F(x) = 0}$$

When $-1 \le x < 0$

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

$$= \int_{-\infty}^{-1} f(u)\,du + \int_{-1}^{x} f(u)\,du$$

$$= 0 + \int_{-1}^{x} (u+1)\,du$$

$$= \left[\frac{(u+1)^2}{2}\right]_{-1}^{x} = \frac{(x+1)^2}{2} - 0$$

$$\therefore F(x) = \boxed{\frac{(x+1)^2}{2}}$$

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When $0 \le x < 1$, $F(x) = \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{0} f(x)dx + \int_{0}^{x} f(x)dx$

$= 0 + \int_{-1}^{0} (x+1)dx + \int_{0}^{x} (-x+1)dx$

$= \left[\frac{x^2}{2} + x\right]_{-1}^{0} + \left[-\frac{x^2}{2} + x\right]_{0}^{x}$

$= (0) - \left[\frac{1}{2} - 1\right] + \left[\left(-\frac{x^2}{2} + x\right) - \left(-\frac{0}{2} + 0\right)\right]$

$= -\left(-\frac{1}{2}\right) + \left(-\frac{x^2}{2} + x - 0\right)$

$= \frac{1}{2} - \frac{x^2}{2} + x - 0$

$= -\frac{x^2}{2} + x + \frac{1}{2}$

When $x \geq 1$

$F(x) = \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx$

$0 + \int_{-1}^{0} (x+1)dx + \int_{0}^{1} (-x+1)dx + 0$

$= \left[\frac{x^2}{2} + x\right]_{-1}^{0} + \left[-\frac{x^2}{2} + x\right]_{0}^{1}$

$0 - \left(\frac{1}{2} - 1\right) + \left[\left(-\frac{1}{2} + 1\right) - (0)\right]$

$\frac{1}{2} + \frac{1}{2} = \boxed{1}$

Hence the distribution function is

$$F(x) = \begin{cases} 0 & x \le -1 \\ \frac{(x+1)^2}{2} & -1 \le x < 0 \\ -\frac{x^2}{2} + x + \frac{1}{2}, & 0 \le x < 1 \\ 0 & x \geq 1 \end{cases}$$

(ii) $P(-0.5 \le X \le 0.5) = F(0.5) - F(-0.5)$

$= \left[-\frac{(0.5)^2}{2} + 0.5 + 0.5\right] - \left[\frac{(-0.5+1)^2}{2}\right]$

$= -\frac{0.25}{2} + 1 - \left(\frac{(0.5)^2}{2}\right)$

$= -\frac{0.25}{2} + 1 - \frac{0.25}{2} = 1 - 0.25$

$= \boxed{0.75}$

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6) If X is the random variable with distribution function $F(x)$ is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}(x^2+x), & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$ find

(i) the probability density function $f(x)$

(ii) $P(0.3 \le X \le 0.6)$

Given

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}(x^2+x), & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

(i) The probability density function.

Soln:

$$\text{p.d.f } f(x) = \frac{d}{dx}(F(x))$$

∴ probability density function

$$f(x) = \begin{cases} \frac{1}{2}(2x+1), & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(ii) $P(0.3 \le X \le 0.6)$

$= F(0.6) - F(0.3)$

$= \frac{1}{2}((0.6)^2+0.6) - \frac{1}{2}((0.3)^2+0.3)$

$= \frac{1}{2}[(0.36+0.6) - (0.09+0.3)]$

$= \frac{1}{2}[0.96 - 0.39]$

$= \frac{1}{2}(0.57)$

$= 0.285$

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MATHEMATICAL EXPECTATION

Mean: Suppose x is a random variable with probability mass or density function $f(x)$. The expected value or mean or mathematical expectation of x, denoted by $E(x)$ or μ is

$$E(x) = \begin{cases} \sum x f(x) & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} x f(x)\,dx & \text{if } x \text{ is continuous} \end{cases}$$

Variance: The variance of a random variable x denoted by $Var(x)$ or $V(x)$ or σ^2 or σ_x^2 is

$$V(x) = E\left(x - E(x)\right)^2 = E(x-\mu)^2$$

properties:

(1) $E(ax+b) = a\,E(x) + b$

2) (a) $E(ax) = a\,E(x)$

b) $E(b) = b$

2) $Var(x) = E(x^2) - [E(x)]^2$

3) $Var(ax+b) = a^2\,Var(x)$

(a) $Var(ax) = a^2\,Var(x)$

b) $Var(b) = 0$

Ex (11.16) Suppose that $f(x)$ given below represents a probability mass function

x	1	2	3	4	5	6
$f(x)$	c^2	$2c^2$	$3c^2$	$4c^2$	c	$2c$

find (i) the value of c

(ii) Mean and Variance

Given $f(x)$ is probability mass function

$\therefore f(x) \geq 0$ and $\sum f(x) = 1$

i $c^2 + 2c^2 + 3c^2 + 4c^2 + c + 2c = 1$

$10c^2 + 3c - 1 = 0$

$(2c+1)(5c-1) = 0$

$c = -\frac{1}{2},\ c = \frac{1}{5}$

-10

$\frac{5}{10}\ \ \frac{-2}{10}$

$\frac{1}{2},\ -\frac{1}{5}$

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$f(x) \geq 0 \quad \forall x \quad \therefore \boxed{C = \frac{1}{5}}$

Hence the probability Mass function is

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{25}$	$\frac{2}{25}$	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{1}{5}$	$\frac{2}{5}$

(ii) To find mean and variance, use the following table.

x	$f(x)$	$xf(x)$	$x^2 f(x)$
1	$\frac{1}{25}$	$1/25$	$1/25$
2	$\frac{2}{25}$	$4/25$	$8/25$
3	$3/25$	$9/25$	$27/25$
4	$4/25$	$16/25$	$64/25$
5	$1/5$	$5/5$	$25/5$
6	$2/5$	$12/5$	$72/5$

$$\sum f(x) = \frac{1}{25} + \frac{2}{25} + \frac{3}{25} + \frac{4}{25} + \frac{1}{5} + \frac{2}{5}$$

$$= \frac{1+2+3+4+5+10}{25}$$

$$= \frac{25}{25} = \boxed{1}$$

$$\sum xf(x) = \frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25} + \frac{5}{5} + \frac{12}{5}$$

$$= \frac{1+4+9+16+25+60}{25}$$

$$= \frac{115}{25} = \boxed{4.6}$$

$$\sum x^2 f(x) = \frac{1}{25} + \frac{8}{25} + \frac{27}{25} + \frac{64}{25} + \frac{25}{5} + \frac{72}{5}$$

$$= \frac{1+8+27+64+125+360}{25}$$

$$= \boxed{\frac{585}{25}}$$

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$$\text{Mean} = E(x) = \sum x f(x)$$

$$= \frac{115}{25} = \boxed{4.6}$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= \sum x^2 f(x) - \left[\sum x f(x)\right]^2$$

$$= \frac{585}{25} - \left(\frac{115}{25}\right)^2$$

$$= \frac{585}{25} - \frac{13225}{625} \quad ⊛$$

$$= 23.40 - 21.16 = 2.24$$

$\therefore$ mean $= 4.6$

variance $= 2.24$

⊛

Note

$$\begin{array}{r} 115 \times 115 \\ \hline 575 \\ 115 \\ 115 \\ \hline 13225 \end{array}$$

$$\begin{array}{r} 2.34 \\ 25\overline{)585} \\ 50 \\ \hline 85 \\ 75 \\ \hline 100 \\ 100 \\ \hline 0 \end{array} \qquad \begin{array}{r} 21.16 \\ 625\overline{)13225} \\ 1250 \\ \hline 725 \\ 625 \\ \hline 1000 \\ 625 \\ \hline 3750 \\ 3750 \\ \hline 0 \end{array}$$

11.17 Two balls are choosen randomly from an urn containing 8 white and 4 black balls. Suppose that we win Rs 20 for each black ball selected and we lose Rs 10 for each white ball selected. Find the expected winning amount and variance.

white – 8
black – 4
Total – 12

$$\therefore n(S) = {}^{12}C_2 = \frac{12 \times 11}{1 \times 2} = \boxed{66}$$

For each black ball, gain = Rs 20

For each white ball, lose = Rs 10

Let x denote the winning amount

x (both are black) $= 2 \times 20 = ₹40$

x (one black and one white) $= ₹20 - ₹10 = ₹10$

x (both are white) $= -₹10 \times 2 = ₹-20$

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$\therefore$ X takes the values 40, 10, −20

No of ways selecting 2 black balls $= {}^4C_2$

$= \frac{4 \times 3}{1 \times 2} = \boxed{6}$

No of ways selecting one black and one white

$= {}^4C_1 \times {}^8C_1$

$= 4 \times 8 = \boxed{32}$

No of ways of selecting two white balls

$= {}^8C_2 = \frac{8 \times 7}{1 \times 2} = \boxed{28}$

Values of X	40	10	−20	Total
No of elements in inverse image	6	32	28	66

probability mass function

when $X = 40$, $f(x) = P(X = 40)$

$= P$(both are black)

$=$ No of ways of selecting 2 black balls / Total number of ways of selecting two balls

$= \boxed{\frac{6}{66}}$

when $X = 10$, $f(x) = P(X = 10)$

$= \boxed{\frac{32}{66}}$

when $X = -20$, $f(x) = P(X = -20)$

$= \boxed{\frac{28}{66}}$

$\therefore$ probability Mass function

x	40	10	−20	Total
f(x)	$\frac{6}{66}$	$\frac{32}{66}$	$\frac{28}{66}$	1

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Mean : $E(X) = \sum x f(x)$

$= 40\left(\frac{6}{66}\right) + 10\left(\frac{32}{66}\right) + (-20)\left(\frac{28}{66}\right)$

$= \frac{240}{66} + \frac{320}{66} - \frac{560}{66}$

$= \frac{560}{66} - \frac{560}{66} = 0.$

∴ The expected winning amount is $\boxed{0}$

Variance: $E(X^2) = \sum x^2 f(x)$

$= 40^2\left(\frac{6}{66}\right) + 10^2\left(\frac{32}{66}\right) + (-20)^2\left(\frac{28}{66}\right)$

$= 1600\left(\frac{6}{66}\right) + 100\left(\frac{32}{66}\right) + 400\left(\frac{28}{66}\right)$

$= \frac{9600 + 3200 + 11200}{66} = \frac{24000}{66}$

$= \frac{4000}{11}$

$E(X^2) = \frac{4000}{11}$

Variance: $= E(X^2) - [E(X)]^2$

$= \frac{4000}{11} - 0 = \boxed{\frac{4000}{11}}$

Ex 11.18 Find the mean and variance of a random variable X, whose probability density function is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Mean:- $\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_{-\infty}^{0} x f(x)\,dx + \int_{0}^{\infty} x f(x)\,dx$

$= 0 + \int_{0}^{\infty} x\left(\lambda e^{-\lambda x}\right)dx$

$= \lambda \int_{0}^{\infty} x e^{-\lambda x}\,dx$

Gamma Integ: $\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{\lfloor n}{\alpha^{n+1}}$

$= \lambda \times \frac{\lfloor 1}{\lambda^{1+1}} = \frac{\lambda}{\lambda^2} = \boxed{\frac{1}{\lambda}}$

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Variance: $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$= \int_{-\infty}^{0} x^2 f(x)\,dx + \int_{0}^{\infty} x^2 f(x)\,dx$

$= 0 + \int_{0}^{\infty} x^2 (\lambda e^{-\lambda x})\,dx$

$= \lambda \int_{0}^{\infty} x^2 e^{-\lambda x}\,dx = \lambda \frac{L^2}{\lambda^{2+1}} = \boxed{\frac{2}{\lambda^2}}$

$\therefore$ Var(x) $= E(x^2) - [E(x)]^2$

$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$

Hence mean $= \frac{1}{\lambda}$ and

variance $= \frac{1}{\lambda^2}$

EXERCISE 11.4

1) For the random variable X with the given probability mass function as below, find the mean and variance

(i) $f(x) = \begin{cases} \frac{1}{10}, & x = 2, 5 \\ \frac{1}{5}, & x = 0, 1, 3, 4 \end{cases}$

Given,

x	0	1	2	3	4	5
f(x)	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$

Mean $= \sum x f(x)$

$= 0\left(\frac{1}{5}\right) + 1\left(\frac{1}{5}\right) + 2\left(\frac{1}{10}\right) + 3\left(\frac{1}{5}\right) + 4\left(\frac{1}{5}\right) + 5\left(\frac{1}{10}\right)$

$= \frac{1}{5} + \frac{2}{10} + \frac{3}{5} + \frac{4}{5} + \frac{5}{10}$

$= \frac{2+2+6+8+5}{10} = \frac{23}{10} = \boxed{2.3}$

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Variance: $E(x^2) = \sum x^2 f(x)$

$= 0\left(\frac{1}{5}\right) + 1^2\left(\frac{1}{5}\right) + 4\left(\frac{1}{5}\right) + 9\left(\frac{1}{5}\right) + 16\left(\frac{1}{5}\right) + 25\left(\frac{1}{10}\right)$

$= 0 + \frac{1}{5} + \frac{4}{5} + \frac{9}{5} + \frac{16}{5} + \frac{25}{10}$

$= \frac{0+2+4+18+32+25}{10} = \frac{81}{10} = \boxed{8.1}$

$\therefore$ Variance $= E(x^2) - [E(x)]^2$

$= 8.1 - (2.3)^2$

$= 8.1 - 5.29 = \boxed{2.81}$

Hence Mean $= 2.3$

Variance $= 2.81$

(ii) $f(x) = \begin{cases} \frac{4-x}{6}, & x = 1, 2, 3 \end{cases}$

Given

x	1	2	3
$f(x)$	$3/6$	$2/6$	$1/6$

Mean $E(x) = \sum x f(x)$

$= 1\left(\frac{3}{6}\right) + 2\left(\frac{2}{6}\right) + 3\left(\frac{1}{6}\right)$

$= \frac{3}{6} + \frac{4}{6} + \frac{3}{6} = \frac{10}{6} = \frac{5}{3}$

$= \boxed{1.67}$

Variance: $E(x^2) = \sum x^2 f(x)$

$= 1^2\left(\frac{3}{6}\right) + 4\left(\frac{2}{6}\right) + 9\left(\frac{1}{6}\right)$

$= \frac{3}{6} + \frac{8}{6} + \frac{9}{6} = \boxed{\frac{20}{6}}$

Variance $= E(x^2) - [E(x)]^2$

$= \frac{20}{6} - \left(\frac{5}{3}\right)^2$

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$= \frac{20}{6} - \frac{25}{9} = \frac{60-50}{18}$

$= \frac{10}{18} = \frac{5}{9} = 0.555\ldots$

$= \boxed{0.56}$

(iii) $f(x) = \begin{cases} 2(x-1) & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

Mean: $E(X) = \int_1^2 x f(x)\,dx = \int_1^2 x \cdot 2(x-1)dx$

$= 2\int_1^2 (x^2 - x)dx$

$= 2\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$

$= 2\left[\left(\frac{8}{3} - \frac{4}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right]$

$= 2\left[\left(\frac{8}{3} - 2\right) - \left(\frac{2-3}{6}\right)\right]$

$= 2\left[\left(\frac{8-6}{3}\right) - \left(-\frac{1}{6}\right)\right]$

$= 2\left[\frac{2}{3} + \frac{1}{6}\right] = 2\left[\frac{4+1}{6}\right]$

$= 2\left[\frac{5}{6}\right] = \boxed{\frac{5}{3}}$

Variance $= E(x^2) - [E(x)]^2$

$E(x^2) = \int_1^2 x^2 f(x)\,dx$

$= \int_1^2 2 \cdot x^2 (x-1)\,dx = 2\int_1^2 (x^3 - x^2)dx$

$= 2\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2$

$= 2\left[\left(\frac{16}{4} - \frac{8}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right]$

$= 2\left[\left(4 - \frac{8}{3}\right) - \left(\frac{3-4}{12}\right)\right]$

$= 2\left[\frac{4}{3} - \left(-\frac{1}{12}\right)\right] = 2\left[\frac{4}{3} + \frac{1}{12}\right]$

$= 2\left[\frac{16+1}{12}\right]$

$= \boxed{\frac{17}{6}}$

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Variance : $E(x^2) - [E(x)]^2$

$= \frac{17}{6} - \left(\frac{5}{3}\right)^2$

$= \frac{17}{6} - \frac{25}{9} \Rightarrow \frac{51-50}{18} = \boxed{\frac{1}{18}}$

(iv) $$f(x) = \begin{cases} \frac{1}{2} e^{-x/2} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Mean $= E(x) = \int_0^\infty x f(x) dx$

$= \frac{1}{2}\int_0^\infty x e^{-\frac{x}{2}} dx = \frac{1}{2}\left[\frac{1!}{\left(\frac{1}{2}\right)^{1+1}}\right]$

$= \frac{1}{2}\left[\frac{1}{\frac{1}{4}}\right]$

$= \frac{1}{2} \times 4 = \boxed{2}$

Variance: $E(x^2) - [E(x)]^2$

$E(x^2) = \int_0^\infty x^2 f(x) dx$

$= \frac{1}{2}\int_0^\infty x^2 e^{-x/2} dx$

$= \frac{1}{2} \frac{2!}{\left(\frac{1}{2}\right)^{2+1}} = \frac{1}{2} \times \frac{2}{\frac{1}{2^3}}$

$= \frac{1}{2} \times 2 \times 8 = \boxed{8}$

$\therefore$ Variance $= E(x^2) - [E(x)]^2$

$= 8 - 2^2 = 8 - 4 = \boxed{4}$

$\therefore$ mean $= 2$, variance $= 4$

---

2) Two balls are drawn in succession without replacement from an urn containing four red balls and three black balls. Let x be the possible outcomes drawing red balls. Find the probability mass function and mean for X.

Red = 4
black = 3
Total = 7

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Two balls are drawn succession without replacement.

$\therefore n(S) = {}^7C_2 = \frac{7\times6}{1\times2} = \boxed{21}$

Let x denote the drawing red balls clearly x takes values 0, 1, 2.

No of ways of selecting 2 red balls

$= {}^4C_2 = \frac{4\times3}{1\times2} = \boxed{6}$

No of ways of selecting 1 Red 1 black

$= {}^4C_1 \times {}^3C_1 = 4\times3$

$= \boxed{12}$

No of ways of selecting 2 black balls

$= {}^3C_2 = \frac{3\times2}{1\times2} = \boxed{3}$

Value of X	0	1	2	Total
No of elements in inverse image	3	12	6	21

probabilities are

$f(0) = p(X=0) = \frac{3}{21} = \frac{1}{7}$

$f(1) = p(X=1) = \frac{12}{21} = \frac{4}{7}$

$f(2) = p(X=2) = \frac{6}{21} = \frac{2}{7}$

$\therefore$ probability Mass function

x	0	1	2
f(x)	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{2}{7}$

Mean $= E(X) = \sum xf(x)$

$= 0\left(\frac{1}{7}\right) + 1\left(\frac{4}{7}\right) + 2\left(\frac{2}{7}\right)$

$= \frac{4}{7} + \frac{4}{7} = \boxed{\frac{8}{7}}$

3) If μ and σ^2 are mean and Variance of the discrete random Variable X, and $E(X+3) = 10$ and $E(X+3)^2 = 116$ find μ and variance.

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Given

$E(X+3) = 10$

$E(X) + 3 = 10$

$E(X) = 7$

$\therefore$ Mean $E(X) = 7$

$E(X+3)^2 = 116$

$E[X^2 + 6X + 9] = 116$

$E(X^2) + 6E(X) + 9 = 116$

$E(X^2) + 6E(X) = 116 - 9$

$E(X^2) + 6E(X) = 107$

$E(X^2) + 6(7) = 107$

$E(X^2) = 107 - 42$

$= 65$

$\therefore$ Variance $Var(X) = E(X^2) - [E(X)]^2$

$= 65 - 49$

$= 16$

$\therefore$ Mean $\mu = 7$

Variance $\sigma^2 = 16$

---

4) Four fair coins are tossed once. Find the probability mass function, mean and variance for number of head occured.

XII / CH11 / 11.4 / P11

Given four coins are tossed once.

$n(S) = 2^4 = 16$

Let X denote the no of heads.

clearly X can take values 0, 1, 2, 3, 4.

X (No heads) : $4C_0 = 1$

X (one head) = $4C_1 = 4$

X (Two head) = $4C_2 = \frac{4 \times 3}{1 \times 2} = 6$

X (Three heads) = $4C_3 = 4C_1 = 4$

X (4 heads) = $4C_4 = 1$

Value of X	0	1	2	3	4
No of elements in inverse image	1	4	6	4	1

probabilities are

$p(X=0) = f(0) = 1/16$

$f(1) = p(X=1) = 4/16$

$f(2) = p(X=2) = 6/16$

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$f(3) = \frac{4}{16}$

$f(4) = \frac{1}{16}$

probability Mass function

x	0	1	2	3	4
$f(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Mean $E(x) = \sum x f(x)$

$= 0\left(\frac{1}{16}\right) + 1\left(\frac{4}{16}\right) + 2\left(\frac{6}{16}\right) + 3\left(\frac{4}{16}\right) + 4\left(\frac{1}{16}\right)$

$= 0 + \frac{4}{16} + \frac{12}{16} + \frac{12}{16} + \frac{4}{16}$

$= \frac{4+12+12+4}{16} = \frac{32}{6} = \boxed{2}$

Variance: $E(x^2) - E(x)$

$E(x^2) = \sum x^2 f(x)$

$= 0\left(\frac{1}{16}\right) + 1^2\left(\frac{4}{16}\right) + 2^2\left(\frac{6}{16}\right) + 3^2\left(\frac{4}{16}\right) + 4^2\left(\frac{1}{16}\right)$

$= 0 + \frac{4}{16} + \frac{24}{16} + \frac{36}{16} + \frac{16}{16}$

$= \frac{4+24+36+16}{16} = \frac{80}{16}$

$= \boxed{5}$

∴ Variance: $Var(x) = E(x^2) - [E(x)]^2$

$= 5 - 2^2$

$= 5 - 4$

$= \boxed{1}$

∴ Mean = 2, Variance = 1

---

5) A commuter train arrives punctually at a station every half hour. Each morning, a student leaves his house to the train station. Let X denote the amount of time, in minutes that the student waits for the train from the time he reaches the train station. If

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Known that the pdf of X

$$f(x) = \begin{cases} 1/30, & 0 < x < 30 \\ 0 & \text{elsewhere} \end{cases}$$

Obtain and interpret the expected value of the random variable X.

Expected Value of X = E(X)

$$= \int_0^{30} x f(x)\,dx$$

$$= \int_0^{30} \frac{1}{30} x\,dx$$

$$= \frac{1}{30}\int_0^{30} x\,dx$$

$$= \frac{1}{30}\left[\frac{x^2}{2}\right]_0^{30}$$

$$= \frac{1}{30}\left[\frac{30\times 30 - 0}{2}\right]$$

$$= \boxed{\frac{30}{2}} = 15 \text{ minutes}$$

6) The time to failure in thousands of hours of an electronic equipment used in a manufactured computer has density function.

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find the expected life of this electronic equipment.

Given pdf is

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Expected life $E(x) = \int_0^{\infty} x f(x)\,dx$

$$= 3\int_0^{\infty} x e^{-3x}\,dx = 3\,\frac{11}{3^{1+1}}$$

$$= \frac{3\times 1}{3^2} = \boxed{\frac{1}{3}}$$

∴ Expected life is $\boxed{\frac{1}{3}}$

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7) The probability density function of the random variable x is given by

$$f(x) = \begin{cases} 16xe^{-4x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

find the mean and variance of X

Mean $E(X) = \int_{-\infty}^{\infty} x f(x) dx$

$= \int_{-\infty}^{0} x f(x) dx + \int_{0}^{\infty} x f(x) dx$

$= 0 + \int_{0}^{\infty} x \, 16x \, e^{-4x} dx.$

$= 16 \int x^2 e^{-4x} dx$

$= 16 \times \frac{2!}{4^3}$

$= \cancel{16}^{4} \times \frac{1 \times 2}{\cancel{4} \times \cancel{4} \times 4} = \frac{2}{4} = \boxed{\frac{1}{2}}$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = 16 \int_{0}^{\infty} x^3 e^{-4x} dx$

$= 16 \times \frac{3!}{4^4} = \frac{16 \times 1 \times \cancel{2} \times 3}{\cancel{4} \times \cancel{4} \times \cancel{4}_{2} \times 4}$

$= \boxed{\frac{3}{8}}$

Variance $= E(X^2) - [E(X)]^2$

$= \frac{3}{8} - \left(\frac{1}{2}\right)^2$

$= \frac{3}{8} - \frac{1}{4} = \frac{3-2}{8}$

$= \boxed{\frac{1}{8}}$

∴ mean $= 3/8$

Variance $= 1/8$

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8) A lottery with 600 tickets gives one prize of ₹200, four prizes of ₹100, and six prizes of ₹50. If the ticket costs is ₹2, find the expected winning amount of a ticket.

$n(S) = 600$

Let the random variable x denotes the winning amount.

$X(200) = 1$

$X(100) = 4$

$X(50) = 6$

$X(0) = 589$

Value of X	0	50	100	200	Total
No of elements in inverse image	589	6	4	1	600

probabilities are

$f(200) = P(X=200) = \frac{1}{600}$

$f(100) = P(X=100) = \frac{4}{600}$

$f(50) = P(X=50) = \frac{6}{600}$

$f(0) = P(X=0) = \frac{589}{600}$.

∴ probability Mass function

x	200	100	50	0
f(x)	1/600	4/600	6/600	589/600

Expected Value $E(X) = \sum x f(x)$

$= 200\left(\frac{1}{600}\right) + 100\left(\frac{4}{600}\right) + 50\left(\frac{6}{600}\right) + 0\left(\frac{589}{600}\right)$

$= \frac{200}{600} + \frac{400}{600} + \frac{300}{600} + 0$.

$= \frac{900}{600} = \boxed{\frac{3}{2}} = ₹ 1.5$

∴ Expected Value = ₹ 1.50 p

Cost of a ticket = ₹ 2.

∴ Loss = CP − E.V

= 2 − 1.50 = 0.5

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Binomial distribution

A Random variable X is said to follow binomial distribution if its probability mass function is given by

$$f(x) = p(X=x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0,1,2 \cdots n$$

(or)

$$f(x) = p(X=x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0,1,2,3 \cdots n$$

where $q = 1-p$

$p \rightarrow$ probability of success

$n \rightarrow$ number of trails

1) mean $= np$
2) Variance $= np(1-p)$ or npq
3) Standard deviation $= \sqrt{npq}$

Note: In a binomial distribution, mean is always greater than variance.

Ex 11.19 Find the binomial distribution function for each of the following

(i) Five fair coins are tossed once and X denotes the number of heads

(ii) A fair die is rolled 10 times and X denotes the number of times 4 appeared.

Given five coins are tossed once.

Here probability of getting head in single coin is $p = \frac{1}{2}$, $q = 1-p = 1-\frac{1}{2} = \frac{1}{2}$

X denotes the no of heads appears in 5 coins.

X takes values 0, 1, 2, 3, 4 and 5.

Here $p = \frac{1}{2}$, $q = \frac{1}{2}$, $n = 5$

∴ Binomial distribution is

$$f(x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0,1,2 \cdots n$$

$$f(x) = \binom{5}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{n-x}, \quad x = 0,1,2 \cdots 5.$$

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$\therefore f(x) = \binom{5}{x}\left(\frac{1}{2}\right)^n$, $x = 0,1,2,3,4,5$.

(ii) Give a fair die is rolled 10 times x denotes the number of times 4 appeared clearly x takes values $0,1,2 \ldots 10$.

Here $n = 10$, $p = \frac{1}{6}$, $q = 1-p$
$= 1 - \frac{1}{6} = \boxed{\frac{5}{6}}$

$\therefore$ The binomial distn is

$$f(x) = \binom{10}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{10-x}, \quad x = 0,1,2 \ldots 10$$

Ex 11.20 A multiple cerice examination has ten questions, each question has four distractors with exactly one correct answer. Suppose a student answers by guessing and it x denotes the number of correct answers, find

(i) binomial distn.

(ii) probability that the student will get seven correct answers

(iii) the probability of getting atleast one correct answers.

(i) x denotes the number of success.

clearly x can take the value $0,1,2,3 \ldots 10$.

The probability of success $p = \frac{1}{4}$

$\therefore q = 1-p = 1 - \frac{1}{4} = \frac{3}{4}$, $n = 10$

$\therefore$ The binomial distn is

$$f(x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0,1,2 \ldots n$$

$$f(x) = \binom{10}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{10-x}, \quad x = 0,1,2 \ldots 10$$

(ii) probability for seven correct answer

$$P(X=7) = f(7) = \binom{10}{7}\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^3$$

$$= {}^{10}C_3 \; \frac{1}{4^7} \times \frac{3^3}{4^3}$$

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$$\frac{10 \times 9 \times 8}{1 \times 2 \times 3} \times \frac{1}{4^7} \times \frac{3^3}{4^3} = 120 \times \frac{3^3}{4^{10}}$$

(OR)

$$30 \times \frac{3^3}{4^9}$$

(iii) probability for atleast one correct answer is

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - P(X = 0)$$

$$= 1 - \binom{10}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{10-0}$$

$$= 1 - {}^{10}C_0 \times 1 \times \frac{3^{10}}{4^{10}}$$

$$= 1 - \left(\frac{3}{4}\right)^{10}$$ //

Ex: 11.21 The mean and variance of a binomial variate x are respectively 2 and 1.5. Find (i) $P(X=0)$, (ii) $P(X=1)$ (iii) $P(X \geq 1)$

Given mean $np = 2$ — ①

Variance $npq = 1.5$ — ②

using ① in ② $2 \times q = 1.5$

$$q = \frac{1.5}{2} = 0.75 = \boxed{3/4}$$

WKT $p = 1 - q$

$= 1 - 3/4 = \boxed{1/4}$

From ① $np = 2$

i.e. $n(\frac{1}{4}) = 2 \Rightarrow \boxed{n = 8}$

∴ Binomial distn is

$$f(x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, 2 \cdots n$$

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Here $p = \frac{1}{4}$, $q = \frac{3}{4}$, $n = 8$

$\therefore P(X = x) = f(x) = \binom{8}{x}\left(\frac{1}{4}\right)^{x}\left(\frac{3}{4}\right)^{8-x}$

$x = 0, 1, 2 \cdots 8$

(i) $P(X=0) = \binom{8}{0}\left(\frac{1}{4}\right)^{0}\left(\frac{3}{4}\right)^{8-0}$

$= 1 \times 1 \times \left(\frac{3}{4}\right)^{8} = \boxed{\left(\frac{3}{4}\right)^{8}}$

(ii) $P(X=1) = f(1) = {}^{8}C_{1}\left(\frac{1}{4}\right)^{1}\left(\frac{3}{4}\right)^{7}$

$= 8 \times \frac{1}{4} \times \left(\frac{3}{4}\right)^{7}$

$= \underline{\underline{2 \times \left(\frac{3}{4}\right)^{7}}}$

(iii) $P(X \geq 1) = 1 - P(X < 1)$

$= 1 - P(X=0)$

$= 1 - \left(\frac{3}{4}\right)^{8}$

Ex 11.22 On the average, 20% of the product manufactured by ABC company are found to be defective. If we select 6 of these products at random and x denote the number of defective products find the probability that (i) two product are defective (ii) at most one product is defective (iii) atleast two products are defective.

Here $n = 6$, $p = 20\%$ defective

$p = \frac{20}{100} = \frac{1}{5}$

$\therefore q = 1 - p = 1 - \frac{1}{5}$

$= \boxed{\frac{4}{5}}$

Let x denote number of defective product.

clearly x takes values $0, 1, 2 \cdots 6$.

$\therefore$ Binomial distn is

$P(X = x) = \binom{n}{x} p^{x} q^{n-x}$, $x = 0, 1 \cdots n$

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$$P(X=x) = \binom{6}{x}\left(\frac{1}{5}\right)^{x}\left(\frac{4}{5}\right)^{6-x}, \quad x = 0, 1, \ldots 6.$$

(i) probability of two product is defective

(a) $P(X=2) = f(2)$

$$= {}^{6}C_{2}\left(\frac{1}{5}\right)^{2}\left(\frac{4}{5}\right)^{6-2}$$

$$= \frac{6\times5}{1\times2} \times \frac{1}{5^2} \times \frac{4^4}{5^4}$$

$$= \frac{3\times4^4}{5^5} = \frac{3\times4\times4\times4\times4}{5\times5\times5\times5\times5}$$

$$= \frac{64\times12}{25\times25\times5} = \frac{768}{625\times5}$$

$$= \boxed{\frac{768}{3125}}$$

(ii) probability for atmost one defective product is

$$P(X\le1) = P(X=0) + P(X=1)$$

$$= {}^{6}C_{0}\left(\frac{1}{5}\right)^{0}\left(\frac{4}{5}\right)^{6-0} + {}^{6}C_{1}\left(\frac{1}{5}\right)^{1}\left(\frac{4}{5}\right)^{5}$$

$$= 1\times1\times\left(\frac{4}{5}\right)^{6} + 6\times\frac{1}{5}\times\left(\frac{4}{5}\right)^{5}$$

$$= \left(\frac{4}{5}\right)^{5}\left[\frac{4}{5} + \frac{6}{5}\right]$$

$$= \left(\frac{4}{5}\right)^{5}\frac{10}{5} = \left(\frac{4}{5}\right)^{5}\times2$$

$$2\times\frac{2^{10}}{3125} = \frac{1024}{3125}\times2$$

$$= \frac{2048}{3125}$$

(iii) $P(X\ge2) = 1 - P(X<2)$

$$= 1 - P(X\le1)$$

$$= 1 - \frac{2048}{3125} = \frac{3125-2048}{3125}$$

$$= \frac{1077}{3125}$$

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EXERCISE 11.5

1) Compute $P(X=k)$ for the binomial distribution $B(n,p)$ where

(i) $n=6,\ p=\frac{1}{3},\ k=3$

Binomial distn:

$$P(X=k) = \binom{n}{k} p^k q^{n-k},\quad k=0,1,2\ldots n.$$

$$P(X=3) = {}^6C_3 \left(\frac{1}{3}\right)^3 \left(1-\frac{1}{3}\right)^{6-3}$$

$$= \frac{6\times5\times4}{1\times2\times3} \times \frac{1}{27} \times \left(\frac{2}{3}\right)^3$$

$$= \frac{6\times5\times4}{1\times2\times3} \times \frac{1}{27} \times \frac{2\times2\times2}{3\times3\times3}$$

$$= \frac{32\times5}{27\times27} = \boxed{\frac{160}{729}}$$

(ii) $n=10,\ p=\frac{1}{5},\ k=4$

$\therefore q = 1-p$

$= 1-\frac{1}{5} = \frac{4}{5}$

Binomial distn

$$P(X=k) = {}^nC_k\, p^k\, q^{n-k},\quad k=0,1\ldots n$$

$$P(X=4) = {}^{10}C_4 \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^{10-4}$$

$$= \frac{\overset{5}{10}\times\overset{3}{9}\times\overset{2}{8}\times7}{1\times2\times8\times4} \times \frac{1}{5^4} \times \frac{4^6}{5^6}$$

$$= 210 \times \frac{4^6}{5^{10}} \quad //$$

(iii) $n=9,\ p=\frac{1}{2},\ k=7$

Binomial distribution

$$P(X=k) = \binom{n}{k} p^k q^{n-k},\quad k=0,1\ldots n$$

$$P(X=7) = {}^9C_7 \left(\frac{1}{2}\right)^7 \left(1-\frac{1}{2}\right)^{9-7}$$

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$= {}^{9}C_{2} \left(\frac{1}{2}\right)^{7} \left(\frac{1}{2}\right)^{2}$

$= \frac{9 \times 8}{1 \times 2} \times \left(\frac{1}{2}\right)^{9}$

$= 36 \times \left(\frac{1}{2}\right)^{9}$ //

---

2) The probability that Mr. Q hits a target at any trial is $\frac{1}{4}$. Suppose he tries at the target 10 times. find the probability that he hits the target

(i) exactly 4 times

(ii) at least one time.

Here $n = 10$

Let X denote number of hits the target

clearly X takes values $0, 1, 2 \cdots 10$.

Given that

$p = \frac{1}{4}$, $q = 1 - p$

$= 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore$ Binomial distn:

$P(X = x) = \binom{n}{x} p^{x} q^{n-x}$, $x = 0, 1, 2 \cdots n$

$P(X = x) = \binom{10}{x} \left(\frac{1}{4}\right)^{x} \left(\frac{3}{4}\right)^{10-x}$

$x = 0, 1, 2 \cdots 10$.

(i) probability of exactly 4 times

$P(X = 4) = {}^{10}C_{4} \left(\frac{1}{4}\right)^{4} \left(\frac{3}{4}\right)^{6}$

(ii) probability of atleast one time

$P(X \geq 1) = 1 - P(X < 1)$

$= 1 - P(X = 0)$

$= 1 - {}^{10}C_{0} \left(\frac{1}{4}\right)^{0} \left(\frac{3}{4}\right)^{10-0}$

$= 1 - 1 \times 1 \times \left(\frac{3}{4}\right)^{10}$

$= 1 - \left(\frac{3}{4}\right)^{10}$

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3) Using binomial distn find the mean and variance of x for the following experiments

(i) A fair coin is tossed 100 times, and x denotes the number of heads.

(ii) A fair die is tossed 240 times and x denotes the number of times that four appeared.

Given $n = 100$

Let p denotes the probability of getting a head when a coin is tossed.

$p = \frac{1}{2}$, $\therefore q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

mean $= np$

$= 100 \times \frac{1}{2} = \boxed{50}$

Variance $= npq$

$= 50 \times \frac{1}{2} = \boxed{25}$

mean $= 50$, Variance $= 25$

(ii) Given $n = 240$.

p denotes the probability of getting a four when a die is thrown.

$p = \frac{1}{6}$, $\therefore q = 1 - \frac{1}{6} = 5/6$

mean $= np = 240 \times \frac{1}{6} = \boxed{40}$

Variance $= npq$

$= 40 \times \frac{5}{6} = \boxed{\frac{100}{3}}$

$\therefore$ Mean $= 40$, Variance $= \frac{100}{3}$

x x

4) The probability that a certain kind of component will survive a electrical test is $\frac{3}{4}$. Find the probability that exactly 3 of the 5 components tested survive.

Given $p = \frac{3}{4}$ $\therefore q = 1 - p$

$= 1 - 3/4$

$= \boxed{\frac{1}{4}}$

$n = 5$

Binomial distribution

$P(X = x) = \binom{n}{x} p^x q^{n-x}$, $x = 0, 1, 2 \cdots n$

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$$P(X=3) = {}^{5}C_{3}\left(\frac{3}{4}\right)^{3}\left(\frac{1}{4}\right)^{5-3}$$

$$= \frac{5\times\cancel{4}\times\cancel{3}}{1\times\cancel{2}\times\cancel{3}} \times \frac{27}{\cancel{64}\,16} \times \frac{1}{16}$$

$$= \boxed{\frac{135}{512}}$$

---

5) A retailer purchases a certain kind of electronic device from a manufacturer. The manufacturer indicates that the defective rate of the device is 5%. The inspector of the retailer randomly picks 10 items from a shipment. What is the probability that there will be

(i) atleast one defective item.

(ii) exactly two defective item.

$n = 10, \quad p = \frac{5}{100} = 0.05$

$\therefore q = 1-p = 1-0.05 = \boxed{0.95}$

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Let X denote number of defective items.

$\therefore$ Binomial distn

$$P(X=x) = \binom{n}{x} p^{x} q^{n-x}, \quad x = 0,1,\ldots n$$

(i) P(atleast one defective item)

$$P(X\geq 1) = 1 - P(X<1)$$

$$= 1 - P(X=0)$$

$$= 1 - {}^{10}C_{0}\,(0.05)^{0}\,(0.95)^{10}$$

$$= 1 - 1\times 1\times(0.95)^{10}$$

$$= \underline{1-(0.95)^{10}}$$

(ii) P(exactly two defective)

$$P(X=2) = {}^{10}C_{2}(0.05)^{2}(0.95)^{10-2}$$

$$= \frac{10\times 9}{1\times 2}\times(0.05)^{2}(0.95)^{8}$$

$$= \underline{45\times(0.05)^{2}(0.95)^{8}}$$

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6) If the probability that a fluoresent light has a useful life of atleast 600 hours is 0.9, find the probabilities that among 12 such lights (i) exactly 10 will have a useful life of atleast 600 hours. (ii) atleast 11 will have a useful life of atleast 600 hours. (iii) atleast 2 will not have a useful life of atleast 600 hours.

Soln: Given $n = 12$

and $p = 0.9$

$q = 1 - p = 1 - 0.9 = \boxed{0.1}$

(i) $P(X=10) = {}^{12}C_{10} (0.9)^{10} (0.1)^{12-10}$

$= {}^{12}C_{10} (0.9)^{10} (0.1)^{2}$

(ii) $P(X \geq 11) = P(X=11) + P(X=12)$

$= {}^{12}C_{11} (0.9)^{11} (0.1)^{1} + {}^{12}C_{12} (0.9)^{12} (0.1)^{0}$

$= 12C_1 (0.9)^{11} (0.1) + 1 \times (0.9)^{12}$

$= (0.9)^{11} [12 \times (0.1) + 0.9]$

$= (0.9)^{11} [1.2 + 0.9] = (0.9)^{11} (2.1)$

(iii) P(atleast two will not have useful life) $= P(X < 11)$

$= 1 - P(X \geq 11)$

$= 1 - (0.9)^{11} (2.1)$

---

7) The mean and standard deviation of a binomial variate X are respectively 6 and 2. Find (i) the probability mass function (ii) $P(X=3)$ (iii) $P(X \geq 2)$

Given mean $np = 6$ ——— (1)

$SD = \sqrt{npq} = 2$

$\therefore npq = 4$ ——— (2)

Using (1) in (2) $6q = 4 \Rightarrow q = \frac{4}{6}$

$= \boxed{\frac{2}{3}}$

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$p = 1-q$

$= 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$

Sub $p = \frac{1}{3}$ in ①

$n\left(\frac{1}{3}\right) = 6 \Rightarrow \boxed{n = 18}$

(i) The probability mass function is $p(X=x) = \binom{n}{x} p^x q^{n-x}$, $x = 0, 1 \cdots n$

$\therefore p(X=x) = \binom{18}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{18-x}$, $x = 0, \cdots 18$

(ii) $p(X=3) = {}^{18}C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{18-3}$

$= {}^{18}C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{15}$

(iii) $p(X \geq 2) = 1 - p(X<2)$

$= 1 - [p(X=0) + p(X=1)]$

$= 1 - \left[{}^{18}C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{18} + {}^{18}C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{17} \right]$

$= 1 - \left[\left(\frac{2}{3}\right)^{18} + 18 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{17} \right]$

$= 1 - \left(\frac{2}{3}\right)^{17} \left[\frac{2}{3} + 6\right]$

$= 1 - \left(\frac{2}{3}\right)^{17} \left(\frac{20}{3}\right)$

---

8) If $X \sim B(n,p)$ such that $4p(X=4) = p(X=2)$ and $n=6$. Find the distribution, mean and SD.

WKT $p(X=x) = {}^nC_x \, p^x (1-p)^{n-x}$

Given $4\,p(X=4) = p(X=2)$ and $n=6$

$4\left[{}^6C_4 \, p^4 (1-p)^{6-4} \right] = \left[{}^6C_2 \, p^2 (1-p)^{6-2} \right]$

$\Rightarrow 4\left[{}^6C_2 \, p^4 (1-p)^2 \right] = \left[{}^6C_2 \, p^2 (1-p)^4 \right]$

$\Rightarrow 4p^2 = (1-p)^2$

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$4p^2 = 1 + p^2 - 2p$

$4p^2 - p^2 - 1 + 2p = 0$

$3p^2 + 2p - 1 = 0$

$(p+1)(3p-1) = 0$

$\Rightarrow p = -1, \quad p = \frac{1}{3}$

-3

$3 \quad -1$

$\frac{3}{3}, \frac{-1}{3}$

$1, \quad -\frac{1}{3}$

$p = -1$ is not possible $\therefore \boxed{p = \frac{1}{3}}$

$\therefore$ The Binomial distn is

$P(X=x) = {}^nC_x \, p^x \, q^{n-x}, \quad x = 0, 1, \cdots n$

$P(X=x) = {}^6C_x \left(\frac{1}{3}\right)^x \left(1 - \frac{1}{3}\right)^{6-x}, \quad x = 0, \cdots 6$

$= {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}, \quad x = 0, 1, 2 \cdots 6$

Mean: $np = 6 \times \frac{1}{3} = \boxed{2}$

$SD = \sqrt{npq} = \sqrt{6 \times \frac{1}{3} \times \frac{2}{3}}$

$= \sqrt{\frac{4}{3}} = \boxed{\frac{2}{\sqrt{3}}}$

9) In a binomial distribution consisting of 5 independent trials, the probability of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the mean and variance of the distribution.

Given $n = 5$

In Binomial distribution

$P(X=x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, 2 \cdots n$

$P(X=x) = \binom{5}{x} p^x q^{n-x}, \quad x = 0, 1, 2 \cdots 5$

Also given that

$P(X=1) = 0.4096$

i.e. $\binom{5}{1} p^1 q^4 = 0.4096$

$5 p q^4 = 0.4096 \quad \text{——} (1)$

Also $P(X=2) = 0.2048$

i.e. $\binom{5}{2} p^2 q^3 = 0.2048$

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ii $\frac{5\times4}{1\times2} p^2q^3 = 0.2048$

$10p^2q^3 = 0.2048$ —— (2)

$\frac{(2)}{(1)} \Rightarrow \frac{10p^2q^3}{5pq^4} = \frac{0.2048}{0.4096}$

$\frac{2p}{q} = \frac{1}{2} \Rightarrow \frac{p}{q} = 1/4$

$\Rightarrow \boxed{4p = q}$

ii $4p = 1-p$

$5p = 1 \Rightarrow \boxed{p = \frac{1}{5}}$

WKT $q = 1-p$

$= 1 - \frac{1}{5} = \boxed{\frac{4}{5}}$

∴ mean $= np$

$= 5 \times \frac{1}{5} = \boxed{1}$

Variance $= npq$

$= 1 \times \frac{4}{5} = \boxed{\frac{4}{5}}$

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Chapter 12 – Discrete Mathematics

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CHAPTER – 12

DISCRETE MATHEMATICS

Defn:- Any operation $*$ is defined on a non-empty set S is called a binary operation on S if the following conditions are

(i) the operation $*$ must be defined for each and every ordered pair $(a,b) \in S \times S$.

(ii) It assign a unique element $a*b$ of S to every ordered pair $(a,b) \in S \times S$.

This property is known as closure property

EXERCISE 12.1

1. Determine whether $*$ is a binary operation on the sets given below.

(i) $a*b = a \cdot |b|$ on R.

Soln:

$$\forall\, a,b \in R \Rightarrow |b| \in R$$
$$\Rightarrow a \cdot |b| \in R$$

i.e. $a*b = a \cdot |b| \in R$

$\therefore$ $*$ is binary operation

(ii) $a*b = \min(a,b)$ on $\{1,2,3,4,5\}$

$\forall\, a,b \in R$

$\min(a,b)$ is either a or b but $a,b \in R$.

$\therefore$ $\min(a,b) \in R$

$\therefore$ $*$ is binary operation on R

(iii). $a*b = a\sqrt{b}$ where $a,b \in R$

$\forall\, a,b \in R$

Square root of negative number is not defined.

$\therefore$ $a*b = a\sqrt{b} \notin R$

$\therefore$ $*$ is not binary operation on R.

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2) On z, define $\circledast$ by $m \circledast n = m^n + n^m$ $\forall\ m, n \in z$. Is $\circledast$ binary on z?

$\forall\ m, n \in z$

Suppose $m = 2,\ n = -2$

$m \circledast n = m^n + n^m$

$= 2^{-2} + (-2)^2$

$= \frac{1}{4} + 4 = \frac{17}{4} \notin z$

$\therefore\ m \circledast n \notin z \Rightarrow \circledast$ is not binary operation on z.

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3) Let $*$ be defined on R by $a * b = a + b + ab - 7$. Is $*$ binary on R. If so, find $3 * \left(\frac{-7}{15}\right)$.

Given $a * b = a + b + ab - 7$

$\forall\ a, b \in R \Rightarrow a + b \in R$ and $ab \in R$

$\therefore\ a + b + ab \in R$ and $-7 \in R$

$\therefore\ a + b + ab - 7 \in R$

$\therefore\ a * b = a + b + ab - 7 \in R$

So, $*$ is binary operation on R

Now $3 * \left(-\frac{7}{15}\right) = 3 - \frac{7}{15} + 3\left(-\frac{7}{15}\right) - 7$

$= \frac{38}{15} - \frac{7}{5} - 7$

$= \frac{38 - 21 - 105}{15}$

$= \frac{38 - 126}{15}$

$= \boxed{\frac{-88}{15}}$

---

4) Let $A = \{ a + \sqrt{5}b : a, b \in z \}$ check whether the usual multiplication is a binary operation on A.

Let $x = a + \sqrt{5}b \in A$

$y = c + \sqrt{5}d \in A$, where $a, b, c, d \in z$

$\therefore\ xy = (a + \sqrt{5}b)(c + \sqrt{5}d)$

$= ac + ad\sqrt{5} + bc\sqrt{5} + 5bd$

$= (ac + 5bd) + \sqrt{5}(ad + bc) \in A$

$\therefore$ Usual multiplication is a binary on A.

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5 (i) Define an operation $*$ on Q as follows $a*b = \frac{a+b}{2}$, $a,b \in Q$

Examine the closure, commutative, and associative properties satisfied by $*$ on Q.

(i) closure property:-

Let $a,b \in Q$

Now $a*b = \frac{a+b}{2} \in Q$ $(\because a,b,2 \in Q)$.

So, closure property is satisfied.

(ii) Commutative property:-

Let $a,b \in Q$

Now $a*b = \frac{a+b}{2} \in Q$ —— (1)

$b*a = \frac{b+a}{2} \in Q$ —— (2)

From (1) and (2) $a*b = b*a$

∴ commutative property is satisfied.

(iii) Associative property:-

For associative property we have to prove $a*(b*c) = (a*b)*c$.

Now $b*c = \frac{b+c}{2}$

LHS $= a*(b*c) = a*\left(\frac{b+c}{2}\right)$

$= \frac{a+\frac{b+c}{2}}{2}$

$= \frac{2a+b+c}{4}$ — (1).

Now $a*b = \frac{a+b}{2}$

RHS $= (a*b)*c$

$= \frac{a+b}{2}*c = \frac{\frac{a+b}{2}+c}{2}$

$= \frac{a+b+2c}{4}$ — (2)

From (1) and (2)

$a*(b*c) \neq (a*b)*c$.

So associative property is not satisfied.

(ii) Define an operation $*$ on Q as follows $a*b = \frac{a+b}{2}$, $a,b \in Q$

Examine the existence of identity and existence of inverse for the

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operation * on Q

$\forall a \in Q$. Let e be the identity element. Then $a * e = a$

i.e. $\frac{a+e}{2} = a$

$a + e = 2a \Rightarrow \boxed{e = a}$

which is not possible.

∵ Identity element must be unique.

∴ Identity does not exist

Existence of Inverse:-

Identity does not exists

∴ Inverse does not not exists.

---

6) Fill in the following table so that the binary operation * on $A = \{a, b, c\}$ is commutative.

*	a	b	c
a	b		
b	c	b	a
c	a		c

Given, The binary operation is commutative.

Find $a * b$

* is commutative

∴ $a * b = b * a$

From the table $b * a = c$

∴ $\boxed{a * b = c}$

From the table $c * a = a$

∴ $\boxed{a * c = a}$ ∵ * is commutative

From the table $b * c = a$

∴ $\boxed{c * b = a}$ ∵ * is commutative.

Hence

*	a	b	c
a	b	(c)	(a)
b	c	b	a
c	a	(a)	c

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7) Consider the binary operation * defined on the set $A = \{a, b, c, d\}$ by the following table.

*	a	b	c	d
a	a	c	b	d
b	d	a	b	c
c	c	d	a	a
d	d	b	a	c

Is it commutative and associative.

From the table $a * b = c$ and $b * a = d$.

So, the binary operation is not commutative.

For associative, to prove

$a * (b * c) = (a * b) * c$

From the table $b * c = b$

LHS $= a * (b * c) = a * b$

$= c$ ——— (1)

RHS $= (a * b) * c$

$= c * c$

$= a$ ——— (2)

From (1) and (2) $a * (b * c) \neq (a * b) * c$

So, binary operation is not associative.

---

8) Let $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

be any three boolean matrices of the same type. Find

(i) $A \vee B$ (ii) $A \wedge B$ (iii) $(A \vee B) \wedge C$

(iv) $(A \wedge B) \vee C$.

To find $A \vee B$

$A \vee B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

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$$= \begin{bmatrix} 1\vee0 & 0\vee1 & 1\vee0 & 0\vee1 \\ 0\vee1 & 1\vee0 & 0\vee1 & 1\vee0 \\ 1\vee1 & 0\vee0 & 0\vee0 & 1\vee1 \end{bmatrix}$$

$$= \begin{bmatrix} \max\{1,0\} & \max\{0,1\} & \max\{1,0\} & \max\{0,1\} \\ \max\{0,1\} & \max\{1,0\} & \max\{0,1\} & \max\{1,0\} \\ \max\{1,1\} & \max\{0,0\} & \max\{0,0\} & \max\{1,1\} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(ii) $A \wedge B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1\wedge0 & 0\wedge1 & 1\wedge0 & 0\wedge1 \\ 0\wedge1 & 1\wedge0 & 0\wedge1 & 1\wedge0 \\ 1\wedge1 & 0\wedge0 & 0\wedge0 & 1\wedge1 \end{bmatrix}$$

$$= \begin{bmatrix} \min\{1,0\} & \min\{1,0\} & \min\{1,0\} & \min\{0,1\} \\ \min\{0,1\} & \min\{1,0\} & \min\{0,1\} & \min\{1,0\} \\ \min\{1,1\} & \min\{0,0\} & \min\{0,0\} & \min\{1,1\} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(iii) $(A \vee B) \wedge C$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$ //

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(iv) $(A \wedge B) \vee C$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} //$$

9(i) Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in R - \{0\} \right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the commutative and associative properties satisfied by $*$ on M.

a) closure property:

Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$, $B = \begin{pmatrix} y & y \\ y & y \end{pmatrix}$, $x, y \in M$

$$AB = \begin{pmatrix} xy+xy & xy+xy \\ xy+xy & xy+xy \end{pmatrix}$$

$$= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in M \quad \because \begin{matrix} x \neq 0 \\ y \neq 0 \\ xy \neq 0 \end{matrix}$$

$\therefore$ M is closed with respect to matrix multiplication.

b) Commutative property:

$$\forall\ A = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \in M,\ B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \in M$$

$$A * B = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} y & y \\ y & y \end{pmatrix}$$

$$= \begin{pmatrix} xy+xy & xy+xy \\ xy+xy & xy+xy \end{pmatrix} = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix}$$

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$$B*A = \begin{pmatrix} y & y \\ y & y \end{pmatrix}\begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$= \begin{pmatrix} yx+yx & yx+yx \\ yx+yx & yx+yx \end{pmatrix}$$

$$= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \quad \text{—— ②}$$

From ① and ② $A*B = B*A$.

∴ $*$ is commutative property on M.

c) Associative property:-

Matrix multiplication is always associative. $\forall$ A, B, C $\in$ M.

∴ $(AB)C = A(BC)$

∴ $*$ is associative property

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(ii) Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in R - \{0\} \right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the existence of identity, existence of inverse properties for the operation $*$ on M.

Given $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in R - \{0\} \right\}$

(i) closure property:

$\forall$ $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \in M$, $B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \in M$

$$AB = \begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} y & y \\ y & y \end{pmatrix}$$

$$= \begin{pmatrix} xy+xy & xy+xy \\ xy+xy & xy+xy \end{pmatrix}$$

$$= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in M$$

$\because$ $x \neq 0$, $y \neq 0$, $2xy \neq 0$.

∴ M is closed under $*$

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Existence of identity:-

$\forall A = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \in M$ and let

$E = \begin{pmatrix} e & e \\ e & e \end{pmatrix}$ be an element in M such that $A * E = E * A = A$.

Now, $A * E = A$

$$\Rightarrow \begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} xe+xe & xe+xe \\ xe+xe & xe+xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\begin{pmatrix} 2xe & 2xe \\ 2xe & 2xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\Rightarrow 2xe = x \Rightarrow 2e = 1$$
$$e = \frac{1}{2}$$

$\therefore E = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \in M$

Hence E is the required identity element in M.

Existence of Inverse:

Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \in M$ and Let

$X^1 = \begin{pmatrix} x' & x' \\ x' & x' \end{pmatrix} \in M$ such that

$A * X^1 = X^1 * A = E$

Now consider $A * X^1 = E$

$$\Rightarrow \begin{pmatrix} x & x \\ x & x \end{pmatrix} * \begin{pmatrix} x' & x' \\ x' & x' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} xx'+xx' & xx'+xx' \\ xx'+xx' & xx'+xx' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 2xx' & 2xx' \\ 2xx' & 2xx' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Rightarrow 2xx' = \frac{1}{2}$$
$$x' = \frac{1}{4x}$$

$\therefore X^1 = \begin{pmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{pmatrix} \in M.$

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∴ The inverse exist for every element in M.

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10 (i) Let A be $Q \setminus \{1\}$. Define * on A by $x * y = x + y - xy$. Is * binary on A? If so, examine the commutative and associative properties satisfied by * on A.

Given $A = Q \setminus \{1\}$ and * is defined as $x * y = x + y - xy$.

(i) closure property:-

$\forall x, y \in A$ i.e. $x \neq 1, y \neq 1$ ——— (1)

To prove $x * y \in A$.

i.e. to prove $x + y - xy \neq 1$.

Suppose $x + y - xy = 1$

$x + y - xy - 1 = 0$

$x(1-y) + (y-1) = 0$

$x(1-y) - (1-y) = 0$

$(1-y)(x-1) = 0$

$\Rightarrow 1 - y = 0$ or $x - 1 = 0$

$\Rightarrow y = 1$ or $x = 1$

which is contradiction. ∵ x and y are not equal to 1 (from (1)).

∴ $x + y - xy = 1$ is not possible

∴ $x + y - xy \neq 1$ and

$x * y = x + y - xy \in A$.

Hence * is binary operation on A.

(b) Commutative property:-

Let $x, y \in A$

To prove $x * y = y * x$

$x * y = x + y - xy$

$= y + x - yx$

$= y * x$

So, $x * y = y * x \Rightarrow$ commutative property is true on A.

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c) Associative property:-

$\forall\ x, y, z \in A$

To prove $(x*y)*z = x*(y*z)$

Now $(x*y)*z = (x+y-xy)*z$

$= x+y-xy+z-(x+y-xy)z$

$= x+y+z-xy-xy-yz+xyz$ ——— (1)

$x*(y*z) = x*(y+z-yz)$

$= x+y+z-yz-x(y+z-yz)$

$= x+y+z-yz-xy-xz+xyz$

$= x+y+z-xy-yz-xz+xyz$ ——— (2)

From (1) and (2)

$(x*y)*z = x*(y*z)$.

$\therefore$ Associative property is true in A.

(ii) Let A be $Q \setminus \{1\}$. Define $*$ on A by $x*y = x+y-xy$. IS $*$ binary on A? If so, examine the existence of identity, existence of inverse properties for the operation $*$ on A.

$*$ is binary operation (proved in 10(1)(i)).

a) Existence of Identity.

$\forall\ x \in A$, and let e be the element such that $x*e = e*x = x$

consider $x*e = x$

$x+e-xe = x$

$e-xe = 0$

$e(1-x) = 0$

$1-x \neq 0 \ \because x \neq 1$

$\therefore e = 0$ is a rational no and not equal to 1

Hence $e = 0 \neq 1 \in A$.

$\therefore$ Identity exists.

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b) Existence of Inverse:-

$\forall x \in A$ and x' be an element in A such that

$$x * x' = x' * x = e$$

i $\quad x * x' = 0$

$$\Rightarrow x + x' - xx' = 0$$

$$x'(1-x) = -x$$

$$x' = -\frac{x}{1-x} \in A \quad (\because x \neq 1)$$

so $x' \in A$

Hence every $x \in A$ there is an inverse $x' \in A$ such that

$$x * x' = x' * x = e.$$

Hence inverse exists for every element in A.

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Example 12.1 Examine the binary operation (closure property) of the following operations on the respective sets

(i) $a * b = a + 3ab - 5b^2$; $\forall a, b \in z$.

For $a, b \in z$, ie a and b are integers

$\therefore$ $3ab$ and $5b^2$ is also an integer

$\therefore$ $a + 3ab - 5b^2$ is also an integer.

Hence closure property is satisfied

(ii) $a * b = \frac{a-1}{b-1}$, $\forall a, b \in Q$

$\forall a, b \in Q$

Suppose $b = 1$ then $\frac{a-1}{b-1} = \frac{a-1}{1-1} = \frac{a-1}{0}$

is not defined

$\therefore$ $a * b \notin Q$

$\therefore$ $*$ is not binary operation.

$*$ is binary operation for all $a, b \in Q - \{1\}$

Ex 12.2 Verify the (i) closure property (ii) Commutative property (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation + on z.

Given

$z = \{-\infty, \ldots -1, 0, 1, 2 \ldots \infty\}$

(i) **closure property:-**

Sum of two integer is also an integer

$\forall a, b \in z \Rightarrow a + b \in z$.

$\therefore$ closure property is satisfied.

(ii) **Commutative property:-**

$\forall a, b \in z \Rightarrow a + b = b + a$

$\therefore$ Commutative property is satisfied.

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(iii) Associative property:-

Addition is always associative

$\forall\ a, b, c \in z$

$\Rightarrow\ a+(b+c) = (a+b)+c$

$\therefore$ Associative property is satisfied.

(iv) Existence of identity:

$\forall\ a \in z$, and identity element is e

$\therefore\ a+e = a \Rightarrow e = a-a$

$e = 0 \in z$

$\therefore\ e = 0 \in z$ is identity element in z

(v) Existence of Inverse:-

$\forall\ a \in z$, the inverse of a is $\bar{a}^1$ such that

$a * \bar{a}^1 = e$

i.e. $a + \bar{a}^1 = 0$

$\bar{a}^1 = 0 - a$

$\bar{a}^1 = -a \in z$.

$\therefore$ Existence of inverse is satisfied.

$\therefore$ $(z, +)$ is abelian group

Eg 12.3 verify the (i) closure property, (ii) Commutative property (iii) Associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation $-$ on z.

Given

$z = \{-\infty, \ldots -2, -1, 0, 1, 2, \ldots \infty\}$

Closure property:

$\forall\ a, b \in z \Rightarrow a-b \in z$

Hence closure property is satisfied.

(ii) Commutative property:-

$\forall\ a, b \in z$

Let $a = 2$, $b = 3$

$a - b = 2 - 3 = -1$

$b - a = 3 - 2 = 1$

$a - b \neq b - a$.

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∴ Commutative property is not satisfied.

(III) Associative property:-

$\forall\ a, b, c \in z$

Let $a = 2, b = 3, c = 4$

$(a-b)-c = (2-3)-4$

$= -1-4 = -5$ —— ①

$a-(b-c) = 2-(3-4)$

$= 2-(-1)$

$= 2+1$

$= 3$ —— ②

from ① and ②

$(a-b)-c \neq a-(b-c)$

∴ $-$ is not associative in z.

(IV) Existence of Identity:

$\forall\ a \in z$ and e be an identity element.

such that

$a-e = e-a = a$

$a-e = a$ | $e-a = a$

$\Rightarrow a-a = e$ | $e = a+a$

$\boxed{e = 0}$ | $\boxed{e = 2a}$

But identity element is unique.

Hence $e = 0$ and $e = 2a$ is not possible.

Hence identity element does not exist in z with respect to $-$.

(V) Existence of Inverse:-

$\forall\ a \in z$ and $\bar{a}^1$ be the inverse element. such that

$a - \bar{a}^1 = e$ But e does not exist. ∴ Inverse element does not exist in z w.r.to $-$.

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Eg 12.4 Verify the closure property (ii) Commutative property (iii) associative property (iv) Existence of Identity and (v) Existence of inverse for the arithmetic operation + on Z_e = the set of all even integers.

Given Z_e = the set of all even integers.

i $Z_e = \{2k \mid k \in z\}$

i $Z_e = \{-\infty, \ldots -8, -6, -4, -2, 0, 2, \ldots \infty\}$

Let $a, b \in Z_e$.

closure property:-

$a, b \in Z_e$

a and b are even integers.

Sum of two even integers is also an even integer.

$a+b$ is an even integer

$\Rightarrow a+b \in Z_e$

Hence closure property is satisfied.

(ii) **Commutative property:-**

Let $a, b \in Z_e$.

Then $a = 2m$, $b = 2n$, $m, n \in Z$.

$a+b = 2m+2n = 2(m+n)$

$b+a = 2n+2m = 2(n+m)$, $m+n \in Z$

$2(m+n) \in Z_e$

∴ Commutative property is true in Z_e.

(iii) **Associative property:-**

Let a, b, c be any three element in Z_e.

Then $a = 2m$, $b = 2n$, $c = 2p$.

$$\begin{aligned} a+(b+c) &= 2m+(2n+2p) \\ &= 2m+2(n+p) \\ &= 2[m+(n+p)] \\ &= 2[(m+n)+p] \\ &= 2(m+n)+2p \\ &= (2m+2p)+2p \\ &= (a+b)+c. \end{aligned}$$

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Hence $a+(b+c) = (a+b)+c$

$\forall a, b, c \in Z_e$.

(iv) Existence of Identity:

Let $a \in Z_e$ and let e be an element such that

$a + e = e + a = a$

Let $a = 2m$

$\therefore a + e = a \Rightarrow 2m + e = 2m$

$e = 0 \in Z_e$

$\therefore$ existence of Identity is True

(v) Existence of Inverse:-

Let $a = 2m \in Z_e$

Let a' be an inverse element such that

$a + a' = a' + a = e$

$2m + a' = a' + 2m = 0$

$\therefore a' = -2m \in Z_e$

Hence inverse exists for every element in Z_e.

Eg 12.5 verify the (i) closure property, (ii) Commutative property (iii) associative property (iv) existence of identity (v) existence of inverse for the arithmetic operation + on Z_o = The set of all odd integers.

Given Z_o = The set of all odd integers.

i.e $Z_o = \{-\infty, \cdots -3, -1, 1, 3, 5, \cdots \infty\}$

(i) closure property:-

Let $a, b \in Z_o$

The a and b are odd integers. sum of two odd integers is an even integer.

$\therefore a + b$ = even integer

$\therefore a + b \notin Z_o$

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∴ + is not binary operation on Z_0.

∴ The other properties need not be checked.

Eg 12.6 Verify (i) closure property (ii) Commutative property and (iii) associative property of the operation on the given set

$a * b = a^b$; $\forall a, b \in N$ (exponentiation property)

Given $N = \{1, 2, 3, \ldots \infty\}$ and * is defined as $a * b = a^b$, $\forall a, b \in N$.

(i) **closure property:-**

$a * b = a^b \in N \quad \forall a, b \in N$

∴ * is closure on N.

(ii) **Commutative property:-**

Let $a = 2, \; b = 3 \in N$

$a * b = a^b$

i.e. $2 * 3 = 2^3 = 8$ —— ①

$b * a = 3 * 2 = 3^2$

$= 9$ —— ②

From ① and ② $a * b \neq b * a$.

∴ The commutative property is true in N.

(iii) **Associative property:-**

Take $a = 2, \; b = 3$ and $c = 4$ in N

$a * (b * c) = 2 * (3 * 4)$

$= 2 * 3^4 = (2)^{3^4} = 2^{81}$ —— ①

$= 2^{3^4}$

$(a * b) * c = (2 * 3) * 4$

$= 2^3 * 4$

$= (2^3)^4$ —— ②.

$= 8^4 = 2^{12}$

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From ① and ②

$a*(b*c) \neq (a*b)*c$

∴ The associative property is not true.

Eg 12.7 Verify (i) closure property, (ii) Commutative property, (iii) associative property (iv) existence of identity and (v) existence of inverse for the following operation on the given set $m*n = m+n-mn$, $m,n \in Z$

Given

$Z = \{-\infty, \cdots -1, 0, 1, 2 \cdots \infty\}$

$*$ is defined as $m*n = m+n-mn$.

(i) Closure property:-

$\forall m, n \in Z \Rightarrow m+n \in Z$ and $mn \in Z$

$\therefore m+n-mn \in Z$

$\therefore *$ is closure on Z.

(ii) Commutative property:-

$\forall m, n \in Z$

Now $m*n = m+n-mn$ — ①

$n*m = n+m-nm$ — ②

From ① and ② $m*n = n*m$.

$\therefore *$ is commutative property on Z.

(iii) Associative property:-

$\forall m, n, p \in Z$

$(m*n)*p = (m+n-mn)*p$

$= m+n-mn+p-(m+n-mn)p$

$= m+n-mn+p-mp-np+mnp$ — ①.

$m*(n*p) = m*(n+p-np)$

$= m+n+p-np-m(n+p-np)$

$= m+n+p-np-mn-mp+mnp$ — ②.

From ① and ②

$(m*n)*p = m*(n*p)$

$\therefore *$ is associative on Z

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(iv) Existence of identity:-

$\forall m \in z$, and let e be the identity element in z such that

$m * e = m$

i.e. $m + e - me = m$

$e(1-m) = 0$

$e = \frac{0}{1-m} = 0 \in z$.

∴ The identity element in z is 0.

(v) Existence of inverse:-

$\forall m \in z$, and let the inverse of m is m' such that

$m * m' = e$

i.e. $m * m' = 0$

$\Rightarrow m + m' - mm' = 0$

$m + m'(1-m) = 0$

$m'(1-m) = -m$

$m' = \frac{-m}{1-m}$

when $m = 1$, $m' = \frac{-1}{1-1} = \frac{-1}{0}$ (not defined)

∴ when $m = 1$, m' is not defined.

when $m = 2$, $m' = \frac{-2}{1-2} = \frac{-2}{-1} = 2$

when $m > 2$, m' need not be an integer.

Hence inverse does not exist in z.

× ×

Some binary operations on Boolean Matrices:-

Defn:- A Boolean Matrix is a real matrix whose entries are either 0 or 1.

Defn:- Join of A and B.

$A \vee B = [a_{ij}] \vee [b_{ij}]$

$= [a_{ij} \vee b_{ij}] = [c_{ij}]$

where $c_{ij} = \begin{cases} 1, & \text{if either } a_{ij} = 1 \text{ or } b_{ij} = 1 \\ 0 & \text{if both } a_{ij} = 0 \text{ and } b_{ij} = 0 \end{cases}$

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Defn:- Meet of A and B

$$A \wedge B = [a_{ij}] \wedge [b_{ij}] = [a_{ij} \wedge b_{ij}] = [c_{ij}]$$

where $c_{ij} = \begin{cases} 1 & \text{if both } a_{ij}=1 \text{ and } b_{ij}=1 \\ 0 & \text{if either } a_{ij}=0 \text{ or } b_{ij}=0. \end{cases}$

It is clear that:

$a \vee b = \max\{a,b\}$

$a \wedge b = \min\{a,b\}$, $a,b \in \{0,1\}$

Ex: 12.8 Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ be any two boolean Matrices of the same type. Find $A \vee B$ and $A \wedge B$.

Given $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$A \vee B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \vee 1 & 1 \vee 1 \\ 1 \vee 0 & 1 \vee 1 \end{bmatrix} = \begin{bmatrix} \max\{0,1\} & \max\{1,1\} \\ \max\{1,0\} & \max\{1,1\} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A \wedge B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \wedge 1 & 1 \wedge 1 \\ 1 \wedge 0 & 1 \wedge 1 \end{bmatrix}$$

$$= \begin{bmatrix} \min\{0,1\} & \min\{1,1\} \\ \min\{1,0\} & \min\{1,1\} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Modular Arithmatic

Ex: 12.9 Verify (i) closure property (ii) Commutative property (iii) associative property (iv) Existence of identity (v) Existence of inverse for the operation $+_5$ on Z_5 using table corresponding to addition modulo 5.

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Given $Z_5 = \{[0], [1], [2], [3], [4]\}$

We take the remainders as $[0, 1, 2, 3, 4]$.

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

(i) Closure property:

From the table all the elements are members of Z_5.

∴ closure property is satisfied.

(ii) Commutative property:-

Addition Modulus 5 is always commutative. ∴ Commutative is satisfied

(iii) Associative property:

Addition Modulus is always associative.

∴ Associative property is satisfied.

(iv) From the table Identity is zero

(v) From the table,

Inverse of 0 is 0

Inverse of 1 is 4

Inverse of 2 is 3

Inverse of 3 is 2

Inverse of 4 is 1

Ex 12.10 verify (i) closure property (ii) Commutative property (iii) associative property (iv) existence of Identity (v) Existence of inverse for the operation $\times_{11}$ on a subset

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$A = \{1, 3, 4, 5, 9\}$ of the set of remainders $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

The multiplication modulo 11 for A is

$\times_{11}$	1	3	4	5	9
1	1	3	4	5	9
3	3	9	1	4	5
4	4	1	5	9	3
5	5	4	9	3	1
9	9	5	3	1	4

closure property:-

From the table, each box has an unique element of A.

∴ $\times_{11}$ is binary operation on A.

(ii) **Commutative property:-**

The entries are symmetrial about the main diagonal.

∴ The commutative property is true for $\times_{11}$

(iii) **Associative property:-**

$\times_{11}$ is always associative.

Hence associative property is satisfied.

(iv) **Existence of identity:-**

From the table, entries of both the row and column headed by the element 1 are identical.

Hence 1 is the identity element.

(v) **Existence of Inverse**

Inverse of 1 is 1
Inverse of 3 is 4
Inverse of 4 is 3
Inverse of 5 is 9
Inverse of 9 is 5

∴ Inverse exists for every element.

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MATHEMATICAL LOGIC

Statement or proposition:-

A statement or a proposition is a sentence which is either true or false but not both.

Truth Value of a statement

The truth or falsity of a statement is called its truth value. If a statement is true, we say that its truth value is TRUE or T and if it is false, we say that its truth value is False or F

conjuction:- If two simple statement p and q are connected by the word 'and' then the resulting compound statement 'p and q' is called the conjunction of p and q and it is denoted by $p \wedge q$.

Disjunction:- If two statements p and q are connects by 'or' then the resulting compound statement 'p or q' is called disjunction of p and q and is written as $p \vee q$

Negation (NOT)

If p denotes a statement, then the negation of p is opposite of a statement p.

It is denoted by $\neg p$ or $\sim p$ or Np

Logical connectives and their Truth Tables:-

(i) Truth Table for (NOT) $|\neg|$ (Negation)

Truth table or $\neg p$

p	$\neg p$
T	F
F	T

(ii) Truth table for AND $|\wedge|$ (conjunction)

Truth table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
T	T	F
T	F	F

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(iii) Truth Table for OR |V| (Disjunction)

Truth table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Ex 12.12 Write the statement in words corresponding to $\neg q$, $p \wedge q$, $p \vee q$, $q \vee \neg p$ where p is It is cold and q is It is raining.

1) $\neg p$: It is not raining
2) $p \wedge q$: It is cold and raining.
3) $p \vee q$: It is cold or raining
   $q \vee \neg p$: It is raining or it is not cold.

Ex 12.13

How many rows are needed for following statement formulae?

(i) $p \vee \neg t \wedge (p \vee \neg s)$

It contains 3 variables p, s, t.
Hence the truth table contains
$2^3 = 8$ rows.

(ii) $(p \wedge q) \vee (\neg r \vee \neg s) \wedge (\neg t \wedge v)$

It contains 6 variables
p, q, r, s, t, v.
Hence the truth table contains
$2^6 = 64$ rows.

Conditional Statement. The conditional statement of any two statements p and q is the statement, "If p, then q" and it is denoted by $p \rightarrow q$. Here p is called the hypothesis or antecedent and q is called the conclusion or consequence.

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$p \rightarrow q$ is false only if p is true and q is false. otherwise it is true.

Truth Table for $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Ex 12.15 write down the (i) conditional statement (ii) converse statement (iii) inverse statement and (iv) contrapositive statement for two statements p and q given below.

p: The number of prime is infinite.

q: Ooty is in kerala.

conditional statement:- $p \rightarrow q$

If the number of prime is infinite then ooty is in kerala.

converse statement: $q \rightarrow p$

If ooty is in kerala then the number of prime is infinite.

inverse statement: $\neg p \rightarrow \neg q$

If the number of prime is not infinite then ooty is not in kerala.

contrapositive statement $\neg q \rightarrow \neg p$

If ooty is not in kerala then the number of primes is not infinite.

Bi - conditional statement:

Bi conditional statement of any two statements p and q is the statement "p iff q" and is denoted by $p \leftrightarrow q$. Its truth value is T, whenever both p and q have the same truth values, otherwise it is false.

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Truth Table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Exclusive OR (EOR) $|\bar{\vee}|$

Let p and q be any two statements. Then p EOR q is compound statement that its truth value is decided by either p or q but not both.

It is denoted by $p \bar{\vee} q$

The truth table $p \bar{\vee} q$ is T whenever either p or q is T, otherwise it is F.

Truth table for $p \bar{\vee} q$

p	q	$p \bar{\vee} q$
T	T	F
T	F	T
F	T	T
F	F	F

Ex 12.16 Construct the truth Table for $(p \bar{\vee} q) \wedge (p \bar{\vee} \neg q)$

p	q	$\neg q$	$r : (p \bar{\vee} q)$	$s : (p \bar{\vee} \neg q)$	$r \wedge s$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	F

Tautology: Its truth value is T always.

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Contradiction:- Its truth value is always F.

Example for Tautology: $p \vee \neg p$

p	$\neg p$	$p \vee \neg p$
T	F	T
T	T	T

Example for contradiction $p \wedge \neg p$

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Logical Equivalence:

Some laws of equivalence:

1) Idempotent Law

(i) $p \vee p = p$

(ii) $p \wedge p = p$

proof:

p	p	$p \vee p$	$p \wedge p$
T	T	T	T
F	F	F	F

From the table $p \vee p = p$

$p \wedge p = p$

---

2) **commutative laws:**

(i) $p \vee q = q \vee p$ **proof:**

(ii) $p \wedge q = q \wedge p$ No of rows $2^2 = 4$.

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

From the table, 3rd and 4th columns are identical.

Then $p \vee q = q \vee p$.

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(ii) $p \wedge q = q \wedge p$

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

From the table 3rd and 4th columns are identical.

$\therefore p \wedge q = q \wedge p$

—— α ——

Note For ∨ (Disjunction) ⊛.

F and F is F

others — T

3) Associative laws:

(i) $p \vee (q \vee r) = (p \vee q) \vee r$

p	q	r	$p \vee q$	$q \vee r$	$(p \vee q) \vee r$	$p \vee (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

From the table

$(p \vee q) \vee r = p \vee (q \vee r)$

—— ∧ ——.

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(ii) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$

p	q	r	$p \wedge q$	$q \wedge r$	$p \wedge (q \wedge r)$	$(p \wedge q) \wedge r$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

From the table

$p \wedge (q \wedge r) = (p \wedge q) \wedge r$

Note: For $\wedge$ (and) ⊛.
T and T is T
others — F.

4) Distributive laws:-

(i) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

p	q	r	$p \vee q$	$p \vee r$	$q \wedge r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	F	T	F	F	F
F	F	F	F	F	F	F	F

From the table

$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

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4(ii) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

p	q	r	$p \wedge q$	$p \wedge r$	$q \vee r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	F	F	F
F	T	T	F	F	T	F	F
F	T	F	F	F	T	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

From the table

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

Note: ⊛ For $\wedge$ (and) T and T is T, others – F | For $\vee$ (Disjunction) F and F is F, others – T

(5) Identity Laws :-

$p \vee T = T$ and $p \vee F = p$

$p \wedge T = F$ and $p \wedge F = F$

(i) $p \vee T = T$ and $p \vee F = p$

p	T	F	$p \vee T$	$p \vee F$
T	T	F	T	T
F	T	F	T	F

From the table

$p \vee T = T$ and $p \vee F = p$

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5(ii) $p \wedge T = p$ and $p \wedge F = F$

p	T	F	$p \wedge T$	$p \wedge F$
T	T	F	T	F
F	T	F	F	F

From the table $p \wedge T = p$ and $p \wedge F = F$

---

6) Identity Laws:-

(i) $p \vee \neg p = T$ and $p \wedge \neg p = F$

p	$\neg p$	T	F	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F	T	F
F	T	T	F	T	F

from the table 3rd and 5th columns are identical

$\therefore p \vee \neg p = T$

from the table 4th and 6th columns are identical.

$\therefore p \wedge \neg p = F$.

6(ii) $p \wedge T = p$ and $p \wedge F = F$.

p	T	$p \wedge T$	F	$p \wedge F$
T	T	T	F	F
F	T	F	F	F

From the table 1st and 3rd are identical and 4th and 5th are identical.

Hence $p \wedge T = p$ and $p \wedge F = F$

---

7) Involution Law or Double Negative Law: $\neg(\neg p) = p$

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

From the table $\neg(\neg p) = p$

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8) De Morgan's Laws:-

(i) $\neg(p \wedge q) = \neg p \vee \neg q$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$(\neg p \vee \neg q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

From the table 6th and 7th columns are equal.

$\therefore \neg(p \wedge q) = (\neg p \vee \neg q)$

(ii) $\neg(p \vee q) = \neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

From the table 6th and 7th columns are identical.

$\therefore \neg(p \vee q) = \neg p \wedge \neg q$

9) Absorption Laws:-

(i) $p \vee (p \wedge q) = p$ (ii) $p \wedge (p \vee q) = p$

p	q	$p \wedge q$	$p \vee q$	$p \vee (p \wedge q)$	$p \wedge (p \vee q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	T	F	F
F	F	F	F	F	F

From the table 5th and 6th columns are identical.

$\therefore p \vee (p \wedge q) = p \wedge (p \vee q)$

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Example 12.17 Establish the equivalence property $p \to q = \neg p \vee q$.

p	q	$\neg p$	$p \to q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

From the Table 4th and 5th columns are identical.

$\therefore \quad p \to q = \neg p \vee q$

Example: 12.18 Establish the equivalence property connecting the bi-conditional with conditional.

$$p \leftrightarrow q = (p \to q) \wedge (q \to p)$$

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

From the table 5th and 6th colums are identical.

$\therefore \quad p \leftrightarrow q = (p \to q) \wedge (q \to p)$

Ex 12.19 Using the equivalence property, show that $p \leftrightarrow q = (p \wedge q) \vee (\neg p \wedge \neg q)$.

p	q	$\neg p$	$\neg q$	$p \leftrightarrow q$	$\neg p \vee q$	$\neg q \vee p$	$(\neg p \wedge q) \wedge (\neg q \vee p)$
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	F
F	T	T	F	F	T	F	F
F	F	T	T	T	T	T	T

From the table 5th and 8th columns are identical $\therefore \quad p \leftrightarrow q = (\neg p \wedge q) \wedge (\neg q \vee p)$

---

* For $\to$
T and F is F
others — T

For $\leftrightarrow$ (iff)
T and T is T
F and F is T
others — F.

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EXERCISE 12.2

1) Let p: Jupiter is a planet and q: India is an island be any two simple statements. Give verbal sentence describing each of the following statements.

(i) $\neg p$ (ii) $p \wedge \neg q$ (iii) $\neg p \vee q$

(iv) $p \rightarrow \neg q$ (v) $p \leftrightarrow q$.

Given: p: Jupiter is a planet
q: India is an island.

(i) $\neg p$: Jupiter is not a planet.

(ii) $p \wedge \neg q$: Jupiter is a planet and India is not an island.

(iii) $\neg p \vee q$: Jupiter is not a planet (or) India is an island.

(iv) $p \rightarrow \neg q$: Jupiter is a planet then India is not an island.

(v) $p \leftrightarrow q$; Jupiter is a planet iff India is an island

2) Write each of the following sentences in symbolic form using statement variables p and q

(i) 19 is not a prime number and all the angles of a triangle are equal.

(ii) 19 is a prime number or all the angles of a triangle are not equal.

(iii) 19 is a prime number and all the angles of a triangle are equal.

(iv) 19 is not a prime number.

Soln: let p: 19 is a prime number
q: All the angles of a triangle are equal.

(i) $\neg p \wedge q$ (iii) $p \wedge q$

(ii) $p \vee \neg q$ (iv) $\neg p$

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3) Determine the truth value of each of the following statements.

(i) If $6+2=5$, then the milk is white.

(ii) Let p : $6+2=5$

q : The milk is white

The truth value is $p \rightarrow q$ is T

∴ the conclution is true.

(ii) china is in Europe or $\sqrt{3}$ is an integer.

Let p : china is in Europe

q : $\sqrt{3}$ is an integer

The truth value of $p \vee q$ is F.

Since the conclution of both the statements are false.

(iii) It is not true that $5+5=9$ or earth is a planet.

Let p : It is true that $5+5=9$

q : Earth is a planet.

The truth value of $\neg p \vee q$ is T.

Since the conclusion is (q) is true.

(iv) 11 is a prime number and all the sides of a reetangle are equal.

Let p : 11 is a prime number

q : The sides of a reetangle are equal.

The truth value of $p \wedge q$ is F.

Since the conclusion of q is F.

4) Which one of the following sentences is a proposition?

(i) $4+7=12$

$4+7=12$ is flase.

∴ It is a proposition

(ii) what are you doing?

This sentence is an Introgative

∴ It is not a proposition.

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(iii) $3^n \leq 81$, $n \in N$

This is true for $n = 1, 2, 3, 4$

∴ It is a proposition.

(iv) peacock is our national bird.

This sentence is true.

∴ It is a proposition.

(v) How tall this mountain is!

This is an exclamatory.

∴ It is not a proposition.

---

5) write the converse, inverse and contrapositive of each of the following implication.

(i) If x and y are numbers such that $x = y$, then $x^2 = y^2$.

converse statement:-

If x and y are numbers such that $x^2 = y^2$ then $x = y$

Inverse statement:-

If x and y are numbers such that $x \neq y$ then $x^2 \neq y^2$

contrapositive statement:

If x and y are numbers such that $x^2 \neq y^2$ then $x \neq y$.

(ii) If a quadrilateral is a square then it is a rectangle.

converse statement:-

If a quadrilateral is a rectangle, then it is a square.

Inverse statement:-

If a quadrilateral is not a square then it is not a rectangle.

Contrapositive statement:-

If a quadrilateral is not a rectangle, then it is not a square.

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6) Construct the truth table for the following statements.

(i) $\neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(ii) $\neg(p \wedge \neg q)$

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

(iii) $(p \vee q) \vee \neg q$

p	q	$\neg q$	$p \vee q$	$(p \vee q) \vee \neg q$
T	T	F	T	T
T	F	T	T	T
F	T	F	T	T
F	F	T	F	T

(iv) $(\neg p \rightarrow r) \wedge (p \leftrightarrow q)$

p	q	r	$\neg p$	$\neg p \rightarrow r$	$p \leftrightarrow q$	$(\neg p \rightarrow r) \wedge p \leftrightarrow q$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

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1) Verify whether the following compound propositions are tautologies or contradictions or Contingency.

(i) $(p \wedge q) \wedge \neg(p \vee q)$

p	q	$(p \wedge q)$	$(p \vee q)$	$\neg(p \vee q)$	$(p \wedge q) \wedge \neg(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

(ii) $((p \vee q) \wedge \neg p) \rightarrow q$

p	q	$\neg p$	$(p \vee q)$	$(p \vee q) \wedge \neg p$	$((p \vee q) \wedge \neg p) \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

(iii) $(p \rightarrow q) \leftrightarrow (\neg p \rightarrow q)$

p	q	$p \rightarrow q$	$\neg p$	$\neg p \rightarrow q$	$(p \rightarrow q) \leftrightarrow (\neg p \rightarrow q)$
T	T	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

* For $\wedge$ (and): T and T is T, others – F
* For $\vee$: F and F is T, others – F
* For $\rightarrow$: T and F is F, others – T
* For $\leftrightarrow$: T and T / F and F $\rightarrow$ T, others – F

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(iv) $((p \to q) \wedge (q \to r)) \to (p \to r)$

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$(p \to q) \wedge (q \to r) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Since the truth value of $(p \to q) \wedge (q \to r)$ is T.

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8)i) Show that (i) $\neg(p \wedge q) \equiv \neg \vee \neg q$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	F	T

From the table 4th and 7th columns are identical.

∴ $\neg(p \wedge q) \equiv \neg p \vee \neg q$

(ii) $\neg(p \rightarrow q) \equiv p \wedge \neg q$

p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg q$	$p \wedge \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

From the table 4th and 6th columns are identical.

∴ $\neg(p \rightarrow q) \equiv p \wedge \neg q$

9) Prove that $q \rightarrow p \equiv \neg p \rightarrow \neg q$

p	q	$q \rightarrow p$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	T	T

From the table 3rd and 6th columns are identical.

Hence $q \rightarrow p \equiv \neg p \rightarrow \neg q$

10 Show that $p \rightarrow q$ and $q \rightarrow p$ are not equivalent.

p	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

From the table $p \rightarrow q \neq q \rightarrow p$

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11) Show that $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

p	q	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$\neg q$	$p \leftrightarrow \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	F	T	F	T
F	F	T	F	T	F

From the table 4th and 6th columns are identical.

Hence $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

---

12) Check whether the statement $p \rightarrow (q \rightarrow p)$ is a tautology or a contradiction without using the truth table.

Given :- $p \rightarrow (q \rightarrow p)$

$p \rightarrow (q \rightarrow p) = p \rightarrow (\neg q \vee p)$ (∵ by Ex 12.17 $p \rightarrow q = \neg p \vee q$)

$= \neg p \vee (\neg q \vee p)$

$= \neg p \vee (p \vee \neg q)$ Commutative Law

$= (\neg p \vee p) \vee \neg q$ Associative Law

$= T \vee \neg q$ ∵ $\neg p \vee p = T$

$= T$ ∵ $T \vee \neg q$ is a tautology.

Hence $p \rightarrow (q \rightarrow p)$ is a tautology

(Again By 12.17)

---

13) Using truth table whether the statements $\neg(p \vee q) \vee (\neg p \wedge q)$ and $\neg p$ are logically equivalent.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg p \wedge q$	$\neg(p \vee q) \vee (\neg p \wedge q)$
T	T	T	F	F	F	F
T	F	T	F	F	F	F
F	T	T	F	T	T	T
F	F	F	T	T	F	T

From the table 5th and 7th columns are identical.

Hence $\neg(p \vee q) \vee (\neg p \wedge q) \equiv \neg p$.

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14) Prove $p \to (q \to r) \equiv (p \wedge q) \to r$ without using truth table.

We know that $p \to q \equiv \neg p \vee q$ (Ex 12.17) ——— (1)

$p \to (q \to r) = p \to (\neg q \vee r)$ by (1)

$= \neg p \vee (\neg q \vee r)$ by (1)

$= (\neg p \vee \neg q) \vee r$ (by associative Law)

$= \neg (p \wedge q) \vee r$ (by De Morgan Law)

$= (p \wedge q) \to r$ (by (1))

$\therefore p \to (q \to r) \equiv (p \wedge q) \to r$.

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15) prove that $p \rightarrow (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$ using truth table.

p	q	r	$\neg q$	$\neg q \vee r$	$p \rightarrow (\neg q \vee r)$	$\neg p$	$\neg p \vee (\neg q \vee r)$
T	T	T	F	T	T	F	T
T	T	F	F	F	F	F	F
T	F	T	T	T	T	F	T
T	F	F	T	T	T	F	T
F	T	T	F	T	T	T	T
F	T	F	F	F	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From the table 6th column and 8th column are identical.

Hence $p \rightarrow (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$

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Thank You



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